



IEJME

International Electronic Journal of
Mathematics Education

www.iejme.com
July 2007
Volume:2, Number: 2



International Electronic Journal of Mathematics Education

Volume 2, Number 2, July 2007

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International Electronic Journal of Mathematics Education

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Publication Frequency - IEJME is published three times a year in **February, July and October** for every year.

Published by:
**GOKKUSAGI LTD. STI.
TURKEY**

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ISSN 1306-3030

www.iejme.com

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International Electronic Journal of
Mathematics Education

Volume 2, Number 2, July 2007

www.iejme.com

**LINKING SCHOOL MATHEMATICS TO OUT-OF-SCHOOL MATHEMATICAL
ACTIVITIES: STUDENT INTERPRETATION OF TASK,
UNDERSTANDINGS AND GOALS**

John Monaghan

ABSTRACT. This article considers one class of high school students as they worked on a task given to them by a company director of a haulage firm. The article provides details of students' transformation of the given task into a subtly different task. It is argued that this transformation is interrelated with students' understandings of mathematics, of technology and of the real world and students' emerging goals. It is argued that the students did not address the company director's task. Educational implications with regard to student engagement with realistic tasks are considered.

KEYWORDS. Goals, Mathematics, Out-Of-School, School, Tasks, Understandings.

INTRODUCTION

This article reports on the work of one class of high school students in a project¹ which investigated ways of linking school mathematics to out-of-school mathematical activities. The class worked on one task, identifying when a vehicle is in a specific area, given to them by a director of a transport/haulage company. This article presents detail of what the students did and argues that they did not address the company director's task because they transformed the task and that this transformation is interrelated with their understandings and their goals.

The next section briefly considers the literature on the use of mathematics in and out of school and the words 'understanding', 'goals' and 'activity'. This is followed by a description of the wider project and this classroom project. Results for students working in groups are then detailed. The Discussion section attends to student transformation of the task and the interrelations of this with their understandings and goals. The final section considers educational implications.

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ISSN: 1306-3030

¹ See <http://www.education.leeds.ac.uk/research/cssme/outofschool.php> for the project description.

THE USE OF MATHEMATICS IN AND OUT OF SCHOOL

This section serves to position the classwork considered in this article with regard to related research and to introduce constructs employed in the Discussion section. It does not aim to present a comprehensive review of literature in the field.

30 years ago there appeared to be a general acceptance, in mathematics education literature, that the relation between mathematics, school mathematics and out-of-school mathematics was relatively unproblematic. Since then 'school mathematics' has been a focus of inquiry. For example, with regard to 'mathematics' and school mathematics, Chevallard's notion of 'didactical transposition' where "mathematics in research and in school can be seen as a set of knowledge and practices in transposition between two institutions, the first one aiming at the production of knowledge and the other at its study." (Lagrange, 2005, 69). The school mathematics considered in this article, however, is somewhat different from the school mathematics Chevallard considers as the students in the series of lessons were not formally studying mathematics; it is school mathematics by virtue of its situation, a mathematics classroom, but this, I argue later, is important.

With regard to the relation between school and out-of-school mathematics, most research shows a strong discontinuity between school and out-of-school mathematical practice. According to early work on situated cognition, e.g. Lave (1988), this discontinuity is a consequence of learning in and out of school being two distinct social practices. School mathematics, moreover, is often not suited to out-of-school practices: in some cases out-of-school problems are only apparently similar to school mathematics problems, but in reality there is a range of explicit and implicit restrictions which makes school methods unsuitable, and thus other methods are used (Masingila et al., 1996); in other cases (Scribner, 1984) work mathematics may appear to be simple, but there are no simple algorithms or methods to solve the problem and school-learnt procedures are of no use.

Despite this evident discontinuity some authors have observed an interplay between school and out-of-school mathematics. Pozzi et al. (1998) found cases of nurses looking for a mathematical explanation for 'simple' mathematical procedures used in their daily practice. Magajna and Monaghan (2003), in a study of technicians designing moulds for bottles, found evidence that, in making sense of their practice, the technicians resorted to a form of school mathematics. The focus of this article is a sort of converse to the focus of these studies: what sense do school students make of an out-of-school problem within their school practice?

Understanding, goals and activity

In this sub-section I briefly attend to how I use the words 'understanding', 'goals' and 'activity' in this paper. This is important for clarity and because these words are used in different ways in everyday and in academic discourse.

I use the word ‘understanding’ in its everyday sense. Davydov (1990) would refer to this as ‘empirical thought’, as opposed to ‘theoretical thought’, connecting features of reality rather than reproducing reality. When I refer to students’ understandings, I refer to how they connect features of their perceptions of reality rather than their theoretic reproduction of reality.

‘Goals’ is another everyday word which is used with specific but variant meanings in academic discourse. To Leont’ev (1978) goals are the *raison d’être* of an activity but Saxe’s (1991) emergent goals are little ‘must do’ things that come into being during activity and often, but not always, pass away. Emergent goals that do not pass away, however, can and do develop into conscious (and bigger) goals. This development of goals is an important aspect of the task-goal transformation considered in this article.

‘Activity’ is, of course, central to activity theory and, in education-speak, to student activity in classrooms. In this article I use it in the education-speak sense. There is, however, a tentative connection between these two senses of ‘activity’ (and, correspondingly, between Leont’ev’s and Saxe’s goals). Activity theory at present is often viewed with regard to activity systems (see Daniels, 2001, pp.83-94). Activity systems are large scale systems. A school qualifies as an activity system, a sequence of lessons does not. This remark applies to Leont’ev’s version of activity theory too, though Leont’ev acknowledges that activities are nested. There is a sense, however, that continued student work over several weeks, such as the student work considered in this article, represents an activity and that students’ goals, when they shape students’ subsequent work over a period of time, can be considered as the goals of this activity.

THE ‘LINKING SCHOOL MATHEMATICS’ PROJECT

The wider research, of which the classroom work reported on here was one of 20 classroom projects, worked with high school mathematics teachers to explore ways in which school mathematics might be ‘made real’ – be linked to out-of-school mathematical activities. Research questions were formulated around four interrelated research themes: the design of tasks; the use of resources; teachers’ perceptions; and student learning. Participating teachers were selected from a wider group of applicants on the basis that they perceived links between school and out-of-school mathematics to be a problematic issue and on their avowed intentions to undertake substantial classroom activities. Teachers were to work with the author and the project research assistant but were free to pursue activities of their own choice. The project ran for two school years.

This classroom project

The class teacher was involved in the wider project because the project reflected what he wanted to move on to in some of his lessons. He has special status as an ‘advanced skills’ teacher and his school is a high attaining state school which accepts students of all attainments.

The class, 29 students, is a Year 10 set 2, i.e. students are 14-15 years of age and are high, but not the highest, attaining students in mathematics in that year group.

The teacher had carried out two previous classroom projects within the wider project. For both he consulted with people in industry to find real life tasks. For this classroom project he consulted with a company director of a large transport company with a fleet of over 250 large refrigerated trucks which distribute frozen food from wholesalers to retailers. The problem, one the company director was actually considering at the time that he was approached, was how to obtain an automatic computer record of the time when each vehicle reaches and leaves a destination, e.g. a specific supermarket depot. The company had a map of each delivery site, so the problem reduced to knowing when a truck is within a specific geographical location. Each truck is fitted with a global positioning system (GPS) which reported the truck's location at a regular time interval to the company's central computer.

The teacher and I visited the company director at his work where he explained about the company and his problem to us. Following the meeting the teacher and I exchanged (and rejected) various ideas, many quite technical such as transforming latitude and longitude readings into map references, before the teacher suggested "Let's just let him present the problem and let the kids find their own solution." This is what happened.

The teacher allocated seven 60 minute consecutive lessons to this work. The company director would present the problem in the first lesson and the students would present their solutions in the last lesson. The students would work in self-selecting groups. The company director was present for the first, fifth and last lesson. The project research assistant joined the team for data collection. The teacher and the two university researchers were present in all lessons.

The company director made his presentation (an abridged transcript of the presentation is provided in Appendix 1). After the presentation the students formed into seven groups of three-five students. The remainder of the classwork is presented in Results section.

Methodology for this classroom project

The approach to researching this series of lessons was to record participants actions, motives and interactions. Meetings of the teacher and the company director were audio recorded. Meetings between the teacher and the author were written up. The teacher and the company director were interviewed prior to the lessons. Two digital video cameras positioned in different parts of the classroom recorded all the lessons. Six audio recorders were used: five on student group's desks and one with the author. During the lessons teacher, company director and student group interviews were conducted (group interviews in an adjacent room, others mainly in-class). The two researchers kept fieldnotes which were later written up. Students' hand written work

was photocopied and copies of their computer files were made. Still photographs of students' work were taken. The only known gap in data collected was that not every group was audio recorded in every lesson – there were five audio recorders for this and seven groups; missing data was partially reconstructed from fieldnotes. All audio recordings were transcribed. Physical and electronic files were compiled for each group and for whole class activity.

Data analysis was a lengthy and uneven process over many months. It used elements of a grounded theory approach (Strauss and Corbin, 1998) but this research did not attempt to follow all the rubrics of grounded theory. The data collected on these students' work could be analysed to support a number of related foci, e.g. collaborative group work or argumentation or student modelling. The focus on students' goals, understandings and their transformation of the task in this article was chosen because it is closely linked with the overarching aim of the project – to investigate how school mathematics may (and may not) be linked to out-of-school activities.

The main dataset for which detailed analysis was conducted were transcripts of students talking (student work in groups and interviews with students). Other data was re-examined but was not subjected to formal analysis. Selected transcripts were analysed by the researchers, first independently and then results were compared, codings refined and transcripts re-examined using: open coding à la Strauss and Corbin (1998), but stopping short of selective coding; Artzt and Armour-Thomas' (1992) cognitive – metacognitive categories; an interim category system. These transcript codings should be viewed as exercises which helped the two researchers make sense of the data. The interpretation made of this data was discussed with the class teacher but, in the end, is just one interpretation.

RESULTS

This section outlines what the seven groups of students did in these lessons and reports on their work as groups. This is done partly for reasons of economy of reporting and partly because the work of students in groups may be regarded as “taken as shared rather than shared” (Cobb et al., 2001, 119). I outline what each group did, how this approach arose and developed and the accuracy of their work². This section ends with an overview of what the students felt they learnt in these lessons. First, however, I comment on an aspect of ‘time’ and the role of adults in students' work.

Student activity (actions and discourse) can be considered over different time intervals: micro (seconds); meso (minutes); and macro (hours). This article considers all three levels but there is an analytic danger in doing this because micro, meso and macro analyses of student activity are likely to produce different results due to differing levels of adult interference (often not present at the micro level; always present at the macro level). Adult intervention was considerably less in this work, compared to much ‘traditional’ classwork, but I nevertheless guard against this danger by recording adult intervention in accounts of student work. A

² The accuracy of each solution was tested by writing a spreadsheet programme for each group's solution and testing it with four points which were, with regard to the depot's perimeter: ‘well’ outside, ‘well’ inside, ‘just’ outside and ‘just’ inside. The students were aware of this programme but did not know where the test points were.

simultaneous consideration of all levels is regarded as important for the focus of this article because, as shall be argued in section 5.3, of the influence of emergent goals (at the micro level) on goals (at the macro level).

Each group interacted with adults (teacher, company director and researchers). The adults claimed that they did not direct students to adopt specific approaches and attended only to practical problems with student approaches and help with specific mathematical techniques. As shall be discussed, however, adult intervention did appear to influence student work.

Student group work

The first half of lesson 1 was devoted to the company director's presentation. Thereafter the students split into groups. In the remainder of lesson 1 each group 'brainstormed' ideas.

Group A consisted of five girls who covered the map of the depot with five rectangles (see Figure 1). In lesson 1 they asked a number of questions:

Are we allowed to change it if we want?

....

Surely you can't actually do that without having like programmed stuff into the GPS.



Figure 1. Group A, five rectangles

A researcher said that they could make changes and emphasised that anything they did must be able to be programmed on to a computer. He also said that they could "find a simpler problem which is in the same ball park". In lesson 2 they started by covering the map with geometrical shapes that made up the polygon but quickly moved on to covering the map with rectangles and triangles which contained the area of the depot because "we wanna make it simple". Triangles were eventually rejected because they were complicated. Rectangles were orientated so that they were parallel to lines of latitude and longitude "to make calculations similar to each other". This solution strategy formed the basis for the remainder of their work and subsequent lessons developed computational methods based on using rectangles. Lessons 5

and 6 were devoted to preparing a Powerpoint presentation and “just doing some examples and some test points to prove that our theory works”. Technical discussions included how to calculate using Northings and Eastings and the accuracy required (they eventually decided to use “simple numbers just to show the basic idea”). Their approach was straightforward to program although they did confuse Northings and Eastings. Once the coordinates were corrected the solution was successful.

Group B consisted of four boys who covered the map of the depot with four rectangles (see Figure 2). During lesson 1 and the first 20 minutes of lesson 2 their main focus was on polygons and whether polygons had to be used. They started by considering whether they had to use the given polygon. One mentioned finding the “equation for a polygon” but another commented that that would be “horrendously complicated”. One mentioned “separate polygons or rectangles” but another commented “I don’t like polygons, can we use triangles?” and in the ensuing discussion several commented that triangles, rectangles and pentagons were all polygons but that polygons themselves were very complicated. The last reference to a “complicated polygon” was followed by “Can we just have a square that fits here?” After this comment they started working looking at squares, then bringing coordinates into squares and later (but still in lesson 2) widening their consideration to rectangles, settling on four rectangles. This solution strategy formed the basis for the remainder of their work. Subsequent lessons attended to the problem of not including the motorway within a rectangle “that’s such a problem” (lesson 3), developing computational methods and preparing a Powerpoint presentation (the group split into two pairs from lesson 3 onwards, one pair attending to technical issues and the other preparing the presentation). Their approach was straightforward to program although they did calculate some coordinates incorrectly. Once the coordinates were corrected the solution was successful.



Figure 2. Group B, four rectangles

Group C consisted of four girls who constructed a six step algorithm which determined whether a specific square metre was within the boundary of a shape:

- 1) Take the largest of x and y regardless of the sign.
- 2) See if there is a number that is greater than or equal to that number, with the same sign, on the same axis.
- 3) If not, then the co-ordinates are outside the area.
- 4) If yes, then see if there is a number that is less than or equal to the chosen number on the same axis.
- 5) If no, the co-ordinate is outside the square.
- 6) Take the second number in the co-ordinate and repeat steps 2 to 5.

Work on this algorithm started in lesson 2. In lesson 1 they considered “technical tags” at the entrance and exit of every depot to record the arrival and departure of lorries and said “It would be easier not to do it with the GPS”. In lesson 2 they discarded this and related approaches, e.g. barcodes on the sides of the vehicle, because “none of them really used much maths”. Their approach was focused on squares (subdivided into 10000 squares) because the shape can be subdivided into squares and “if you had 10,000 squares I don’t think it would take a computer long to check it, matter of seconds”. A lot of classroom time in lessons 3, 4, 5 and 6 was spent attending to the logic of the algorithm: the word ‘work’, as in “does it work” appears on every page of the transcripts of these lessons; when asked during their presentation “what was the main thing that directed you towards that solution?” the reply was “Just logic”. This theoretical solution could not be programmed as no coordinates were calculated. The students realised this but were satisfied with their method.

Group D consisted of three girls who enclosed the map of the depot with a quadrilateral (see Figure 3). In lesson 1 they had two ideas: having a box by the entrance to a depot where the lorries waited until the company computer registered their arrival; and covering the map of the depot with enclosing rectangles. In lesson 2 the rectangles idea was pursued and gradients were mentioned (earlier mathematics lessons had covered the equation of a straight line). In lesson 3 they considered the diagonals of their rectangles and focused on the equations of these diagonal lines. This led them to look at how to tell which side of a straight line (given by an equation) a given point was on. This determined the focus for subsequent work. Their reasoning was:

We have four equations of the lines and using an equality to it we can find out if it’s, which side of each line it’s on and so you work out an equation and you fit the corners to the four equations and if it works for every single one of them then it’s inside the shape.

Interview, lesson 5

Adults provided mathematical assistance but the method of determining which side of the line a point was on came from the students. They did not prepare for the presentation until lesson 6. Their solution contained an error with the equation of one of the lines but once this was corrected it was straightforward to program and was successful.

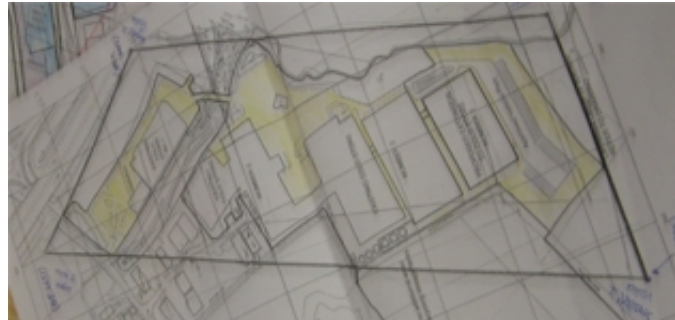


Figure 3. Group D, quadrilateral

Group E consisted of four boys who considered an octagon (see Figure 4). In lesson 1 they considered “something near the entrance of one so that when it drove past it registered there” but rejected it because “maybe the lorries wouldn’t be in the entrance long enough for it to register”. In lesson 2 they consider a square, “from (1, 1) to (2, 2) and if both points are between 1 and 2 then we know it’s inside the square”. The teacher asked them how they would know if the square was within the depot. They considered ways that the shape of the depot could be split up. The group focused on regular octagons, which they referred to these as “right angled triangles and squares”, because “any polygon can be made up of right angled triangles and squares”. Subsequent lessons focused on a specific octagon with simple coordinates which were not related to map references of the depot. In lesson 4 the teacher encouraged the group to start considering the presentation and two students took this as their main focus of activity. The solution used equations of lines to represent the sides of the octagon. It was possible to program their solution but: (i) simple coordinates rather than map references had to be used; (ii) all points within the square enclosing the octagon (produced from extending the hypotenuses of the triangles) registered as being within the shape.

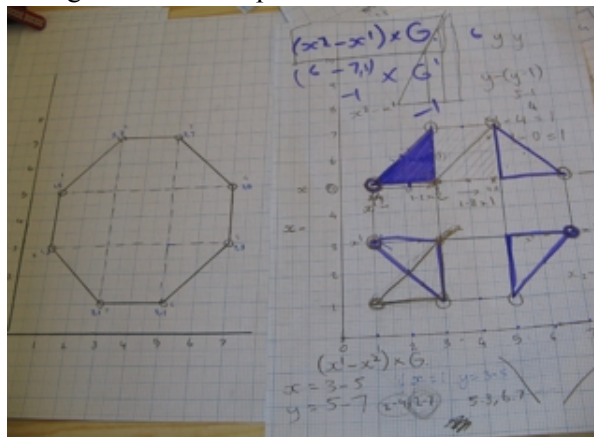


Figure 4. Group E, octagon

Group F consisted of five boys who covered the map of the depot with seven circles (see Figure 5). The group did not initially see why the company director had a problem “Can the lorry drivers just like phone say when they’re in the depot?” and “When the GPS reports back ... it

kinda lights up the area and makes a big beeping noise so we know it's in there". A researcher eventually persuaded the students that neither solution was desirable because 24 hour automation was required. The students then wondered if there was "formula" for the shape but did not know how to proceed. In lesson 2 they started thinking about "four different shapes ... it would be easier to work out if a point was in each shape ... it needs more development thought 'cos it's like complicated because latitude and longitude lines are not like on the rectangle". This led to their focus on circles (which is, mathematically, quite sophisticated³) because "you don't have the problem of it being over, like crossed over, where I mean however you turn a circle it's always gonna be the same shape on the grid", i.e. a circle does not need to be orientated to be parallel to latitude/longitude. Thereafter circles were the focus of their work. The group struggled with relevant school circle maths (pi, arc, area, 360 were mentioned) and the teacher provided mathematical assistance, $(x-a)^2 + (y-b)^2 = r^2$, which he felt took its cue from student-generated ideas. This was new mathematics to the students but they were quickly able to apply it to map coordinates. The teacher encouraged the group to prepare for the presentation at the start of lesson 4 but it was not until midway through lesson 5 that they began this as they were engrossed in working out the details, to 6 significant figures, of what they proudly called their "360 theory". Their solution was tailored to the company depot but they made recommendations for further use, "We suggest that if this theory is used, one or two sizes of circle should be used instead of several to save time and complications." The solution provided by this group was mathematically correct and easy to program. It required no alterations to ensure it worked correctly.

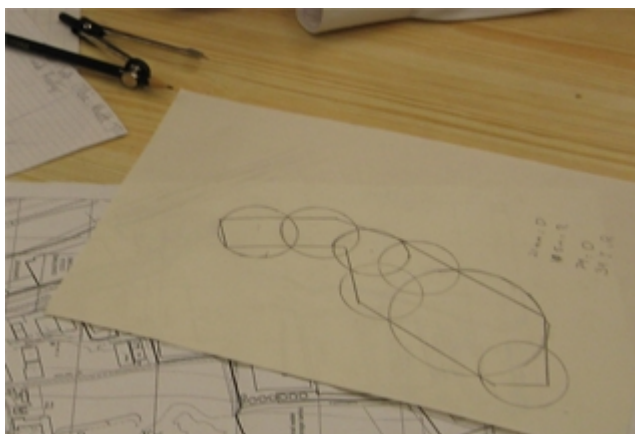


Figure 5. Group F, seven circle

Group G consisted of four girls who covered the map with lots of thin rectangles (see Figure 6). They had two ideas in lesson 1, split the polygon into simpler shapes and to have an oval shape with a central point which was somehow symmetrical. But in lesson 2 they stated that they did not really see what the problem was "you've already got the polygon ... points so can't you just use the polygon ... the points that you've got?" An extended dialogue with a researcher led the students to see the problem. Nevertheless the group remained uncertain as to how to

³ I think several of the students realised that I was impressed when they explained this to me. I mention this because this could have influenced their goals/motives. This is a very difficult matter to state with any certainty but "this guy thinks we're clever" could have encouraged them to continue along these lines.

proceed until lesson 3, where ‘straight lines’ were considered and developed as an approach:

Like just have a simple straight line around the edge ... Or you could like have lines across the actual thing
...

.....

... lots of lines all over it just so that you could like see how [tape unclear]

.....

So you put the co-ordinates of the line in

This is taken further but is then dropped as an idea. They reconsider their oval shape but know another group is using circles and reject their oval approach because it is not original. In lesson 4 they still do not have a plan. They discuss using rectangles and this connects with their lesson 3 ‘straight lines’ idea – instead of using straight lines they could use “loads of thin rectangles” and this idea is pursued for the remaining lessons. The group works on the details but time is short and although it should be programmable (and overall correct) they are far from completing calculating the coordinates for all the rectangles by the end of lesson 6 and do not make a presentation.

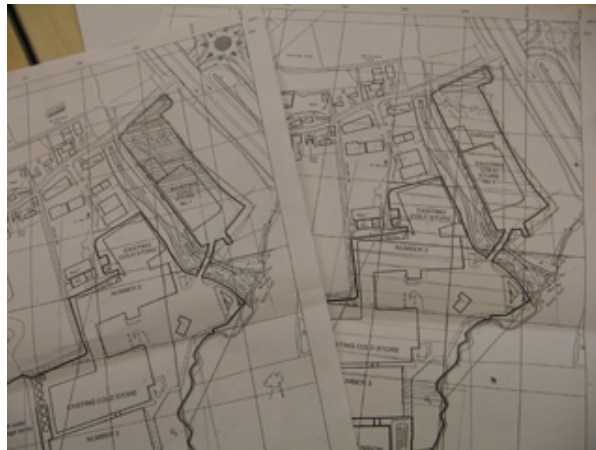


Figure 6. Group G, thin rectangles

What the students felt they learnt in these lessons

I briefly comment on what the students felt they learnt in these lessons as I comment on this later. In the lesson following the presentation 25 students (all who were in that lesson) were asked to write what they felt they learnt in these lessons. The mean word length of their responses was 40 words. Their responses were coded using open coding. All students listed more than one thing they learnt. Their responses are summarised below.

The largest category (19) was ‘use of mathematics’. Comments to the effect that they learnt that mathematics can be used, or arises, in real life. In two cases students seemed a little surprised at this:

I have learnt that mathematics can be used in everyday situations and still be helpful.

I never really thought that coordinates could be used in a real life situation.

Then came learning about business (9):

given me an insight into the way that businesses work

including mathematics can be used in business

how closely connected it can be in the efficiency of running a business.

Other categories with frequencies greater than 1 were: GPS (5); team work (5); different ways to attack a problem (4); specific mathematics learning (4), e.g. “equation for a circle”; about mathematics (3), e.g. “mathematics can be computerized”; problem solving (2); and understanding over time (2).

DISCUSSION

In this section I consider the following three-part proposition.

- 1) These students transformed ‘the task’ into ‘their task’.
- 2) What students wanted to do (their goal) was interrelated with where they were (a mathematics classroom), their interpretation of the task and their understandings (of mathematics, of technology, of everyday life).
- 3) Students’ goals and their interpretation of the task interacted and developed over time; there was a task-goal trajectory.

Students’ interpretations of the task set

I restate the task the company director gave the class. He concluded his presentation with,

What we need to be able to do is to be able to identify for a particular latitude and longitude whether it is within the boundary of that shape The best solution ... so far is to use a polygon ... it very difficult to identify when a point is within a polygon.

He identifies a point being within the boundary of a shape with a point being within a polygon and the other adults present interpreted his task as determining when a point is inside a polygon. None of the final student solutions, however, used polygons. With the exception of the group that focused on octagons, all the groups, in one way or another, enclosed a polygon in a set of simpler shapes. In an in-class interview during lesson 5 the company director commented that it “will be easier if we just, if we can approach it on the basis of a polygon as opposed to breaking it down into shapes”.

Why did the student solutions not address the polygon problem? Well, polygons are

somewhat complex shapes and students from all groups and several adults mentioned, at one time or another, the need to simplify the problem. Only Group B pursued polygons for some time: during lesson 1 and the first 20 minutes of lesson 2. They did not mention polygons again except during the lesson 2 interview (after the focus on squares had emerged) where they rationalised that the company director “didn’t want complex shapes so if you break it down into simple ones instead of having a big polygon which is complex”. Such development in interpretation is an example of what I mean by *they transformed ‘the task’ into ‘their task’*.

The complexity of polygons also featured in the considerations of Group A but this was not the only reason for task transformation. Appendix 2 summarises the development of the groups highlighting their ideas, foci, goals and interpretations of the task. To further understand these transformations of the task I turn to students’ goals and understandings.

Goals and understandings

Motives and goals are inextricably linked. Goals are what one wants to do and motives are why one wants to do it. Goals were easier to locate in students’ discourse than motives. For this reason I focus on goals. In this sub-section I consider students’ goals in relation to the situation of the activity (a mathematics classroom) and their understanding of mathematics, of technology and of everyday life.

In a mathematics class

This out-of-school activity was conducted in-school and this matters. These students were in a mathematics class and they were, obviously, aware of this. Students expect to do ‘school mathematics’ in a mathematics class. This was implicit in some cases, e.g. Groups D and E used equations of straight line graph, a topic studied in mathematics classes just prior to this work. In the discourse of Groups A and C the expectation to use school mathematics was explicit, e.g.:

Shall we draw this in a graph?

Why?

Cos that’s normally what you do with co-ordinates

Group A, in-class discourse, lesson 2

We had lots of ideas but then we realised none of them really used much maths

Group C, interview during lesson 2

To spell out what may be obvious I draw attention to “Shall we draw this in a graph” suggests a goal and “none of them really used much maths” suggests that some goals were not

pursued and that suggesting or avoiding these goals was related to students' understandings of mathematics and of being in a mathematics class. There was, however, no clear indication that what Groups B, F and G did was related to their work being done in a mathematics class.

In a workplace, however, when mathematics is used, it is not used for the sake of using mathematics, it is used to get something done (a goal). There is also the issue of 'the mathematics' used: in the workplace 'invisible' mathematics may be more relevant in an activity than 'visible' mathematics (Noss et al., 1996) but it is likely that the Group C student (see quote immediately above, "none of them really used much maths") thought of 'mathematics' as 'visible mathematics'.

Understandings of technology

Technological considerations were central to the task set as a central computer would receive a GPS reading and register whether a vehicle was in a depot and students' understandings of technology interrelated with their developing goals. I focus on two student understandings of technology which impinged on their goals/solution strategies, perceptions that computers 'know' and perceptions of their calculating prowess.

Many students in England are familiar with technology which will tell them, unprompted, say, if there is a wireless network in the area, and doing things for them, e.g. automatic correction of spelling. Without knowledge that there are programs behind such features it is not unreasonable for students to assign intelligence to technology to do things beyond than that which it has been programmed to do. Two groups, C and G, did not initially see what the problem was because they thought a computer would 'know' when a vehicle was in a depot:

We didn't see why they had a problem because it shows on the board ... there's the little dot⁴. Can't you just have like a shape of the depot marked out and then you know when it's gone in.

Group C, interview during lesson 2

We were wondering what the actual problem is because you've already got the polygon ... surely they just need to update it so that they can just type in the co-ordinates and it can know.

Group G, interview during lesson 2

One possible goal in such cases is to find out what the problem is. Another possible goal is to appear to be doing something (because you are supposed to be doing something). It is difficult to tell from the transcripts quite what the goals of Groups C and G were in this period.

None of the 29 students in the 30+ hours of audio recordings for these lessons appeared to consider data processing limitations of the company's central computer. This is not unreasonable for the company director did not mention this in his presentation and it is popular

⁴ The "little dot" is the dot representing a vehicle in the company director's real-time computer display (see Appendix 1).

mythology that ‘computers can do it in a nanosecond’. This is a case of technological considerations not influencing students’ goals when, to the company director, such considerations would have been apposite. Indeed, during lesson 5 he commented, regarding all student solutions, that “it takes a lot of computing power to get through that”⁵.

Everyday understandings

There is a case for students employing their knowledge of the world in real life problems but this assumes that this knowledge is pertinent to the problem at hand. This is not always the case as the following exchange between students shows:

They could have heat sensors or maybe pressure actually then [sentence unfinished]

Like on my dad’s car where they could tell whether someone’s sat on the seat or not

Group C, in-class discussion, lesson 1

Or the following student who was responding to a researcher asking why other ideas of the group had been rejected:

We were thinking about it but they weren’t really, I dunno they weren’t original and they didn’t kind of appeal to us Group G, interview, lesson 5

Originality may be useful in, say, art and design but it was not important to the solution of this problem. This desire for originality appears to be linked to the situation, a school, where competition between the groups may be expected.

Understandings of technology were often intertwined with understandings from everyday life. For example, an early idea of Groups B, C and E was to have an electronic scanner at a depot’s entrance (electronic scanners are common in clothes shops in England):

We had an initial idea of putting a single line with two co-ordinates over the entrance and then when it passed it would register Group B, interview during lesson 2

like having an infra red scanner like you have those tabs on clothes in shops and if you walk through the barriers it beeps at you Group C, interview during lesson 2

This is a sensible idea but, as the company director pointed out, he cannot insist that his clients have a scanner at their depots’ entrances and there may be more than one entrance.

Of course a person’s prior understanding will always influence their problem solving (in any domain) and these students engaged on this task were not exceptional in this. The point of this sub-section, however, is that the understandings of these young people (with no in-depth knowledge of the company, GPS or the working of a central computers) in a mathematics classroom interacted with their goals in ways that did not appear appropriate to a solution to the task given by the company director.

⁵ The extreme case, with regard to the number of calculations, was Group C whose solution strategy considered each square metre of a depot. Imagine this information being processed by a computer every three minutes, for every depot for over 250 lorries and you may begin to see that it is not feasible.

Task-goal trajectory

In this sub-section I suggest that a relation sometimes holds between emergent goals and the goal(s) of a task and that the former can develop into the latter. I first attempt to clarify the relationship between task and goal.

With regard to school mathematics a task is something the students do, such as doing the questions in a section of a textbook. Chevallard (1999) views tasks as related to techniques⁶ (used to solve tasks), technology (the discourse used to explain and justify techniques) and theory. Mathematics educators may judge a given task as ‘good’ or ‘bad’ but to the student it is something they do (because they are in a mathematics class). Why, their motive, and how students engage in a task may vary considerably, from wanting to complete the exercise before the end of the lesson to avoid homework, to having genuine interest in the mathematics in the task. Students with such extreme motives will have different goals and these will determine different actions. Out-of-school tasks are similarly things to do but in workplace settings (such as the director’s company) it is expected that one does it quickly and correctly and this is an important and sometimes the only goal, though related goals such as ‘finishing by 5.00’ may co-exist. So tasks and goals may be extensionally identical or distinct.

The summary of the development of the groups’ ideas, foci, goals and interpretations of the task in Appendix 2 suggests that, with each group, goals emerged which became the focus of subsequent work, i.e. emergent goals contributed to the transformation of the task (and when this happened the emergent goal was no longer an emergent goal but became the goal of the transformed task). I relate, below, the development of Group D with regard to these ideas but a similar narration is possible for the other groups.

In lesson 2 gradients were mentioned and the students returned to gradients in lesson 3 where they considered the equation of the diagonal line of a rectangle. This led them to consider how one can determine which side of a straight line (given by an equation) a specific point is on. This was an emergent goal but it became the goal of a new task which in their words was:

we can find out if it’s, which side of each line it’s on and so you work out an equation and you fit the corners to the four equations and if it works for every single one of them then it’s inside the shape.

There is a sense in which redefining tasks and setting new goals is unexceptional in student work such as this: problem solving, as meta-cognitive activity, can be characterized by an implementing-evaluating-refining cycle (Artzt and Armour-Thomas, 1992) and investigative school mathematics in England has been influenced by texts such as Mason et al. (1982) which encourage ‘open’ approaches. But work-based problems expect solutions that solve the given problem and, in my opinion and in the opinion of the company director, none of the student solutions solved the given problem; they transformed the problem/task into related but different tasks.

⁶ The Group A student quote on graphs in section 5.2 “Cos that’s normally what you do with co-ordinates” suggests that these students see school mathematics in a similar way to Chevallard.

EDUCATIONAL IMPLICATIONS

The upshot of the discussion above is that these students did not address the company director's task because their task-goal trajectories resulted in related but transformed tasks. This was only one school-based out-of-school project and care must be taken to not generalise from one case but student work in other activities in the wider project resulted, in my opinion, in similar task-goal trajectories. This raises the question as to whether out-of-school activities can be conducted in school as they would be in the work place. If the answer to this question is 'no', then does this mean there is no worth in doing such tasks in the classroom? I think not. The students clearly enjoyed the work⁷ and most came up with reasonable solutions, in several cases exceptionally good school-based solutions. Further to this they felt they learnt many different things and most of them (19 out of 25) felt they learnt that mathematics can be used in out-of-school. Real tasks like this can also encourage 'rich equal exchange' (Sahlberg and Berry, 2003) in that all can contribute and the team outcome is likely to be much better than any one student could have done.

My final thoughts concern the role of adults/teachers in school work of this nature. Adults in these lessons, though they sometimes pointed out problems with students' ideas, were quite tolerant with regard to students' ideas. The adults let the students develop their task transformations. Perhaps adults/teachers in lessons of this kind should be less tolerant of students' ideas which are not seen as leading to practical solutions.

⁷ 12 of the 25 students stated this in their comments on 'what have you learnt' even though they were not asked to comment on whether they enjoyed it.

ACKNOWLEDGEMENTS

This work was supported by the Economic and Social Research Council, Award Ref. No. RES-000-22-0739. I would like to thank Louise Sheryn, the students and the (sadly anonymous) teacher and company director for giving me the chance to examine a fascinating series of lessons.

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APPENDIX 1

The problem given by the company director. Extracts from a 22 minute presentation. ‘...’ indicates missing text.

We have a business problem which is mathematically related and which we hope you will be able to come up with some good ideas about. ... We distribute frozen food on behalf of other people ... to supermarket depots all over the country ... we run 24 hours a day ...

A trailer tracking project ... to identify the positions of every one of our vehicles on the road ... The location of vehicles helps with our planning ... it reads positions constantly using GPS ... needs to record information about the time you arrived and departed from depots, how far we have travelled, the temperatures in the vehicle and other events as well ...

At this point the company director presents a real-time computer display on a large screen which shows a road map with a dot. The dot moves, indicating a new GPS reading.

So it is moving down the A1 [a major road], let’s see where it goes now, ... right, it is here. One of the key things that this system has to be able to do is to identify when a vehicle is at a particular location, so the problem is to develop a method for the computer system to calculate from a GPS position when the vehicle is at a particular depot, and the example that I have picked out is our depot at ...

The company director brings the map below onto the large computer screen.



This is a map of the site and our land is highlighted. You see how irregular a shape it is. What we need to be able to do is to be able to identify for a particular latitude and longitude whether it is within that shape within the boundary of that shape ... we have tried various things ... we put a rectangular boundary around it and if you make it a rectangle ... but the problem is it is a very limited method of working it out, so we have a particular problem because

we have the highway running very close by and if a vehicle gives us a point as it is passing by we don't want a false reading, we don't want it to say it has arrived in the depot when it hasn't. ... Another possibility is a circle which is a slightly better solution, it is more complicated to work out when it is in a circle but it is not impossible but again you are limited, we might take in the highway with a circle. The best solution ... so far is to use a polygon which is perhaps the most sophisticated of the solutions, very flexible and will give us a reliable reading I hope, but the problem is that we are finding it very difficult to identify when a point is within a polygon.

APPENDIX 2

Group A initially attended to the polygon, then split the polygon into triangles and rectangles. Triangles were rejected because they were complicated. Rectangles were examined in lesson 2 and formed the basis for the remainder of their solution orientated work. Lessons 5 and 6 were largely devoted to preparing their presentation.

Group B's development was very similar to that of Group A. Considerable time was spent on "complicated" polygon. The focus then shifted to squares, which are easier, and then rectangles (lesson 2). From lesson 3 onwards they divided 'technical' and presentation preparation work between themselves.

Group C started with technical tags in lesson 1, rejected this and moved on to their algorithm, which formed the basis for all their subsequent work.

Group D had two ideas, a box by the entrance and enclosing rectangles. In lesson 2 the rectangles idea was pursued; gradients were mentioned. In lesson 3 consideration of the equation of the diagonals of rectangles led them to focus on which side of a line a given point was on, which determined the focus for subsequent work. Preparation for the presentation was in lesson 6.

Group E initially considered "something near the entrance" and moved on to consider squares and splitting the shape of the depot up in lesson 2. These two ideas led them to consider octagons which, together with preparing for the presentation from lesson 4, formed the basis for subsequent work.

Group F initially considered an electronic register. A consideration of 'four shapes' in lesson 2 led them to focus on circles which formed the basis for subsequent work. They resisted teacher encouragement to prepare for the presentation until lesson 5.

Group G had two initial geometric ideas but rejected these. They spent lessons 3 and 4 trying and rejecting ideas. The 'thin rectangles' idea forms as a fusion of two earlier ideas in lesson 4. Presentation considerations were minimal. Originality was cited as a factor in their foci.

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International Electronic Journal of
Mathematics Education

Volume 2, Number 2, July 2007

www.iejme.com

**DIDACTIC EFFECTIVENESS OF MATHEMATICAL DEFINITIONS
THE CASE OF THE ABSOLUTE VALUE**

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ABSTRACT. Quite often a mathematical object may be introduced by a set of equivalent definitions. A fundamental question is determining the “didactic effectiveness” of the techniques for solving a kind of problem associated with these definitions; this effectiveness is evaluated by taking into account the epistemic, cognitive and instructional dimensions of the study processes. So as to provide an example of this process, in this article we study the didactic effectiveness of techniques associated with different definitions of absolute value notion. The teaching and learning of absolute value are problematic; this is proven by the amount and heterogeneity of research papers that have been published. We propose a “global” study from an ontological and semiotic point of view.

KEYWORDS. Definition, Partial and Holistic Meanings, Cognitive Effectiveness, Absolute Value, Implicative and Hierarchic Statistical Analysis.

**MATHEMATICAL EQUIVALENCE VS. DIDACTIC EQUIVALENCE OF
DEFINITIONS**

One goal for the teaching of mathematics is to channel everyday thinking towards a more technical-scientific thinking at an earlier stage, as a mean for overcoming the conflicts between the mathematics (formal) structure and cognitive progress. Defining mathematical objects involves “more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition” (Vinner, 1991, p.65). This fact justifies the great number of papers in Mathematics Education focussed on mathematical definition (Linchevsky, Vinner & Karsenty, 1992; Mariotti & Fischbein, 1997; De Villiers, 1998; Winicki-Landman & Leikin, 2000; etc.). These researchers refer to specific aspects of the definition: *contexts of use* (geometric, analytical, algebraic, etc.), *mathematical objects* (differentiation, tangent, absolute value, etc.), *properties of the definition*

(minimalism, elegance, consistency, etc.) or relations to other mathematical processes (description, metaphor, model, theorem). We are interested in justifying the large gap between the mathematical equivalence of two definitions of the same object and their epistemic, cognitive or instructional equivalence, that is to say, their didactic equivalence. This theoretical analysis will be carried out for the specific case of the absolute value notion (AVN).

From the viewpoint of Mathematics Education, one fundamental question consists of determining the *didactic effectiveness* of problem-solving techniques associated with a mathematical definition; this effectiveness is assessed by taking into account the *epistemic*¹ (field of applicability of the techniques and mathematical objects involved), cognitive (effectiveness and cost in the use of the techniques by the individuals) and instructional (amount of material resources and time required for its teaching) dimensions. Hence, with the expression *didactic effectiveness* we refer to the articulation of these partial types of effectiveness in an educational project. However, this concept poses us difficult research questions, such as the following:

- Is there a definition that minimises the cognitive and instructional cost of use of resources, maximises the individuals' effectiveness in the specific problem field and facilitates adaptation to new problems?
- Is it possible to classify the definitions according to their scope or generality (field of applicability), their *mutual implication* (one definition may be obtained deductively from another one) or their role *within the institutional practices* (social, cultural, conventional)?

The first step in answering these questions for the case of AVN is determining the nature of this notion and accepting the complexity of objects and meanings that explicitly refer to it. In this article we present a systematic analysis of AVN nature, emphasising the role of its definitions in mathematical work, which is based on the ontosemiotic approach to mathematics cognition (Godino, 2002). Below we present a synthesis of this theoretical framework.

The ontosemiotic approach to mathematics cognition

This is a theoretical framework developed by Godino, Batanero and collaborators for jointly analysing mathematical thinking, the ostensives that support it and the situations and factors conditioning its development. This approach is named ontosemiotic given the essential role it attributes to language and the categorisation of the different types of objects that emerge in mathematical activity. It conceives mathematical language in a wide sense, that includes a variety of expressions and consider as mathematical object any kind of real or imaginary entity to which we refer when performing, communicating or learning mathematics.

This theory starts from the notions about institutional and personal meaning of mathematical objects (Godino and Batanero 1998), where, basing on pragmatic assumptions, they focused on institutional mathematical knowledge, without forgetting the individuals, which

¹ The term 'epistemic' refers to institutional knowledge in this work.

are the focus of educational effort. In that work, they conceive the meaning of a mathematical object (e.g. real number, function, etc.), in terms of the “systems of practices carried out to solve certain types of problems”. A practice is “any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems” (Godino and Batanero, 1998, p. 182). The systems of practices are composed by operative (actions, problems) and discursive (notions, propositions, arguments) facets, which are mutually interrelated and complementary. These two facets are articulated by language that allows us to create new mathematical objects and communicate the procedures and meanings involved in mathematical activity.

These operative and discursive practices can either be attributed to individuals – and then Godino and Batanero speak of personal meaning of the object, or be shared in an institution - and then they consider the corresponding institutional meaning. An institutional (personal) object is an emergent of the system of social (personal) practices linked to a problem field.

The theory of institutional and personal meanings is extended in Godino, Batanero and Roa (2005) including the notion of semiotic function and a category of mathematical objects. They propose as primary entities to describe mathematical activity the following:

- (1) *Language* (terms, expressions, notations, graphics).
- (2) *Situations* (problems, extra or intra-mathematical applications, exercises,...).
- (3) *Subjects' actions* when solving mathematical tasks (operations, algorithms, techniques, procedures).
- (4) *Concepts*, given by their definitions or descriptions (number, point, straight line, mean, function,...)
- (5) Properties or attributes, which usually are given as statements or propositions.
- (6) *Arguments* used to validate and explain the propositions (deductive, inductive, etc.).

The main purpose of this article is remarking that the ontosemiotic relativity of mathematical objects and meanings applies both to the institutional and personal frames and to the internal mathematics contexts of use. With this aim we introduce the notion of *partial meaning* for a mathematical object, which is interpreted as the “subsystem of practices” linked to a particular definition thereof. We also introduce the notion of *holistic meaning*, which is understood as the ontosemiotic network made of the different partial meanings. These new theoretical notions will be developed for the particular case of AVN and will allow to progress towards a coherent articulation of the pragmatic and realistic visions of mathematics.

In the following section, a set of research problems are described to show the difficulties of teaching and learning the absolute value notion (AVN). From these investigations we deduce the psychological and pedagogical complexity of the AVN, but none of them tries to integrate the meanings attributed to this notion in the different contexts of use and characterise its ontological

and semiotic complexity. Therefore, in relation to the AVN, the first objective of this paper is to show how the different meanings (associated with the contexts of use) are synthesized in different definitions and how these definitions and those meanings can be structured.

Then, we introduce a sample of different definitions of the AVN and, backed by the calculation of the solutions of a linear equation with an absolute value, we indicate how these definitions condition mathematical practices. After, we present a way to structure the objects and meanings associated with the AVN and we describe its holistic meaning. Next, we report a teaching experiment whose purpose is to analyse, from a cognitive viewpoint, the heuristic power of two of the different partial meanings of absolute value identified. Later, we highlight some micro and macrodidactic implications and, finally we provide a brief synthesis and some theoretical implications.

NATURE OF THE NOTION OF ABSOLUTE VALUE

The teaching and learning of the AVN are problematic. This is proved by the amount and heterogeneity of the research papers that have been published. Gagatsis and Thomaidis (1994) describe succinctly the historical evolution of the knowledge about absolute value. They also determine the processes for adapting that knowledge in Greek schools and interpret the students' errors in terms of *epistemological obstacles* (linked to the historical study) and *didactic obstacles* (related with the processes of transposition). More recently, Gagatsis (2003, p.61) argues, on the basis of empirical data that the "obstacles encountered in the historical development of the concept of absolute value are evident in the development of students' conceptions".

From a professional point of view, Arcidiacono (1983) justifies an instruction of the AVN based on the graphic analysis on the Cartesian plane of piece-wise linear functions and Horak (1994) establishes that graphic calculators represent a more effective instrument than pencil and paper for performing this teaching.

On the other hand, Chiarugi, Fracassina & Furinghetti (1990) carried out a study on the cognitive dimension of different groups of students faced with solving problems that involve the AVN. The study determines the need for research that will allow the errors and *misconceptions* to be overcome. On her part, Perrin-Glorian (1995) establishes certain guidelines for the institutionalisation of knowledge about the AVN in arithmetical and algebraic contexts; so she argues the central function of the teacher's didactic decisions in the construction of the AVN, that must take into account the students' cognitive restrictions and must highlight the instrumental role of the AVN.

All these research papers implicitly consider the AVN to be transparent, i.e. they do not see this object as problematic from the epistemological viewpoint. From an ontological and semiotic approach of mathematical cognition and instruction (Godino, 2002; Godino, Batanero

and Roa, to appear) it is necessary to theorise the notion of meaning in didactics. These authors start off with the elements of the technological discourse (notions, propositions, etc.) and conclude that the meaning of a mathematical object is inseparable from the pertinent systems of practices and contexts of use.

DEFINITIONS OF ABSOLUTE VALUE

Mathematical definitions and propositions are the most visible part of the “iceberg” that ultimately constitutes the anthropological and cultural reality of mathematics. Mathematical definitions are a discursive component of the systems of mathematical practices, which from an ontogenetic point of view depend on and are “subsequent” to operative practices. But the definitions interact with problems, techniques, and previously established rules in a complex and recursive way, and then pose new questions and new systems of practices.

In this section, we introduce some definitions of the AVN associated with different contexts of use, showing different mathematical texts and describing the relationships that are established between the definitions given. We also briefly indicate how these definitions, as objects emerging from the different subsystems of practices, condition the discursive rules for stating and validating properties and the operative instruments for problem solving, as well as the type of actions and arguments that are acceptable². We exemplified this fact by means of the resolution of a linear equation with absolute value ($|x - 2| = 1$). This equation can be solved of diverse forms. In this section, we shall show how the definitions of absolute value determine specific techniques for resolution. This brief study will allow us to justify the structuring of the objects (definitions, partial meanings, system of practices elements, etc.) associated to the AVN and the relevance of the “holistic meaning” notion.

Arithmetical definition

In the arithmetical context, the AVN represents a rule that *leaves the positive numbers unchanged and changes the negative numbers into positive ones*.

“The absolute value of x , denoted by $|x|$, is defined as follows:

$$|x| = x \text{ if } x > 0; \quad |x| = -x \text{ if } x < 0; \quad |0| = 0$$

Thus, the absolute value of a positive number or zero is equal to the number itself. The absolute value of a negative number is the corresponding positive number, since the negative of a negative number is positive.” (Leithold, 1968, p.10).

Then, in order to solve the equation $|\dots - 2| = 1$, we reason as follows: “the absolute value of a number is 1, then this number is 1 or -1 ; What number, when subtracting 2 from it, gives 1?, What number, when subtracting 2 from it, gives -1 ?”. The formalisation of this method may

² Another essential question is justifying each definition, that is, finding the systems of operative and discursive practices that have generated the culturally accepted definitions and the language used to represent the networks of objects with which the notion is related. Given the length restriction, we do not enter in this reciprocal question in this article.

be expressed in the following way:

$$|x-2|=1 \Rightarrow \begin{cases} x-2=1 \Rightarrow x=3 \\ x-2=-1 \Rightarrow x=1 \end{cases}$$

Piece-wise function definition

In an analytical context, formally, the AVN is often introduced using a piece-wise function in \mathbf{Q} and, by extension, in \mathbf{R} . Then, in order to solve the equation $|x-2|=1$, we reason as follows: In general, the definition of AVN is

$$|()|= \begin{cases} () & \text{if } () \geq 0 \\ -() & \text{if } () < 0 \end{cases}$$

Then:

$$|(x-2)|= \begin{cases} (x-2) & \text{if } x \geq 2 \\ -(x-2) & \text{if } x < 2 \end{cases}$$

Therefore:

$$|x-2|=1 \Rightarrow \begin{cases} x-2=1 (x \geq 2) \Rightarrow x=3 \\ -(x-2)=1 (x < 2) \Rightarrow x=1 \end{cases}$$

Maximum function definition

The classic definition of absolute value, as a basic notion for the foundations of mathematical analysis, is sometimes reformulated in terms of the maximum function: $|x| = \max\{x; -x\}$. This definition of the absolute value in terms of the maximum function can be extended to complete ordered vectorial spaces in terms of the most general notion of “supreme”. In fact, it is possible to define functionally in any ordered field³ (non empty) the absolute value or, more precisely, the abstract notion of “measurement or norm”.

Then, in order to solve the equation $|x-2|=1$, since $|x-2| = \max\{x-2; -(x-2)\}$, the proof is reduced to the process performed according to the piece-wise function definition.

Compound function definition

Finally, it is easy to demonstrate that: $|x| = +\sqrt{x^2}$ (Mollin, 1998, p.47). Then, in order to solve the equation $|x-2|=1$, we reason as follows:

$$\begin{aligned} |x-2|=1 &\Rightarrow \sqrt{(x-2)^2}=1 \Rightarrow (x-2)^2=1 \Rightarrow x^2-4x+3=0 \Rightarrow \\ &\Rightarrow x = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} = \begin{cases} x=3 \\ x=1 \end{cases} \end{aligned}$$

³ In the number sets \mathbf{Q} and \mathbf{R} an order relation can be defined.

As we have previously pointed out, we only present a sample of all the possible definitions. Other characterizations of absolute value can be based, for example, on the notion of vector norm (Mollin, 1998, p.47). What is essential in this paper is not the number or the particular types of definitions used, but the fact that several definitions for the same mathematical notion coexist, that each of them involves a network of related objects and that all these definitions interact. All of this, together, lead to a high semiotic complexity.

Short analysis of calculation solution techniques of $|x - 2| = 1$

The aforementioned definitions are mathematically equivalent, but their use conditions the mathematical activity. In particular, their use conditions the resolution techniques of the equation $|x - 2| = 1$, as we have shown. For this reason, we can affirm that the definitions are not equivalent from the epistemic point of view: they do not involve the same mathematical objects in the resolution of a problem and, therefore, they condition the operative and discursive practices in relation to the AVN.

The proofs we have carried out involve applying some general techniques for solving linear equations with an absolute value (namely, equations of the type $|x - k| = m$; $k, m \in \mathbf{R}$) to a particular case. In the epistemic analysis of a mathematical object, it is necessary to answer a basic question: What is the applicability field for the techniques? For example: Do they allow equations of the form $|r.x - k| = |s.x - t|$ ($r, k, s, t \in \mathbf{R}$) to be solved? Is it necessary to make substantial adaptations or simple adaptations to be able to solve these problems? What instruments of evaluation do we have to judge if these technical adaptations are substantial or simple?

Nevertheless, if we restrict the analysis to solving linear equations with absolute value of the form $|x - k| = m$ ($k, m \in \mathbf{R}$), the shown techniques must solely be analysed in terms of their effectiveness, students' cognitive cost depending on their mathematical background and the necessary resources for teaching and learning. The analysis of the educational effectiveness of the techniques involved, particularly, the determination of the type of relations that exist between these techniques. Next we shall show the onto-semiotic complexity of the AVN, that is deduced from the diversity of contexts of use, from the definitions associated with them and the operative and discursive practices that these definitions condition. Then, backed by the theoretical notion of holistic meaning, we shall organise the partial meanings of absolute value, whilst showing the relations that are established between them.

ONTO-SEMIOTIC COMPLEXITY OF ABSOLUTE VALUE

Structure of definitions, objects and meanings associated with the notion of absolute value

The professional mathematician identifies the same formal structure in the variety of objects and (operative and discursive) practices; a structure that he/she considers to be “the mathematical object”. This formal structure represents the implicit reference in the resolution of types of problems associated with the variety of systems of practices and objects emerging in the different contexts of use.

As suggested by Ullmann (1962, p. 76), researchers should first gather an appropriate sample of contexts and approach them later with an open mind, thus allowing the meaning or meanings to emerge from these contexts. Once this phase has been concluded, we can safely enter into the “referential” phase and try to formulate the meaning or meanings identified in this way. Our meaning begins by being pragmatic, relative to the context, but there are typical uses that allow us to guide mathematical teaching and learning processes. These types are objectified by language and constitute the referents of the institutional lexicon.

Figure 1 schematically shows the diversity of objects associated with the AVN. Each definition represents an object emerging from a system of practices in a given context of use. No definition may be privileged *a priori*. Each “emergent object - system of practices” binomial determines a *partial meaning* of the AVN. The partial meanings are then a coherent form for structuring the different contexts of use, the mathematical practices relating to them and the objects emerging from such practices; so forming a network or *local epistemic configuration* (associated with a specific context of use).

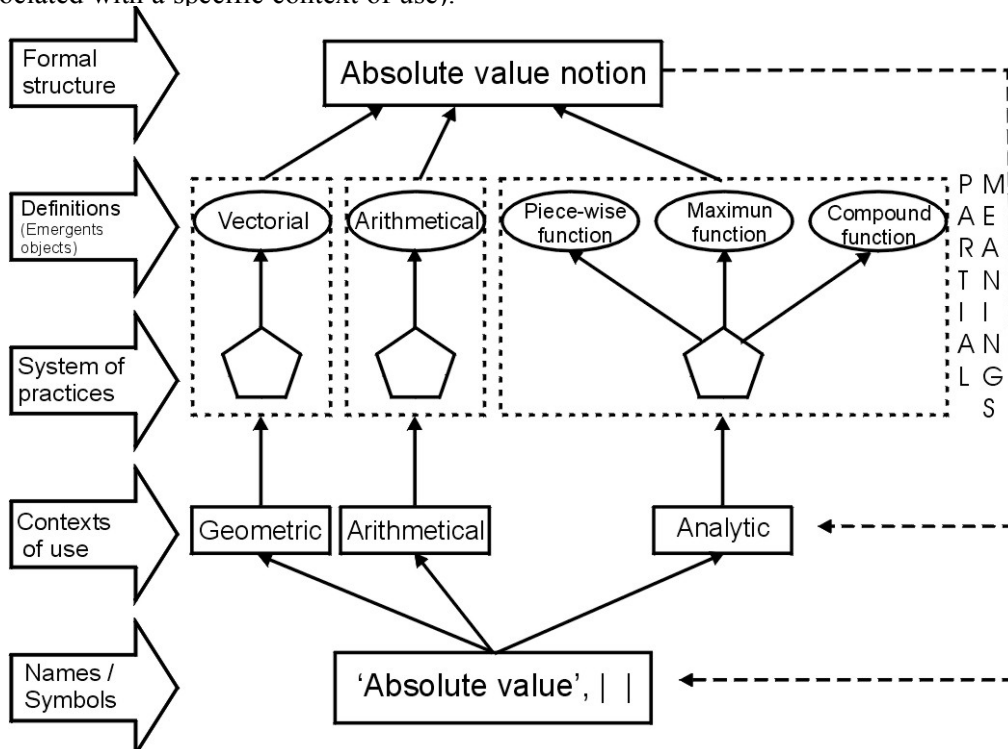


Figure 1. Structure for the objects and meanings associated with the absolute value.

In Figure 1 we do not represent a chronological order in the process of conceptualising AVN, but the structure of the different aspects that intervene in this process. Names and symbols adopt a specific sense in each context of use and the formal structure provides a coherent reference for the different partial meanings.

In the ontosemiotic approach to mathematics cognition *meaning* is the content of any semiotic function, that is to say, the consequent in the correspondence we establish between two mathematical objects in a communicative act. The antecedent is usually, but not restricted to a linguistic object. The answer proposed by the ontosemiotic approach to the question, *what does “absolute value” means?* is diverse, according to the contexts, the language games, people or institutions involved. Meaning might be one definition, the system of practices associated to a given context, or even a student’s personal interpretation (his/her personal meaning of absolute value). Meaning of absolute value (derivative, function, etc.) is also dynamic, since it “grows” when the problems and practices are generalised.

The definitions described represent the objective elements of these systems of practices and, therefore, their relational description will determine an explicit partial meaning for these systems. Below we introduce the idea of *holistic meaning* to represent a relational structuring of the associated mathematical partial meanings for the AVN.

Holistic meaning of the notion of absolute value

From the strictly *formal and official* viewpoint (Brown, 1998), it is accepted that the definition of a mathematical object constitutes its meaning. The description of the system of partial meanings associated with a notion is obtained from the statement and demonstration of a characterisation theorem: usually this privileges one definition and requires the justification of the equivalence of the other definitions.

Based on empirical data, Leikin and Winicki-Landman (2000) emphasize different strategies used by teachers to analyse equivalent definitions of a mathematical notion. The application of these strategies is not restricted to the epistemological dimension. “The teachers usually discussed both pedagogical and mathematical characteristics of the definitions, and often used didactical considerations in order to explain their mathematical preferences” (Leikin y Winicki-Landman, 2000, p.27). In fact, the empirical data provided by Leikin & Winicki-Landman allow to be stated that the equivalence of mathematical definitions cannot be assessed just from the epistemological viewpoint, it is necessary take into account the *cognitive* (What strategies for action generate each one of the definitions?), *instructional* (What definition is the most suitable within a given project for teaching?) and *didactic* (What relationship is established between the personal meaning learnt and the institutional meaning intended?) dimensions. The *holistic meaning* comes from the coordination of the partial meanings associated with AVN and the tensions, filiations and contradictions that are established between them and is relative to the

“ideal mathematical-didactics institution”, where absolute value has a unique “global” reality.

Hence, the justification of the mathematical equivalence of the definitions of absolute value does not represent a suitable instrument for structuring the partial meaning associated to that notion from the teaching and learning point of view. The study of the partial meanings of the absolute value (with their associated definitions) and the application for solving the linear equations show the need for a flexible transit between the different partial meanings. Tall (1991) defines the *reflective abstraction* as a mechanism whereby the subject encapsulates a process in a concept. It is precisely this encapsulation that enables the professional mathematicians to use, in a flexible way, a symbol that represents a process and a concept at the same time. That is to say, reflective abstraction is the basis of so-called “flexible mathematical thought”, that allows passing through both states of a mathematical object in a way that is not abrupt: *operational dynamic* (when the object is used as an instrument, that is to say, understood as a process) and *structural static* (when the object is related to other objects in a theory, that is to say, understood as a concept).

The definition introduced by Tall is not operative (Williams, 1999): how can we determine whether a subject performs a mathematical activity using the dual character process-concept of a mathematical object in a flexible way? What observable actions, gestures, arguments constitute evidence of the fact that a subject operates by means of a flexible mathematical thought in a process of mathematical study? Is flexible mathematical thought a *metaphor* (Barrow, 1997) for determining an optimal behaviour with regard to a system of operational and discursive practices linked to a mathematical object?

Our view is that flexible mathematical thought represents an action carried out by a subject that allows the routine transit between different associated partial meanings of a mathematical object (recognizing the limitations of each partial meaning). The subject also establishes, by means of flexible mathematical thought, links between these partial meanings and one or several mathematical fields. Those links determine an effective control of the activity and enable the subject to take mathematical responsibility for the results that he/she produces. The holistic meaning is the result of the mathematician’s activities in the coordination process for the partial meanings and, also, the tensions, connections and contradictions that he/she establishes between the partial meanings. The mathematician controls this coordination process using flexible mathematical thought.

COGNITIVE EFFECTIVENESS OF THE ARITHMETIC AND “PIECE-WISE FUNCTION” PARTIAL MEANINGS OF THE ABSOLUTE VALUE

In our theoretical framework, the epistemic dimension for a didactic problem is always institutional and might refer to the teaching process (*implemented meaning*, which is the immediate reference for students’ learning), the planning of that teaching process (*intended*

meaning), or a global reference wider than the planned teaching (*holistic meaning*). The cognitive dimension in this framework refers to the students' personal objects and meanings. This institutional–personal facet serve to explain the students' learning (as well as its limitations and conflicts) in function of the relationships between each student's personal meaning and the implemented institutional meanings.

In this section we explore the cognitive dimension in a teaching process of AVN from the students' responses to a questionnaire. This was part of a wider investigation that included revision of syllabuses, preparation of a textbook and teaching materials, observation of teaching sessions and interviews with students. The analysis made in previous sections of AVN holistic meaning served in that research as the reference in the design of teaching and evaluation of learning.

As mentioned earlier, from the viewpoint of mathematics education, a fundamental question consists of determining the *didactic effectiveness* of a mathematical process for problem-solving. In section “Definitions of absolute value” we showed how the techniques associated to the definitions of absolute value allowed the “linear equations with absolute value” type of problems to be solved. This fact did not mean an epistemic equivalence, since the techniques do not involve the same mathematical objects. In this section we aim to analyse the *cognitive* dimension (effectiveness and cost in the use of the techniques by individuals) of the problem-solving techniques associated with the “arithmetical” definitions and “piece-wise function”. To do so, we use an experimental study with a group of 55 students (secondary trainee teachers) solving a set of elemental exercises that require the AVN (table 1). The *implicative* and *hierarchical* statistical method (Gras, 1996) was used to analyse the responses given by the students.

Table 1. Questionnaire.

1. Complete, when possible, the following equalities:			
$ -2 =$	$ 2 =$	$ 0 =$	$ \sqrt{-2} =$
$ \sqrt{2} =$	$ \sqrt{-2} =$	$ 2 - \sqrt{2} =$	$ \sqrt{2} - 2 =$
2. Replace the dots, when possible, in the following expressions to make them correct:			
$ \dots - 2 = 1;$	$ \dots + 2 = 1;$	$ \dots - 2 = 0;$	$ (\dots)^2 - 4 = 0;$
$ (\dots)^2 + 4 = 0;$	$ (\dots)^2 - 1 = 1;$	$ (\dots)^2 - 3 = 1$	
3. Represent the function $f(x) = x+1 $ in a graphical way.			
4. Let a be a real number. Complete, when possible, the following equalities:			
$ -a =$	$ a =$	$ a - 2 =$	
$ -a - 2 =$	$ 2 - a =$	$ a + 2 =$	

Institutional implemented meaning in the study process

The main goal of the empirical study was to reveal the agreements and mismatches between the institutional intended meaning and the achieved personal meaning, which were determined after an in-depth personal interview of the teacher carried out by a researcher. It was complemented with content analysis of syllabuses, the textbook and materials used in the teaching.

The intended meaning was reduced to the relationship between the “arithmetic”, “piece-wise function” and “metric” partial meanings. In the institution, the metric partial meaning also plays a merely descriptive role of the action “remove the minus sign from a negative number”. Therefore, the intended meaning is structured for achieving the following objectives: 1) introduce the absolute value notion using the arithmetic partial meaning; 2) develop this partial meaning for the formulation of the piece-wise function partial meaning; 3) accept that the piece-wise function partial meaning is a formalization of the arithmetic partial meaning that is most effective in the problem-solving process. Nevertheless, the cognitive and mathematical students’ restrictions and the didactical teachers’ restrictions limit the implemented meaning to a collection of manipulations with particular numbers and the evaluated meaning to routine activities for obtaining the absolute value of real numbers ($|-2|$, $|2|$, $|0|$, $|2 - \sqrt{2}|$, $|\frac{1}{2} - \frac{1}{\pi}|$, etc.), as well as to the graphical representation of the absolute value of linear functions by means of the “point to point technique”.

Effectiveness in problem-solving

Generically, we affirm that a person understands the AVN if he/she is capable of distinguishing its different associated partial meanings, structuring the said partial meanings in a complex and coherent set and meeting the operative and discursive practices needed in relation to the AVN in the different contexts of use. Bills and Tall (1998) establish that a definition is operative for a subject when he/she can use this definition in a pertinent way in a logical-deductive demonstration process. From our viewpoint, the operating capacity of a mathematical object in a study process needs a balance between the operating role and the theoretical or discursive function. In a global education project, the analysis of the operating capacity should also take into account, in addition to the cognitive dimension, the epistemic and instructional dimensions.

The notion of “didactic effectiveness” introduced earlier takes into account these three dimensions and incorporates the operative and discursive functions of mathematical activity. Then, in relation to the arithmetic partial meaning and piece-wise function partial meanings we ask ourselves: Is it possible to discriminate those partial meanings by their effectiveness and cost in the use of the techniques by the subjects?

The only means for distinguishing the meaning attributed by an individual to an object is through a situation or a set of problems that may be solved by using different partial meanings capable of generating pertinent and useful actions.

The experimental work performed has allowed us to classify the students according to the partial meaning of absolute value associated with the operative and discursive practices in relation to the problems proposed (that determines a certain level of effectiveness). So as to be able to classify the students, it is necessary to interrelate a collection of tasks and determine (with a level of approximation) the tasks that allow the performance of other tasks to be assured.

Analysis of a questionnaire

The main purpose of the teaching experiment is to empirically support or not the thesis according to which the partial meanings “arithmetical” and “piece-wise function” associated with the AVN are quite similar. The a priori analysis provides criteria to select variables to carry out the implicative and hierarchical study (Gras, 1996).

A priori analysis

The section “Onto-semiotic complexity of absolute value” provides the a priori analysis of the AVN institutional meanings. The contrast between this a priori analysis and the students’ effective realizations is described in terms of suitability of a study process, that is, fitting of intended, effectively taught and achieved meaning. Some suitability criteria, which can be synthesized taking into account the epistemic, cognitive and instructional dimensions, will be introduced below.

The evaluation of epistemic suitability consists assessing the agreements and disagreements between the reference institutional and effectively taught meanings. Cognitive suitability assesses whether the proposal took into account the students’ cognitive restrictions and whether the material and temporal resources were enough to overcome gaps between initial personal meanings and the institutional meanings we wanted to teach. Finally, instructional suitability refers to teaching mechanism, to identify semiotic conflicts and solve them by negotiation of meanings.

The validity of the questionnaire given in Table 1 should be assessed from the epistemic, cognitive and instructional dimensions. For example, in question 1 we expect the answer “it is a complex number”, “non sense”, “you cannot take out square root of a negative number”, when asking students to determine $|\sqrt{-2}|$. The institutional restrictions determine the absolute value function to be defined in the real numbers set (epistemic dimension); a problem have one, none or several solutions (cognitive dimension); and, finally, the whole study process has been focused on the absolute value of real numbers (instructional dimension).

Implicative and hierarchical study

In the implicative analysis (Gras, 1996), the aim is to find whether, in the sample, the fact of having answered a question correctly statistically implies the response to another question. In particular, it is admissible to expect that any individual who is capable of performing a task that is more complex than another (and that generalises it in a certain way), then he/she will also be capable of performing the second one. However, this is not always so: the notion of “difficulty” is associated to the naturalized practices in a particular institution and the personal meaning that is not always specified; therefore, the resolution expectation can be “contradicted” in a particular sample (by an ample group of subjects). That is why, in many circumstances, it is necessary to compare certain hypotheses for implementing a hierarchy for performing tasks.

A wider question that may be posed is whether the fact of having answered a set of questions correctly implies (in a preferential manner) the right answer in another set of questions. The hierarchical analysis, defined on the basis of the cohesions between classes, allows the implicative relationships between the kinds of questions to be described in a more “dynamic” way and, therefore, constitutes a response to the question posed.

Results

Implicative and hierarchical analysis provide the following results:

- The arithmetical partial meaning of absolute value is understood as a rule that operates on the “numbers”, that is to say, numbers “in decimal format”.
- The students who have a “good” behaviour in the course mostly operate the absolute value “symbolically” ($|\sqrt{2} - 2| = 2 - \sqrt{2}$) and understand the $f(x) = |x + 1|$ function analytically and graphically.
- The group of students can be structured in two subgroups, each of them with stable answers in relation to the tasks they carry out (and the meanings of absolute value applied).
- The students performing symbolically the tasks and capable of systematically and effectively applying the piece-wise function partial meaning form one group; the other being characterised by application of the arithmetical partial meaning.
- The piece-wise function partial meaning is essential for the effective accomplishment of tasks related to the AVN. More specifically, we expect an implication from tasks requiring analytical partial meaning to tasks needing the arithmetic partial meaning.
- The analysis of the questionnaire, observation of teaching sessions and interviews with the teacher and students, showed that the teaching carried out before applying the questionnaire, might be suitable from the cognitive point of view since the time and material resources allowed student to solve the tasks proposed during this study process. However, the epistemic and

instructional suitability might be low. On one hand, the taught institutional meaning did not allow most students the formalization of AVN in terms of the “piece-wise function” partial meaning (intended institutional meaning). On the other hand, the tasks proposed in the study process did not allow the identification of conflicts of meaning related to AVN (as it was shown in the students’ answers to the questionnaire).

Brief discussion

The questionnaire has related two partial meanings of the AVN: the arithmetical and the analytical “piece-wise function”. The emergence of one or another partial meaning conditions the performance of the tasks proposed in the questionnaire (Table 1), at least for the sample studied. In fact, despite the fact that a priori (mathematically) the first two exercises and the last one may be solved “by the systematic application of the rule associated with the arithmetic partial meaning”, what is true is that the students who only have this partial meaning encounter serious problems for solving the tasks. Just referring to the first two exercises and the last one, it may be affirmed that the analytical partial meaning is preferred due to its greater heuristic power, despite the fact that “in principle” both partial meanings allow the “complete” solution of the exercises.

The students with the “piece-wise function” partial meaning are capable of using the absolute value symbolically and of systematizing the use of the arithmetical partial meaning. However, the students who only have the arithmetical partial meaning associate the AVN with the algorithmic action:

$$[a \in \mathbf{R} \rightarrow \text{decimal expression of } a \rightarrow \text{sign}(a)] \rightarrow |a|$$

The arithmetical partial meaning is transformed in the statement: “the absolute value is a rule that removes the minus sign from the negative numbers”; i.e., a rule for annotation: the ostensives “3” and “|-3|” represent the same object, they constitute a case of synonymy. The students perform symbolic manipulations with “hidden” positive numbers. If the goal of teaching is that the students learn this partial meaning, the mathematical work is centred on the technical use of the absolute value. The “good” student is the one who accepts the rules imposed by the teacher, who accepts playing the game with the symbols and acquires great mastery in this game.

Hence, it is necessary to make a didactic contract with severe restrictions (which alters the original mathematical interest of the object, with it being relegated to a simple game of symbols). The situation is similar to when a child “sings” the numbers or recites the alphabet, like a learned sequence of sounds, ordered for him/her by an arbitrary criterion; the only reason by which the child sings the numbers “in an ordered way” or recites the alphabet correctly is affective: he/she receives the adult’s praise and, possibly, a material prize. Then, it is clear that the same old story taught in a different order would be recited with the same rash enthusiasm.

MACRO AND MICRO DIDACTIC IMPLICATIONS

Cognitive difficulties (Chiarugi, Fracassina & Furinghetti, 1990) and the incapacity of educational institutions to draw up a pertinent curriculum to introduce and develop the AVN (Perrin-Glorian, 1995; Gagatsis and Thomaidis, 1994) has led to merely technical teaching based on the arithmetical partial meaning (as a rule that “removes the minus sign”). The arithmetical partial meaning of the AVN proves to be a didactic obstacle that restricts, in many cases, the personal meaning to a mere game of symbols. This obstacle is shown in different ways; for example: $|a| = a$ and $|-a| = a$, for any $a \in \mathbf{R}$; $|\sqrt{2} - 2| = \sqrt{2} + 2$, etc.

Macrodidactic implications

As Winicki-Landman & Leikin (2000, p.17) affirm “one of the more important questions in mathematics education is: ‘What is the best way to introduce a new mathematical concept to a learner?’” Teaching a mathematical notion by means of an associated partial meaning, it is necessary to ensure its representativeness in relation to the institutional referential meaning. The introduction of the absolute value by means of the arithmetic partial meaning is not representative: any analytical partial meaning cannot be dealt with guarantees (the theory of functions is beyond the students’ knowledge); the vectorial partial meaning can only be described in natural language (not formalized); and, finally, the geometric partial meaning is understood as a simple rule “to delete the minus sign”. Hence, the introduction of the absolute value in the arithmetical context represents an unfortunate decision in modern-day school institutions: it means the inclusion in the curriculum of the notion of “absolute value” for merely cultural reasons. However, the curricular structure is not ready at the present to properly cope with the study of this notion in an exclusively arithmetical context. It would be advisable to “temporarily” remove the notion. This would be temporary, either until a pertinent didactic transposition, or until the students start to study the theory of functions, central in relation to the notion of absolute value (Arcidiacono, 1983; Horak, 1994).

Microdidactic implications

For the AVN, the partial meaning “piece-wise function”, using the graphic representation of the function in the Cartesian plane, pertaining to the theory of functions. Hence, it is necessary to establish a didactic engineering for developing the AVN in the setting of theory of functions. This engineering will have to articulate the epistemological analysis with the methodological and time restrictions within each specific institution. In relation to the AVN, the objective consists of establishing a system of practices that will make the explicit interaction of the arithmetical partial meaning with the rest of the partial meanings possible and, most particularly, with the analytical partial meaning.

SYNTHESIS AND THEORETICAL IMPLICATIONS

In this paper we have contributed a new perspective about two theoretical problems of interest for the epistemology of mathematics and mathematics education:

1. Global and partial meanings of mathematical objects. Even when a mathematical object (AVN, tangent, real number, function, etc.) can be characterized by diverse definitions that are formally equivalent, some of these definitions might be not equivalent from the epistemic, cognitive or instructional viewpoints. Each definition is linked to a configuration or network of other objects and relationships, which efficiently allow the solving of a certain class of problems. Different definitions are not equally effective, that is, the possibilities of generalizing their associate techniques and their ontosemiotic complexity involved in their study vary.

In consequence, the didactic analysis of a mathematical object should characterize the diverse configurations composing its global or holistic meaning, in order to adopt decisions related to representative and effective selection of meanings in each specific educational circumstance.

2. Coherent articulation between the pragmatic and realist visions of mathematics. Building and communicating the meanings of mathematical objects require, on one hand, recognizing an institutional and contextual relativity for the same and, on the other hand, accepting the existence of the realistic-referential viewpoint for these objects usually assumed by professional mathematician. Articulating these two visions supposes the recognition of an intra-mathematics context of use, where the description of the formal structure (common to other contexts of use) and the foundation of mathematics as a body of knowledge is carried out.

These two theoretical contributions have implications about curricular design questions, as well as on the teaching and learning mathematics. The notion of holistic meaning of a mathematical notion provides an instrument for controlling and assessing the systems of practices implemented and an observable response for the analysis of personal meanings. More precisely speaking:

- The notion of holistic meaning (network of partial meanings) represents the structuring of the knowledge targeted and may be used to determine the degree of representation of a system of practices implemented in relation to the institutional meaning intended.
- The notions of partial and holistic meanings provide a response to the questions: What is a mathematical notion? What is understanding this notion?; in particular, What is the AVN? What does understanding the AVN mean?

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ACKNOWLEDGEMENT

Article prepared in the framework of the projects: Resolution No. 1.109/2003 of 13th October of the UPNA and MCYT-FEDER SEJ2004-00789.

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International Electronic Journal of
Mathematics Education

Volume 2, Number 2, July 2007

www.iejme.com

**EMPIRICISM, CONTINGENCY AND EVOLUTIONARY METAPHORS:
GETTING BEYOND THE “MATH WARS”**

Kurt Stemhagen

ABSTRACT. This article provides a philosophical conceptualization of mathematics *given the particular tasks of its teaching and learning*. A central claim is that mathematics is a discipline that has been largely untouched by the Darwinian revolution; it is a last bastion of certainty. Consequently, mathematics educators are forced to draw on overly absolutist or constructivist accounts of the discipline. The resulting “math wars” often impede genuine reform. I suggest adopting an evolutionary metaphor to help explain the epistemology/nature of mathematics. In order to use this evolutionary metaphor to its fullest effect in overcoming the polarization of the math wars, mathematical empiricism is presented as a means of constraint on the development of mathematics. This article sketches what an evolutionary philosophy of mathematics might look like and provides a detailed descriptive account of mathematical empiricism and its potential role in this novel way of thinking about mathematical enterprises.

KEYWORDS. Philosophy of Mathematics Education, Empiricism, Evolution, Math Wars, Philip Kitcher, John Dewey, Imre Lakatos, John Stuart Mill.

INTRODUCTION

As a middle school mathematics teacher, I was frequently frustrated by what went on in the classroom. Theorists and practitioners in other subject areas have worked to explicitly link the role of human agency to their respective disciplines and to find ways to apply school knowledge in reasonably realistic contexts. In language arts, science, and history, emphasis on traditional conceptions of subject matter as given and inert has shifted to an increased focus on the role people play in the creation and perpetuation of the disciplines. Furthermore, there has been at least a tacit change in the way knowledge within each discipline is conceived of, as the social dimensions of each are now recognized, if not embraced¹. For a number of reasons, some of which will be the focus of this article, similar changes in the way mathematics is regarded has not taken place. I submit that the teaching and learning of mathematics has suffered as a result.

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ISSN: 1306-3030

¹ In science, the work of philosophers of science, such as Kuhn and Popper, coupled with a post-Sputnik concern for relevant, applicable science education facilitated shifts in the way science is taught and learned in school. Increasingly, science class has become a place where students play the role of fledgling scientists. History class, with the advent of computers and other technology, has undergone a similar metamorphosis. Traditionally a place where students were expected to memorize names, dates and places; it has become a potentially dynamic forum for students to act as mini-historians, using the newfound ease of access to primary source data as a means to work as historians, discovering and interpreting historical material.

Part of the problem is that when educators turn to professional mathematicians for inspiration regarding conceptualization of the nature of mathematical enterprises and of the place of mathematics in the lives of humans, the results are often confusing. Many mathematicians hold the belief that mathematics bears no inherent ties to the physical world and possibly even to human life beyond the mathematics department (Hersh, 1997; Kline, 1980). Philosophical stances that tend to perpetuate this point of view often are forms of mathematical absolutism, that is, they describe mathematics as certain, objective, wholly logical, and based on objects or truths outside of human and earthly influence. Bertrand Russell provides a classic example of this way of thinking, referring to mathematics as a “beautiful world; it has nothing to do with life and death and human sordidness, but is eternal, cold and passionless...(it has) an immense dignity derived from the fact that its world is exempt from change and time...The only difficulty is that none but mathematicians can enter this enchanted region” (Griffin, 2002, p. 224)².

Recent reform efforts that have sought to include social facets of mathematical knowledge in mathematics education have worked against these particularly entrenched understandings of the mathematical enterprise. Reformers position themselves against these rigid conventions by offering subjective, relative, and fallible versions of mathematics. The resulting “math wars” position rigor-oriented, “back-to-basics” traditionalists against child-centered, application-focused constructivists. This “either/or” forced choice has made it so that many of the reform-oriented mathematics educators’ work is rejected a priori by those who self-identify as traditionalists. Likewise, this polarization often makes it difficult for reformers to acknowledge that mathematics does seem to be an especially stable form of knowledge. My concern is primarily with the former situation, as resistance to the high quality work of many thoughtful reformers seems unnecessary and counterproductive to mathematics education³.

There appear to be similar conflicts occurring in other areas of the curriculum. The phonics versus whole language debate seems, at first blush, to be analogous to the polarized confrontation at the heart of the math wars. However, there is at least one important difference. The whole language-phonics debate is primarily about teaching methods, whereas the math wars-while certainly concerned with teaching methods-also possess philosophical dimensions. In other words, while phonics-whole language debates tend to focus on ways to teach the subject, mathematical traditionalists and constructivists, while arguing over curriculum and methods on the surface, often do so from differing positions as to the very nature of mathematics. Lerman’s three “levels” of mathematics education provides a useful model in this regard: “At the first level are the surrounding (sometimes called foundation) disciplines of psychology, sociology, philosophy, anthropology, (in our case) mathematics, and perhaps others. At the second level are mathematics education and other curriculum areas of educational research. At

² Responses to a recent article on mathematics education and social justice (Stemhagen, 2006) suggest that contemporary attitudes share much with Russell’s. One mathematics/computer science graduate student, presumably parroting talk within his discipline, claimed: “if there’s one thing I’m sure of, it’s that mathematics has nothing to do whatsoever with...any aspect of the physical world...math lives in its own separate Platonic world and we mere mortals can only hope for an occasional peek inside” (Kontorovich, 2007).

³ The competing websites *Mathematically Correct* (2007) and *Mathematically Sane* (2007) demonstrate this sharply oppositional position taking, each encouraging the cultivation of a perspective that fundamentally rules out the other.

the third level are curriculum and classroom practice” (Lerman, 2000, p. 20).

My claim is that underlying philosophical positions (Lerman’s first level) can, at times, influence mathematics educational practices (his third level). In the context of the math wars, traditionalists tend to draw on absolutism—viewing mathematics as certain, permanent, given, and independent of human activity. Reformers, on the other hand, generally draw on philosophical constructivism and focus on the ways in which humans actively create mathematical knowledge, rendering mathematics contingent on human activity. While having much to offer mathematics education, both provide incomplete accounts of mathematical knowledge and mathematical activity. Absolutism suggests an understanding of mathematics that captures its unique stability and universality but that does not acknowledge its human dimensions. Conversely, constructivism tends to encourage understandings of mathematics that feature human involvement but, in doing so, seems to lose the ability to explain the remarkable stability and universality of mathematical knowledge⁴.

Rorty’s distinction (1999) between those who view phenomena as fundamentally “found” versus those who see it as “made” provides an interesting analogy, as mathematical absolutists can be thought of as seeing mathematical knowledge as found, whereas constructivists often conceptualize it as made. A la Rorty (drawing on Dewey), I argue that a way beyond this either/or stalemate is needed if theoretical considerations are to have a role in improving mathematics education and I seek to provide a philosophical understanding of mathematics that is able to account for both its stability–universality and its contingent human–influenced qualities. I do this by conceiving of the discipline in evolutionary terms.

Evolutionary models run the risk of not being adequately sensitive to the strengths of the claims of mathematical absolutists (namely regarding the stability and universality of mathematics), and I argue that the constraints on mathematics posed by an empirical conceptualization of its development and even facets of contemporary practice are consistent with an evolutionary account and sufficiently restrictive to depict the stability of mathematics. Thus the bulk of the paper is devoted to providing an account of a particular form of mathematical empiricism and a defense of the merits of acknowledging empirical dimensions of the discipline as a means to develop ways of thinking about mathematics that retain the “best of both worlds”—the absolutist focus on mathematical stability and the constructivist contribution of illuminating the human and contingent dimensions of mathematics.

By way of disclaimer, my goal is not to settle any larger metaphysical or ontological scores as to the ultimate nature of mathematics. Instead, my work is situated in the philosophical tradition of American pragmatism and, accordingly, in this article I work to develop a functional conceptualization of mathematics *given the task of its teaching and learning*. It is possible, even likely, that adoption of other understandings of mathematics is more fruitful for those with other

⁴ For more on the many varieties of constructivism, see Phillips (2000).

purposes. For example, the non-empirical even Platonic philosophical understandings often employed by pure mathematicians might serve very well as they engage in their disciplinary practices (Hersh, 1997). My pragmatic argument is that, because many working mathematicians and philosophers of mathematics find consideration of the empirical dimensions of mathematics distasteful and perhaps even fruitless, it does not follow that such consideration is not useful for those who teach and learn mathematics. In the words of Larry Hickman, for the pragmatist, “function trumps ontology” (2001, p. 32).

EVOLUTION AND MATHEMATICS

My claim is that mathematics is a discipline that has been largely untouched by the Darwinian revolution. In essence, it is a last bastion of certainty. Take, for example, the work of Daniel Dennett, respected philosopher and avowed evolutionist. In *Darwin's Dangerous Idea*, he develops an argument for adopting the process of evolution by natural selection as an overarching explanatory principle for almost all phenomena. He patiently explains how evolutionary biology works and then develops the thesis that our socio-cultural objects can also be thought of as an evolutionary development. He places cars, libraries, and political freedom on the “tree of life.” In this way, Dennett recognizes that just as species could have turned out otherwise (he talks of the vast design space and how small a portion of it is taken up by the tree of life), so could have our socio-cultural artifacts.

Dennett places two things beyond the pale of natural selection: the laws of physics and logic/mathematics (Dennett, p. 129). Dennett forwards a fundamentally ahistorical and extra-human version of mathematics. To Dennett, while most things are changing over time, mathematics does not evolve – it just is. In what follows, I present the work of several thinkers in an effort to make philosophy of mathematics somewhat more amenable to evolutionary thinking, all the while being careful to consider the ways in which a focus on the empirical dimensions can provide a sufficient explanation of the constraints on evolutionary developments. Philip Kitcher's empiricism provides the map for this new direction. The mathematical thought of J. S. Mill, John Dewey, and Imre Lakatos are used to flesh out Kitcher's framework.

KITCHER'S MATHEMATICAL NATURALISM: A POINT OF DEPARTURE

Kitcher describes his project as offering a much-needed alternative to mathematical absolutism, as most work in philosophy of mathematics has featured arguments between factions within the absolutist school of thought (1988, p. 3). Kitcher explains that one reason why absolutism is so prevalent is that philosophers of mathematics typically have not had much choice, other than absolutism or some overly simple version of empiricism (p. 294). He attempts to develop a more careful version of empiricism as a critical component of his overall philosophy

of mathematics. Kitcher claims that the physical manipulation of objects plays a role in actually creating mathematics (p. 5).

Kitcher's Debt to Mill's Empiricism

While it is beyond the scope of this paper to go into much detail on J. S. Mill and Gottlob Frege, a bit of background is in order, as Kitcher's empiricism draws on what Kitcher calls a formulation of: "Mill's optimal position" (1980, p. 215). Furthermore, their confrontation has much to do with the subsequent relative disrepute of mathematical empiricism (Kessler, 1980). Mill theorized that mathematics is not to be understood as the study of abstract objects, but instead that it consists of truths that are demonstrated by empirical observations, inductions from experience: "All numbers must be numbers of something; there are no such thing as numbers in the abstract. Ten must mean ten bodies, or ten sounds, or ten beatings of the pulse" (Mill, p. 167).

Mill was not content to relegate arithmetic only to relations between sets of specific instances. While he argued for the empirical origins of mathematics, he also recognized its stability and generalizability: "...though numbers must be numbers of something, they may be numbers of anything. Propositions, therefore, concerning numbers have the remarkable peculiarity that they are propositions concerning all things whatever..." (Mill, p. 167).

Frege's opposition to Mill's empiricism was intense and largely successful, as Frege's lasting influence and the relative obscurity of Millian mathematics attests. Frege understood Mill to say that number is a quality of groups of objects. Frege used color as an example of a quality that an object either possesses or not and he pointed out that number is a fundamentally different phenomenon. As for Mill's contention that number comes from arranging objects, Frege stingingly responded: "if Mill is right, we are very lucky that not all objects in the world are nailed down, for otherwise it would be false that $2 + 1 = 3$ " (1997, p. 94). Frege's critique of Mill effectively kept empiricism out of the mainstream in philosophy of mathematics discussions for the better part of the last century.

Links between the History and Epistemology of Mathematics

Kitcher states: "most philosophers of mathematics have regarded the history of mathematics as epistemologically irrelevant" (1983, p. 5). Kitcher's version of mathematics is very nearly the opposite of such mainstream accounts, as he argues that mathematics is a product of its past and that it can be better understood if this history is taken into account (1988, p. 298). Kitcher's ideas can be broken down into two main parts, the origins and development of mathematics (1983, p. 96).

Kitcher asks: "How do we come to know mathematics?" (1988, p. 297). Even though at this point he is working primarily within the context of the community of professional mathematicians and the portions of mathematics on which he is focused are the established

axioms, rather than turn to proof as the sole means by which to “know” the veracity of mathematics, Kitcher turns to education. He proclaims: “In almost all cases...they were displayed on a blackboard or discovered in a book, endorsed by the appropriate authorities, and committed to the learner’s memory” (Kitcher, 1988, p. 297). In short, according to Kitcher, our understanding of mathematics is based in large measure on the mathematics that we were taught in school. Kitcher points out that mathematics education makes it so that: “mathematical knowledge is not built up from the beginning in each generation” (p. 298). Implicit in this idea is that, without the common historically-funded version of mathematics that students take in during their school years, understandings of what mathematics is would be different.

Kitcher’s second observation concerns the specific ways in which we come to understand mathematics: “some of this knowledge is acquired with the help of perceptions” (1983, p. 92). Kitcher points out that early in our mathematical development we use manipulatives, such as rods and beads and that later we use diagrams to understand more complex mathematics (geometry is one obvious example). Still more advanced mathematics has less in common with “everyday” physical objects but can be, Kitcher argues, linked to its empirical roots through a succession of transformations⁵. The idea that an individual can learn mathematics in an isolated and relatively synchronic manner is, to Kitcher, wrongheaded: “... the community supplements primary source (authorities) with local justifications, providing the student with ways of looking at mathematical principles which seem to make them obvious. So it comes to appear that the mathematician, seated in his study, has an independent, individual means of knowing the basic truths he accepts” (1983, p. 93).

Kitcher enriches the notion of a mathematical community by postulating that the mathematical practices in which the community engages can be conceived of as consisting of five components: a language unique to mathematicians, a set of accepted statements, a set of questions that are taken to be important and not currently settled, a set of reasonings used to justify accepted statements, and a set of views regarding how mathematics is to be done (1988, p. 299). After presenting his idea of mathematical practice, Kitcher is able to summarize his theory of mathematical development: “I claim that we can regard the history of mathematics as a sequence of changes in mathematical practices, that most of these changes are rational, and that contemporary mathematical practice can be connected with the primitive, empirically grounded practice through a chain of interpractice transitions, all of which are rational” (1988, p. 299). So Kitcher envisions mathematical development as taking place through a chain of knowers, each loosely bound by the mathematics they learned as well as the conventions particular to communities of mathematicians.

In terms of the origins of mathematics, Kitcher is a bit less clear. He certainly offers an account that is at least somewhat in line with Mill’s empiricism, albeit a reconsidered version. At times, he sounds very Millian, such as when he explains how this “chain of knowers” began:

⁵ Elsewhere, I use the work of C. S. Peirce to make the case that even the highly abstract endeavors of today’s professional mathematicians have empirical dimensions. In fact, to Peirce, a primary role of the mathematician is to create manipulatable models of phenomena. For more on this, see Peirce (1898).

Here I appeal to ordinary perception. Mathematical knowledge arises from rudimentary knowledge acquired by perception. . .our ancestors, probably somewhere in Mesopotamia, set the enterprise in motion by learning through practical experience some elementary truths of arithmetic and geometry. From these humble beginnings mathematics has flowered into the impressive body of knowledge which we have been fortunate to inherit. (1983, p. 5)

Kitcher intimates that the originators of mathematics made their mathematical observations in the midst of trying to solve practical problems. This seems a step in the direction toward the avoidance of the development of a version of the earliest mathematicians that has much in common with the solitary thinker sitting in his study that he dismissed as an unacceptable way to conceive of the contemporary mathematician.

This sketch of Kitcher's philosophy of mathematics depicts an unconventional theory that draws on a form of empiricism, history, and community in its effort to provide an alternative to absolutism. However, there are problems with Kitcher's work, including an overly passive account of how we use mathematics and that in spite of references to a chain of knowers, he does not render clear a connection between the empirical origins of mathematics and what he calls its current highly abstract state.

Kitcher's Work as the Basis for an Evolutionary Framework

Although I argue that Kitcher's theory is not truly evolutionary, it is fairly revolutionary. In bringing empiricism back into the conversation through his notion of perceptions as a critical component of the origins of mathematics, Kitcher has reintroduced psychological explanations of mathematics. Additionally, by linking contemporary practice to the origins of mathematics through a chain of groups of knowers, Kitcher has brought both the history of mathematics and the influence of community into play. Both accomplishments are important. In what follows, I use Dewey's pragmatic and psychological philosophy of mathematics to strengthen Kitcher's claims regarding the empirical origins of mathematics. Kitcher's reliance on Mill leads to an account of the empirical origins that does not sufficiently recognize the functional role of mathematics in human activity. The result is that Kitcher has trouble explaining how empirical origins have led to the highly abstract nature of contemporary mathematics, and Dewey's functional version of how human employ empirical objects is one way to connect today's mathematics to its origins, given Kitcher's general framework. Lakatos' historical explanation of the development of mathematics is next employed in order to flesh out Kitcher's notion of a "chain of knowers" and to help make the connection between the origins of mathematics and its contemporary state.

DEWEY'S PSYCHOLOGY OF NUMBER

In spite of John Dewey's massive influence within the philosophy of education, his work is rarely invoked in contemporary conversations within the philosophy of mathematics

education. His non–presence is particularly vexing in that, Dewey’s co–authored mathematics education book, *The Psychology of Number and its Applications to Methods of Teaching Arithmetic* (1900) featured a philosophical approach to both educational psychology and mathematics education. In what follows, I introduce Dewey’s thought in order to argue that, in spite of an aversion to purely empirical or historical approaches, his philosophy of mathematics is not unfriendly to Mill’s empiricism and Kitcher’s historical conception. Additionally, Dewey’s exploration of the psychological processes involved in an individual’s coming to know mathematics provides a point of entry for human elements into a discipline (philosophy of mathematics) that has frequently worked to explain mathematics in non–human, anti–psychological terms.

Psychology has often been viewed largely as something to be overcome or ignored in philosophical work, as there is fear that mental processes can be a serious impediment to understanding how the world “really is.” According to this traditional philosophic conception of psychology there is a sharp line between the mental and the physical. Dewey sought to mediate between those tendencies that focus disproportionately on either the mental or physical aspects of existence. Dewey saw this polarization as overly static and inaccurate, and one way he combated this way of thinking was to employ “psychology” in an unorthodox manner.

Dewey’s inclusive and activity–sensitive psychology is at the core of his more general pragmatic beliefs. According to his pragmatic conception of how we know and what there is to know; knowledge, belief, and psychology are inextricably linked:

I start and am flustered by a noise heard. Empirically, that noise is fearsome; it really is, not merely phenomenally or subjectively so. That is what it is experienced as being. But, when I experience the noise as a known thing, I find it to be innocent of harm. It is the tapping of a shade against the window, owing to movements of the wind. The experience has changed; that is, the thing experienced has changed not that an unreality has given place to a reality, nor that some transcendental (unexperienced) Reality has changed, not that truth has changed, but just and only the concrete reality experienced has changed... This is a change of experienced existence effected through the medium of cognition. (1910, p. 230)

In Deweyan terms, the world is as it is experienced. According to his immediate empiricism, or pragmatism, philosophers are obligated to employ psychology in understanding experience. This is radical, as to many philosophers, psychology is seen as a barrier to logic, obscuring the contents of the logical, a priori realm. The Deweyan reconception of logic and psychology in light of human activity posits psychology and even logic merely as modes by which we undertake the act of figuring out how to live our lives. Psychology encompasses the mental processes by which we actually think (and live) and Dewey goes so far as to characterize logic as empirical in origin, coming about as fruitful methods of inquiry are recognized, emulated, and eventually formalized (1967, p. 138).

With regard to mathematics, Dewey conceived of number as transactional in nature – residing within the processes of mathematical activity. We come to use number only after a great deal of rational and abstract thought. In support of his claim that the sensory input with which we work, while rich in raw information as to the multiplicity of things in nature offers

no insights regarding the notion of number, Dewey wryly noted: “There are hundreds of leaves on the tree in which the bird builds its nest, but it does not follow that the bird can count” (McLellan & Dewey, 1900, p. 23).

Dewey developed a technical descriptive account of how children come to know mathematical concepts. This account centered on the mental activities of children as they encountered various empirical situations. Without the psychological processes he detailed, there would only be the ideas of “much” and “many” but not the more refined notions of “how much?” and “how many?.” Dewey understood this simple sense of quantity coming about in light of the human need to measure in order to live more efficient and better lives (p. 42).

Dewey sought to blur the commonly understood distinction between counting and measuring. Traditionally conceived, counting is related to determining how many of something there are and measuring involves determination of how much of something there is. In other words, the question is whether some phenomenon is a series of parts of one whole, or a related group made up of individual units. Dewey’s pragmatic answer was that they may be either and that the context and the needs of the counter/measurer must be taken into account when answering the question.

So, to Dewey, mathematics can best be defined and understood by its use. The concept of a particular number (say three) does not reside within a group of three apples, beanbags or any other objects any more than it does in the symbol “3.” The concept of three emerges from the activities in which we engage that utilize quantification (measuring) as a means to an end. The accompanying pedagogy suggested by this philosophy of mathematics focused on measurement. This is a reasonable suggestion, as, to Dewey, all counting is measuring and all measuring is counting. Making measurement the vehicle for mathematical explorations ensured, according to Dewey, that number symbols will always be linked to concrete units and encouraged active, empirically-oriented, and contextualized conceptions of mathematical enterprises.

With his work in philosophy of mathematics and mathematics education, Dewey created an alternative to what he considered to be overly empiricist and rationalist options of his day. Thus, it would be easy to conclude that Mill’s empirical explanation of mathematical knowledge did not have much in common with Dewey’s. While Dewey’s version of mathematics emphasized the interplay between empirical objects and our actions, I argue that Mill’s philosophy of mathematics contained more than just simple empiricism; it showed a nascent acknowledgement of the role of human intent in the construction of mathematical knowledge⁶. Although Mill never fully articulated the pragmatic notion that mathematical knowledge is actually created by the interaction of our activities and the physical world, he did come fairly close. For example, Mill stated: “... Two pebbles and one pebble are equal to three pebbles... affirms that if we put one pebble to two pebbles, those very pebbles are three” (Mill, p. 168). Although not the major point that Mill is attempting to make, here he is clearly affording the

⁶ While pragmatism and Millian empiricism might seem strange bedfellows, note that William James dedicated *Pragmatism*, to: “the memory of John Stuart Mill from whom I first learned the pragmatic openness of mind and whom my fancy likes to picture as our leader were he alive to-day” (James, 1978, p. 4).

human act of “putting” the pebbles a place in his understanding of the nature of the number 3. About Mill, Kitcher makes a similar point:

We have seen that Mill suggests that apparent references to classes can be parsed away. His idea is to say that objects belong to a class is to assert that we regard those objects as associated... Thus the root notion in Mill’s ontology is that of a collecting, an activity of ours, rather than that of a collection, an abstract object... At times, Mill seems to come very close to an explicit proposal of this kind. (Kitcher, 1980, p. 224)

The similarities between Kitcher’s formulation of Mill’s “optimal” position and Dewey’s account of mathematics are striking. Dewey’s general tenet of conceiving of mathematics as arising from human activity (measuring) certainly is friendly to Mill’s account. There are also parallels between Dewey and Mill’s ideas of how children come to know number. Mill’s semi-recognition of the active role of the child in establishing the concept of number is somewhat sympathetic to Dewey’s version. Kitcher says of Mill’s version of learning arithmetic: “Children come to learn the meanings of ‘set’, ‘number’, ‘addition’ and so forth by engaging in activities of collecting and segregating” (1980, p. 224).

Dewey is helpful in fleshing out Kitcher’s intriguing but under-developed idea that the first mathematical ideas were empirical. Whereas Mill’s work aids the project of developing an evolutionary understanding of mathematics through its recognition of the role of the physical world in shaping the development of mathematics, Dewey—while similarly acknowledging the role of the physical world—goes farther. He makes clear the critical component of the role that human activity plays in this development.

While the physical world provides some constraint on the potential directions the development of mathematics may take, so too do the ways in which we choose to use mathematics. According to Dewey’s pragmatic account, our mathematics is what it is – in addition to empirical limitations on how mathematics “work”—because of the ways in which we live our lives. If we lived our lives differently (due to chance, choice, accident, or other circumstance) it is conceivable that our mathematics would be different. For example, had our ancestors not needed to use geometry to organize their experiences as farmers, the ways in which we systematized spatial relations might be different than they are. Just as Kitcher warns about Mill, it would be a mistake to view Dewey as a simple constructivist, as Dewey’s focus on function over structure makes clear. To Dewey, the development of mathematics is driven by the ways in which we use it (i.e., its functions). A common feature of most different brands of constructivism is their structuralism, as most constructivist accounts posit some underlying structure that can account for the development of mathematics. Dewey’s functionalism suggests a means to consider whether the constructions are good ones. In other words, the functional approach tests constructions by acting on them. If the results are satisfactory, then the construction is more than mere belief. The beginnings of an evolutionary philosophy of mathematics is beginning to emerge, as the functionality of mathematical ideas leads to their acceptance and subsequent influence on future mathematical ideas.

LAKATOS AND MATHEMATICAL DEVELOPMENT

Imre Lakatos developed what he described as a “quasi–empiricist” philosophy of mathematics. He compared the discipline of mathematics to a Popperian version of science, putting forward a conception of mathematics that rejects absolute certainty as a goal and acknowledges the ways in which mathematicians develop mathematical knowledge through an unending series of hypotheses and critiques. Lakatos coined the term quasi–empiricism to show how mathematics and science are similar in method but differ as to content. Science uses the physical world as a basis for its experiments while mathematics uses mathematical ideas.

Although the comparison between mathematics and science is interesting and useful, I contend that Lakatos’ work is pertinent to this project for another reason. Dewey’s work offered a functionalist account of how lay people develop mathematical understandings. Lakatos, on the other hand, tended to focus on the ways in which mathematics is done within organized professional (or at least academic) communities of mathematicians where strictly absolutist versions of mathematics are the norm. Rather than thinking of proof as the way mathematicians uncover the immutable Truth that is mathematics, Lakatos conceived of proof more pragmatically. As Kitcher explains:

His central thesis is that the role of proofs in mathematics is misunderstood, and he suggests that this crucial misunderstanding will affect the quality both of mathematical research and of mathematical education. The mistake is to regard proofs as instruments of justification. Instead we should see them as tools of discovery, to be employed in the development of mathematical concepts and the refinement of mathematical conjectures. (1977, p. 782)

Dewey thought of mathematics as a set of tools we have developed in order to solve problems in an effort to lead better lives. Lakatos shared an understanding of mathematics that employed the tool metaphor, but his focus was different than Dewey’s. Lakatos’ work was primarily concerned with the use of mathematics as a tool for inquiry within the discipline of mathematics, be it the professional or school varieties of mathematics. Lakatos describes the purpose of his work, *Proofs and Refutations*, as an attempt to cast doubt on the idea that formalist accounts of mathematics are sufficient⁷. He sets the stage for his own historically–oriented version: “Formalism disconnects the history of mathematics from the philosophy of mathematics, since, according to the formalist concept of mathematics, there is no history of mathematics proper” (1976, p. 2). He goes on to point out that formalism fails to accept most of what is typically thought of as mathematics as part of the discipline and that it offers a version of mathematics in which nothing meaningful can be said about its development: “None of the ‘creative’ periods and hardly any of the ‘critical’ periods of mathematical theories would be admitted into the formalist heaven, where mathematical theories dwell like the seraphim, purged of all the impurities of earthly uncertainty” (1976, p. 2).

Proofs and Refutations is the fictional story of a discussion set in a mathematics classroom. As Lakatos explains: “The class gets interested in a PROBLEM: is there a relation

⁷ Formalism is a particular type of absolutism that posits mathematics as a set of formal rules.

between the number of vertices V , the number of edges E and the number of faces F of polyhedra – particularly of regular polyhedra – analogous to the trivial relations between the number of vertices and edges of polygons (they are equal)?” (p. 6)⁸. The class uses trial and error and comes up with a formula: $V - E + F = 2$ ⁹. Students surmise (conjecture) that it is likely that this formula will be true for all regular polyhedra. Attempts to falsify the conjecture fail, thus suggesting that a proof will demonstrate the truth of the statement. The bulk of the work is dedicated to what happens once a teacher offers the proof. What starts as a reasonably simple exploration of a particular mathematical problem becomes an increasingly complex discussion about the nature of mathematical inquiry and also of proof itself.

Proofs and Refutations would be an impressive achievement if it simply presented an account of a group of students struggling with the notion of proof, but there is another dimension to the work. A series of detailed footnotes explains how each development in the class discussion has an analog in the historical development of this particular piece of mathematics. One interpretation of his argument could be that the history of mathematics is relevant to its teaching and learning. While this is clearly a point of his, it seems likely that Lakatos had something more radical in mind. According to Kitcher: “I think that Lakatos has demonstrated that there are important issues about mathematical discovery that should not be neglected. The process of mathematical discovery cannot be dismissed (as it so often has been) as a series of ‘happy guesses’” (1977, p. 196). In fact, I take a primary contribution of the work to be that however we choose to characterize mathematics, its development ought to be a consideration.

TOWARD AN EVOLUTIONARY PHILOSOPHY OF MATHEMATICS AND MATHEMATICS EDUCATION

Dewey’s insistence of the import of individual psychology to the development of mathematics strikes a blow against the absolutist conceptions of mathematics as unsullied by the imperfect and psychology-ridden human mind. In grounding mathematical experiences in the context of our activities, Dewey’s work also counters relativistic and idiosyncratic accounts of mathematics forwarded by radical constructivists, as mathematical knowledge is tied to how we relate to our physical world and can be judged by how well they help us live our lives. Dewey’s detailed explanation of how humans come to develop the concepts of number also serves to enrich Kitcher’s account of the empirical origins of mathematics. Likewise, Lakatos enriches the historical dimensions of Kitcher’s account through an exploration of the ways in which formal mathematics develops, both within the discipline and the classroom. Additionally, Lakatos strengthens Kitcher’s notion that the nature of mathematics and its teaching and learning are connected.

⁸ A polyhedron is a three-dimensional object made up of plane faces, a regular solid.

⁹ This formula means that for any regular solid, subtracting the number of edges from the number of vertices and adding the number of faces will come to 2. A cube, for example, has 8 vertices (corners). Subtract its 12 edges (where two faces (sides) meet) to get -4. Finally adding the 6 faces does yield 2. It is likely that trial and error refers to the class looking at cubes, prisms and other polyhedra in hopes of finding the pattern that they eventually found.

The understanding of mathematics presented in this paper grounds mathematical activity in the empirical world, while still recognizing the importance of mental activity in mathematical enterprises. Thus, this perspective is sensitive to the remarkable stability and universality of mathematics without describing it in absolute or otherworldly terms. At the same time, by grounding constructions in the empirical world, this perspective recognizes the role humans play in the creation of mathematical knowledge and the judgment of the worth of such constructions. This reconsideration of mathematical empiricism lays a promising foundation for the development of an evolutionary philosophy of mathematics.

Acknowledging the ways in which human interaction with the physical world affects the development of mathematics can help to get beyond the current philosophical stalemate between absolutist and constructivist philosophers of mathematics. Perhaps more significantly for this project, an evolutionary conception of mathematics – particularly one nourished by a rich and nuanced understanding of mathematical empiricism – offers a promising conceptual base camp from which mathematics educators might be able to find their way out of the morass of the math wars. The practical upshot of this work is not so much to seek to create a new pedagogy or curriculum based on the evolutionary account, although elsewhere I have presented classroom activities that draw on such a conception (Warnick & Stemhagen, 2007). Instead I hope that reintroducing empirical constraints on the development of mathematical knowledge will help make the constructivist/reformer versions of mathematics more palatable to some who currently see themselves allied with the traditionalist side of the math wars. That is, the reintroduction of empiricism helps to soften the sharp either/or forced choice of conceiving of mathematics as either fundamentally precast or as simply the product of however we choose to think about it.

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ISSN: 1306-3030