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Using learning progression-based interactive dynamic geometry instruction to develop quadrilateral reasoning in preservice teachers

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ABSTRACT

This study investigated the learning of K–8 preservice teachers (PTs) while enrolled in an in-person class that was integrated with an online, individualized, and interactive Dynamic Geometry curriculum (iDGi), built from empirically derived learning progressions (LP). The research examined PTs' developing understanding of the prototypical defining properties of quadrilaterals—formal geometric attributes that align with the visually salient features students often use when identifying and reasoning about quadrilateral types and their relationships. It also explored how the curriculum supported PTs' transition toward more precise and abstract forms of reasoning. The curriculum's effectiveness was evaluated with 541 PTs across multiple courses, all of whom completed identical online multiple-choice LP-based assessments before and after engaging with iDGi. Findings indicated that iDGi significantly improved PTs' reasoning about quadrilaterals. Analyses of three whole-class discussions illustrated how related in-class activities complemented and deepened PTs' conceptual understanding.

Keywords: geometry, learning progressions, elementary preservice teachers, quadrilaterals, dynamic geometry

INTRODUCTION

Research has found that elementary teachers' knowledge of mathematics has a significant relationship to their students' mathematics achievement (Hill et al., 2005). As Ball et al. (2005) argued "How well teachers know mathematics is central to their capacity to use instructional materials wisely, to assess students' progress, and to make sound judgments about presentation, emphasis, and sequencing" (p.14). Unfortunately, numerous research studies have found that elementary teachers lack the deep mathematical understanding and reasoning needed to teach mathematics meaningfully (e.g., Ball, 1990; Fennema & Franke, 1992; Ma, 1999). One of the major components for preparing elementary (K-8) teachers to teach mathematics well is deepening their conceptual understanding of the mathematics they are expected to teach. In this research project, we examined how an in-person/online geometry instructional unit supported pre-service elementary and middle school teachers' learning and understanding of geometric content that they will teach their K-8 students.

The study is an extension of a previous project that produced and tested an online, individualized, interactive Dynamic Geometry (DG) learning system for grades 3-10 that can be used by students independently or by teachers in classrooms (Battista, 2019). The resulting *Individualized Dynamic Geometry Instruction* (iDGi) online learning system integrates the use of DG, formative-assessment, research-based Learning Progressions (LP), sequencing that interactively adapts to students' locations in LPs, and built-in student guidance. iDGi focuses on 2D geometry and geometric measurement for grades 3-10 as described in the Common Core State Standards for Mathematics (2010). Although iDGi can be used by students as an independent learning system, it was designed to be most effective if its use is strongly linked to appropriate classroom discussions of students' reasoning within iDGi.

In the present research, we investigated the use of iDGi with pre-service elementary and middle school teachers (PTs) using its instructional/assessment modules focused on quadrilaterals. All assessments and modules were created based on research and development of the LP (Battista, 2012ab) on how students' learning develops and progresses for quadrilaterals. Prior school-based iDGi research and development supports the validity of these assessments for testing the impact iDGi has on student learning (Battista, 2019). In the present research, we focus on PTs' learning of the properties of quadrilaterals and the following research questions:

1) How does instruction with iDGi Quadrilateral modules and interrelated classroom instruction affect PTs' knowledge of, and reasoning about, properties of, and interrelationships between, quadrilaterals?

2) How can whole class discussions about iDGi module content be used to help and encourage PTs to transition to higher levels of reasoning in the LP?

THEORETICAL FRAMEWORK

Learning Progressions

LPs are playing an increasingly important role in mathematics education (NRC, 2001, 2007; Smith et al., 2006). According to the National Research Council, "Learning progressions are descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic" (2007, p. 214). Battista (2019) stated, "A LP for a topic (a) starts with informal, everyday-cultural, pre-instructional reasoning typically possessed by students; (b) ends with related formal mathematical concepts; and (c) indicates cognitive plateaus reached by students in moving from (a) to (b)" (p.1). Within a constructivist framework, LPs are used in assessment, standards, curriculum design, teaching, and research (Sztajn et al., 2012). Research indicates that, in addition to having a firm understanding of the mathematics they are teaching, teachers should understand the mathematical thought processes of their students, the general stages that students pass through in acquiring concepts and reasoning about a topic, and the reasoning students use to solve different problems at each stage—that is, to use detailed, research-based LP in teaching (Battista, 2019; Carpenter & Fennema, 1991; Cobb et al., 1990; Fennema et al, 1996; Steffe & D'Ambrosia, 1995). The purpose of this study is to demonstrate how a LP serves as an effective lens for assessment, curriculum design, and instruction by examining how iDGi Quadrilateral modules affect preservice teachers' (PTs) knowledge and reasoning, and how whole-class discussions can be leveraged within the LP framework to facilitate their transition to higher levels of geometric reasoning.

The iDGi Learning Progression for Geometric Reasoning about 2D Shapes

Battista's (2012b) LP was the basis for the iDGi curriculum, in-class PT instruction, and assessment.

Empirical derivation of the iDGi LP

The iDGi LP is based on a long line of research on students' geometry learning evolved from the van Hiele theory (1986), and dealing with geometric reasoning for students in K-12, and preservice elementary and middle school teachers (e.g., Battista, 2007a; Burger & Shaughnessy, 1986; Clements & Battista, 1992; Erdogan & Durmas, 2009; Fuys, Geddes, and Tischler 1988; Mayberry, 1983). However, the iDGi LP represents a significant, research-based elaboration and clarification of the original van Hiele theory (Battista, 2007a, 2009, 2012b; Borrow, 2000)." The iDGi LP utilized in iDGi instruction and assessment is based on research obtained through:[1]

- A thorough review of the extensive research on students' reasoning about shapes, much of it based on the van Hiele levels;
- Numerous individual clinical interviews and teaching experiments;
- Long-term teaching experiments with pairs of students;
- Long-term whole-class teaching experiments;
- Analysis of students' written work.

iDGi LP summary

Briefly, the iDGi LP levels are as follows[2][see Battista, 2012b for expanded descriptions and examples].

Level 1: Students identify shapes as visual-wholes

"Students identify, describe, and reason about shapes according to their appearance as visual wholes—what the shapes 'look like.' For instance, students often refer to visual prototypes, saying, for example, that a figure is a rectangle because 'it looks like a door,' or they might decide that two figures are the 'same shape' because they 'look the same'" (Battista, 2012b, p. 9).

1. Incorrect Identification of Shapes
2. Correct Identification of Shapes

Level 2: Students describe parts and properties of shapes

"Students recognize, specify, and analyze shapes by identifying their parts and describing spatial relationships between these parts – that is, by describing the shapes' properties. However, students' descriptions and conceptualizations vary greatly in

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[1] *Clinical interviews* investigate students' thinking by posing tasks that uncover the conceptualizations and reasoning that students use to make sense of particular mathematical topics. Typically, several interviews are conducted in sequence, with initial tasks based on published research, curriculum materials, or the interviewer's conjectures about the nature of students' knowledge. Subsequent sessions use new tasks that are based on students' responses to initial tasks. The goal of a clinical interview is to discover what goes on in a student's mind, to understand the student's understandings. Interviewers seek to determine where the student is in constructing meaning for a particular mathematical topic. They attempt to determine not just what students do but why they do it.

[2] In *teaching experiments,* researchers attempt to promote students' construction of new knowledge as a way of investigating the processes by which that knowledge is constructed and the common learning trajectories students take for learning particular mathematical concepts. In teaching experiments, after the researcher has developed a model of the student's current thinking about a topic, researchers hypothesize ways to guide the student's conceptualizations towards adult competence. The "experimental" part of teaching-experiment methodology consists of developing and testing ways of guiding students' construction of mathematical meaning for a topic.

sophistication, starting with visual, informal, imprecise specifications in sublevel 2.1 and ending with completely formal geometric specifications in sublevel 2.3" (Battista, 2012b, p. 19).

1. Strictly Informal Description Parts and Properties of Shapes
2. Informal and Inadequate Formal Description of Parts and Properties of Shapes
3. Complete and Correct Formal Description of Parts and Properties of Shapes

Level 3: Students interrelate properties and categories of shapes

"Students explicitly interrelate and make inferences about geometric properties of shapes. For example, a student might say 'If a quadrilateral has opposite sides equal in length (property 1), then it has opposite sides parallel (property 2).' ... The major factor that varies at Level 3 is the sophistication of students' justifications for statements about relationships between properties. These justifications start with empirical associations (when property X occurs, so does property Y), progress to construction based analyses that explain why one property 'causes' another property, move to logically inferring one property from another, and end with using inference to organize shape conceptualizations into a hierarchical classification system" (Battista, 2012b, p. 37).

1. Using Empirical Evidence
2. Using Analysis of Shape Construction
3. Using Logical Inference
4. Using Hierarchical Classification and Minimal Definitions

Level 4: Students understand and create formal deductive proofs

"Students can understand and construct formal geometric proofs. That is, within an axiomatic system, they can meaningfully produce a sequence of statements that logically justifies a conclusion as a consequence of the 'givens'" (Battista, 2012b, p. 45).

For the remainder of this paper, we will refer to the iDGi LP simply as the LP.

Multimedia-Based Digital Content and Dynamic Geometry Environments

Multimedia-based instruction combines diverse file formats—including text, audio, video, and imagery—to create an immersive educational environment that captures student interest and improves academic outcomes (Aldalalah, 2021; Aldalalah et al., 2025). By engaging multiple senses, these tools accommodate diverse learning speeds, foster classroom collaboration, and optimize instructional time (Aldalalah et al., 2025; Burgos, 2022). Rather than relying on traditional, lecture-based methods, teachers can pivot to mentoring/facilitating roles while students leverage multimedia for autonomous, self-paced learning. Empirical research underscores that using multimedia digital content significantly boosts student motivation and enhances information retention across various subjects (e.g., Aldalalah et al., 2025; Krishnasamy & Ling, 2020; Linder et al. 2021; Shenouda, 2022).

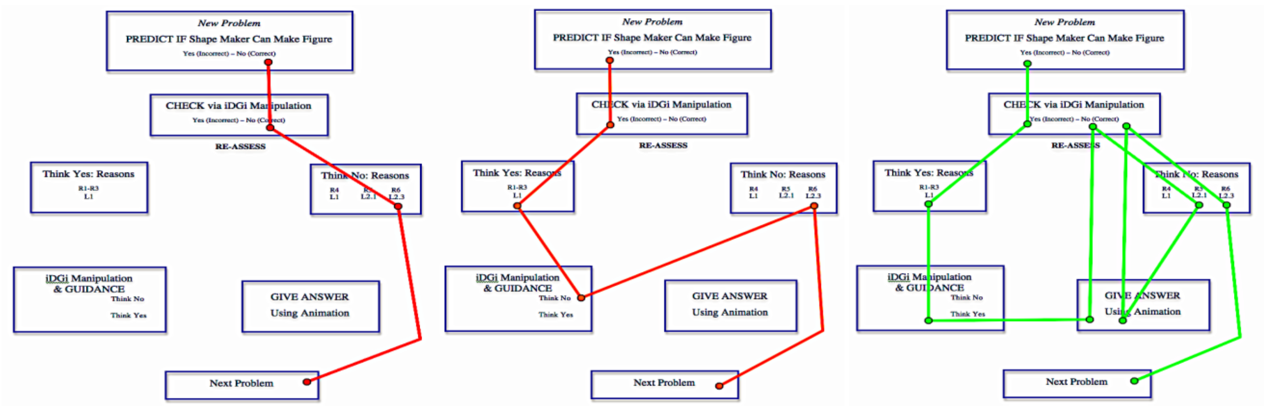
Building upon these multimedia components, educational digital content represents a sophisticated blend of interactive learning activities using audio-visual materials, rather than a simple digital conversion of printed textbooks. Grounded in learning theory, this content is systematically structured yet flexible enough to be updated easily and accessed by students and teachers anytime online via computers or mobile devices (Yasar et al., 2019). Ultimately, well-designed digital content cultivates student self-reliance, bridges individual learning gaps, and provides an error-free repository of knowledge that enhances both cognitive development and practical skills (Aldalalah et al., 2025; Al-Khayyat et al., 2016; Al-Shammari & Al-Shammari, 2019).

One example of such advanced digital content in mathematics education is Dynamic Geometry (DG) environments, where static shapes are replaced by manipulable, property-constrained screen objects. Researchers note that DG provides "a revolutionary means for developing geometrical understanding" (Mariotti, 2001, p. 257), earning strong endorsements from major U.S. educational organizations like the NCTM, NRC, CCSS, and NCATE. For instance, the Common Core State Standards for Mathematics emphasize that these environments "provide students with experimental and modeling tools that allow them to investigate geometric phenomena" (CCSSM, 2010, p. 74). One specific environment showing significant promise at the elementary and middle school levels is Battista's (2001; 2012c) *Shape Makers* microworld and curriculum. Battista (2001) found this microworld to be an extremely powerful technological tool for teaching critical geometric concepts and cultivating advanced reasoning techniques. Synthesizing these fields, the overarching purpose of this study is to examine how iDGi Quadrilateral modules and interrelated classroom instruction affect preservice elementary teachers' understanding of quadrilateral properties and interrelationships, while exploring how whole-class discussions can encourage PTs to transition to higher levels of reasoning in a LP.

iDGi

iDGi was created as an expansion and elaboration of Battista's (2012c) *Shape Makers* DG curriculum by extending it into an individualized online format. Its major focus is to move students forward through Levels 1 and 2, and into the Level 3, in the LP. iDGi research with students from 5th, 8th, and 10th grades shows dramatic gains in student understanding of quadrilaterals (Battista, 2019). We investigated if similar gains in achievement are made with PTs, a population that is known to struggle with learning and teaching mathematics. Moreover, research indicates that few in-service elementary teachers possess the required knowledge of mathematics or the inclination to teach in ways consistent with learning progressions (Carpenter et al., 1996; Charles & Carmel, 2005; NRC, 2003; Wilson et al., 2013).

To achieve the major goal of the iDGi Quadrilaterals unit to move students to Level 2.3, and for older students into sublevels of Level 3, iDGi uses a LP-based two-pronged instructional sequencing: (a) inter-module sequencing *between* modules that



(a) (b) (c)

Figure 1. Concepts maps that shows the different branching that occurs in iDGi modules. (a) Path for student who already knows the content; (b) Path for student who figures out the content with iDGi guidance; (c) Path for student who struggles to find answer and needs to be given the answer eventually (Source: Authors' own elaboration)

encourages movement through LP levels; (b) intra-module sequencing *within* modules, that interactively branch students according to the LP level of reasoning they are exhibiting at the time (more details on this below).

iDGi focuses, first, on developing prototypical defining properties of shapes, and second, on property-based interrelationships between shapes (e.g., Battista, Frazee, & Winer, 2018). As an example, the prototypical defining properties of rectangles are opposite sides congruent and parallel, and four right angles. These properties express in formal geometric terms the most visually salient spatial characteristics that students use in identifying rectangles. In the iDGi Quadrilaterals unit, students investigate, and meaningfully develop reasoning about, properties of squares, rectangles, parallelograms, rhombuses, kites, and trapezoids. We will discuss the details of our quantitative assessment tool of the iDGi Quadrilateral Tests 1 and 2 and the iDGi Quadrilateral modules in the Methodology and Methods section below.

Within-module structure. Continuous monitoring and instructional branching

iDGi software continuously monitors both students' answers and their reasoning and branches to various places in the module based on this information. The branching is implemented by locating students in the LP, then using the LP to decide on subsequent instructional goals and activities. For this study, to move on to the next module in iDGi, PTs had to score 80% in the current module and to get access to the iDGi Quadrilaterals posttests 1 and 2, PTs needed to complete all iDGi Quadrilateral modules. If PTs did not receive a score of 80% on an iDGi module, they had to repeat the module until they did. In this study, all PTs were eventually able to pass all modules with 80% and progress to the iDGi Quadrilateral Posttest 1 and 2.

Repeating iDGi module structure

Many iDGi modules are based on a repeating instructional-sequence structure that our research indicates (Battista, 2019) is extremely effective in teaching geometry and geometric measurement. This structure is outlined below.

- Student **PREDICTS** an answer to a geometric problem.
- Student **CHECKS** answer using iDGi dynamic-object manipulation.
- Student **EXPLAINS** reasoning that produced answer using Learning-Progression-Based Multiple Choice (LPBMC) items[3].
- Student is **BRANCHED** to various locations in the module based on LP-interpretation of answer and reasoning.
- One branch provides **GUIDANCE** on iDGi manipulation (what to manipulate and what to look for while manipulating).
- After guidance, if a student gives an incorrect answer or LP level of reasoning is not optimal, iDGi **GIVES ANSWER** (often with animation or a movie).
- Student is **RE-ASSESSED** to see if the answer and reasoning make sense to the student.

Through this branching students flow within iDGi modules was *individualized*. **Figures 1a-c** illustrate how student individualization was accomplished within iDGi by providing different paths that students might take through a module, guided by answer correctness and LP level exhibited. There are, of course, many other actual paths that students followed.

Teachers Knowledge of Geometry

To situate our study within the existing research, we reviewed relevant literature as it related to the van Hiele levels and K-8 PTs' content knowledge of geometry in teacher training programs. This literature yielded many findings about PTs' geometric content knowledge and subsequent instruction used to enhance that knowledge.

[3] The multiple-choice items have distractors that are generated from previous research using clinical student interviews, and that correspond to specific levels in learning progressions (Briggs et al., 2006, have labeled this format "Ordered Multiple-Choice [OMC]").

Although all K-8 PTs, *ideally* should have attained a Level 4 (deduction/proof) reasoning by the end of a high school geometry course, many researchers have argued that realistically PTs should at minimum be at least reasoning at Level 3 (informal deduction) to effectively teach geometry to K-8 students (Van der Sandt & Niewoud, 2003; Knight, 2006; Zilkova, Gunagam, & Kopacova, 2015). However, many studies have found that the majority of elementary PTs are reasoning lower than Level 3 in the van Hiele levels (e.g., Van der Sandt & Niewoudt, 2003; Milsaps, 2013; Robichaux-Davis & Guarnio, 2016; Knight, 2006). For example, Marchis (2012) assessed elementary PTs (n=36) van Hiele levels and found that two-thirds of participants were at Level 2.1 or 2.2 in the LP. Many of these PTs were able to identify quadrilaterals by their properties, but most times they were unable to sufficiently write a minimal definition that only used the prototypical properties needed to define that specific type of quadrilateral. A similar study done by Zilkova et al. (2015) found that many PTs could only identify types of triangles and quadrilaterals in prototypical configurations but did not accurately understand the properties of these shapes. They argued that it is troubling that teachers who were teaching primary students about shapes had misconceptions typical of Levels 1 and 2, when teachers need to be thinking minimally at Level 3 to be effective at teaching shapes. Moreover, Knight (2006) wondered if teachers can teach geometry effectively if the target reasoning for their students is higher than their own level of reasoning.

Another study done by Cunningham and Roberts (2010) found that their participating PTs had only a procedural understanding of geometry in which many resorted to rote memorization of the definitions of shapes and their properties. As a result, they found that their PTs lacked the conceptual understanding of the interrelationships of the properties both within shapes and among shapes, as well as could not recognize a minimal set of properties used to define those shapes. Similar studies have also found that most PTs struggled to understand the hierarchical relationships between the types of quadrilaterals (Duatepe-Paksu, Pakmak, & Iymen, 2012; Fujita, 2012; Millsaps, 2013; Pickreign, 2007). For example, Fujita (2012) found that most elementary PTs only identified prototypical examples of parallelograms, seeing that both pairs of opposite sides are parallel, but they did not recognize rectangles, rhombuses, and squares as special types of parallelograms.

There has been considerable research on developing curriculum and instructional strategies based on the van Hiele theory to teach and assess its impact on K-8 PTs' geometry content knowledge (e.g., Armah, Cofie, & Okpoti, 2018; Yi, Flores, & Wang, 2020). Most of this research has demonstrated a positive effect of such instruction on PTs' understanding of geometry content (e.g., Armah, Cofie, & Okpoti, 2018; Yi et al., 2020; Erdogan & Durmas, 2009). Many of these studies employed activities in which PTs explored geometric concepts using physical materials or manipulatives. For instance, Yi et al. (2020) engaged 111 elementary PTs in instructional activities aimed at developing: 1) their own geometry content knowledge, 2) an understanding of the van Hiele levels for K-8 students, and 3) the ability to select appropriate instructional activities aligned with K-8 students' van Hiele levels for 2D shapes. These activities involved instructional tasks such as sorting familiar and unfamiliar shapes based on geometric attributes, listing properties of different quadrilaterals, and using Venn diagrams to explore the interrelationships between various types of quadrilaterals in terms of their properties. While most studies, including Yi et al. (2020), have shown positive outcomes in enhancing PTs' understanding of 2D geometric shapes, the activities often relied on static images of shapes that lacked dynamic manipulability. In contrast, in the iDGi program, PTs engage with preconstructed Shape Makers for special quadrilaterals that are manipulable in real-time, allowing them to explore geometric properties of quadrilaterals interactively. This approach, which is embedded within a LP sequenced set of learning modules (more details to follow), enables PTs to better discover and understand the properties and interrelationships of quadrilaterals

Numerous studies have demonstrated the positive impact of DG environments on K-12 students' van Hiele levels (e.g., Battista, 2019; Clements & Battista, 1990; Breen, 1999; Kutlaca, 2013). However, fewer studies have examined how DG-based instruction affects PTs' van Hiele levels in geometry (Karakus & Peker, 2015; Lee, 2015; Stols, 2012). Moreover, many of these studies have focused on secondary PTs (e.g., Lee, 2015; Stols, 2012), rather than the K-8 PTs that are targeted participants in this study. Nonetheless, a few studies have included K-8 PTs as participants. For example, Karakus and Peker (2015) conducted a quasi-experimental design comparing two groups of K-8 PTs: one group engaged in DG-based instructional activities (using GeoGebra and Cabri 3D), while the other group worked with physical manipulatives and drawing activities. The DG activities included worksheets that guided PTs in constructing shapes and figures within the DG environment, which they then used to explore various geometric concepts. The results indicated significant gains in van Hiele levels for both the DG and physical manipulative groups, as measured by pre- and posttest scores. A key distinction between this study and iDGi, in terms of DG environment, is that iDGi does not require PTs to create their own geometric constructions in the DG environment. Instead, iDGi provides instructional modules, with all shapes already constructed and manipulable for exploring geometry concepts. These modules are designed to progressively guide PTs through the LP levels, reaching up to Level 3. In the current research, we investigated how iDGi supports PTs' understanding of geometric content they will teach, and later in the Discussion section, we compare their knowledge to that of K-12 students.

METHODOLOGY

We hypothesized that using the iDGi Quadrilaterals unit, along with subsequent PT class activities and discussions supplementing these modules, would significantly increase PTs' knowledge of properties of quadrilaterals and their interrelationships. We tested this hypothesis using the online multiple choice, LP-based, pre and posttests that are built into the iDGi Quadrilaterals unit. We collected the following types of data: iDGi pre and posttests, completion of iDGi modules, written homework related to iDGi coursework, and video recordings of all classes related to iDGi instruction. We performed statistical tests to analyze the PTs' pre and posttests scores and we analyzed three vignettes of a typical whole-class discussion to illustrate how iDGi related class activities and discussions enhanced PTs' learning from the iDGi modules (more on this in the Findings section).

Participants' Role in the Study and Data Collection

Participants for this study were 580 volunteer K-8 PTs enrolled in a required 10-week course for their majors at a university in the Pacific Northwest. The purpose of this course is to help PTs develop a conceptual understanding of the geometry and measurement topics found in grades K-8. Data collection occurred from April 2018 to March 2024, with 26 courses taught using iDGi (n=541) and one control group course (n = 39). Data collection for six courses occurred during COVID-19 and were taught synchronously on Zoom. The remaining 21 courses were taught in a classroom. The iDGi online units that were used in this study were already a part of the PTs' regular coursework. All iDGi related data was de-identified by the course instructor assigning each PT a username (e.g., S1Class1) that the PTs used to complete all their iDGi activities and assessments and allowed researchers to track PTs' work throughout the course.

The course met four times a week for 50 minutes for a 10-week quarter of instruction. Prior to the iDGi Quadrilaterals unit, PTs had completed a unit on spatial visualization and polygons and the iDGi Triangles unit. The iDGi Quadrilaterals unit was completed in ten to twelve 50-minute class periods. Courses were taught by 4 different instructors Dr. W (23 courses), Dr. G (1 course), Dr. B (1 course), and Dr. N (2 courses). All four instructors implemented the same lesson plans and activities and were trained in how to use iDGi applications. The same instructor, Dr. W, taught all the synchronous Zoom courses and tried to teach the course with as much congruence to the face-to-face instruction as possible. For both the iDGi modules and class activities each PT had access and used iDGi through their own electronic device.

The control group consisted of 39 PTs in an in-person section of the course. While the content and instructional time remained identical to the experimental group, these PTs did not complete any iDGi module assignments during the Quadrilaterals unit. Following instruction, the control group course completed iDGi Quadrilaterals Posttests 1 and 2 as the unit assessment. This course was randomly selected as the control group and instruction occurred during a regular academic quarter (i.e., Winter) in which only one section of the course was offered taught by

Dr. W, who taught the majority of the experimental group courses. The PTs in both control and experimental groups had passed the same prerequisite mathematics course, thus, showing a baseline equivalence between the control group and the experiment group.

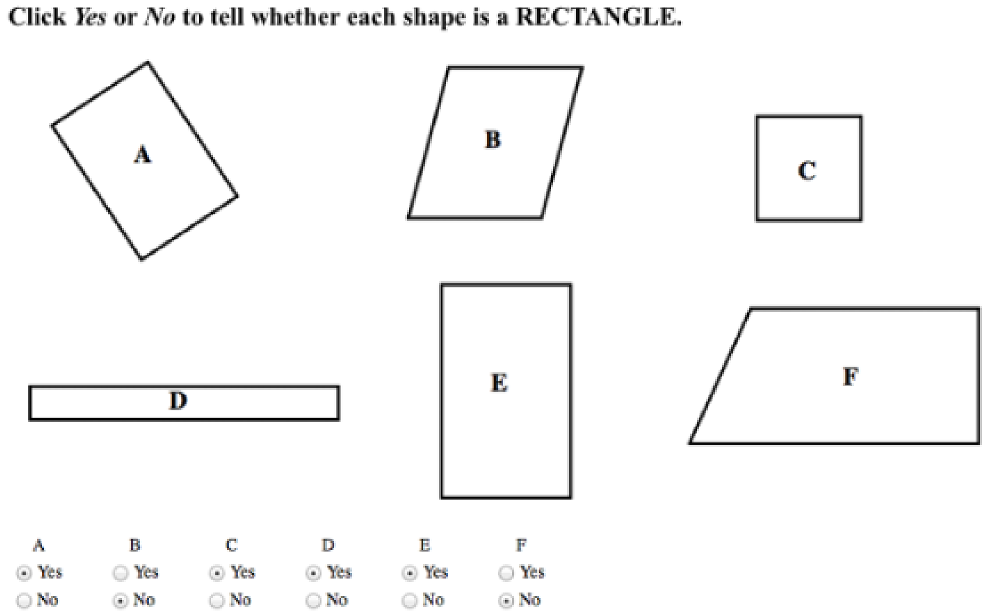
All enrolled PTs were eligible for the study, and those who signed an IRB-approved consent form were included as participants. PTs who declined consent still completed all course activities, assignments, and assessments; however, their data was excluded from the research. For the data analysis, we also removed PTs who were taking the course for more than their first time or who failed to complete any of the pre- or post-tests to have a full set of data for each participant, without any duplicates, to preserve equivalence in backgrounds of participants. More specifically, we excluded 17 PTs (not included in our final n = 541 treatment sample) from the quantitative analysis because they dropped the course before completing the treatment (the iDGi Quadrilaterals unit) and posttests, preventing us from tracking their pre-to-posttest progress to measure the iDGi curriculum's effect.

iDGi Quadrilateral Tests 1 and 2

We used a pretest-posttest assessment model to collect our quantitative data. Prior to the beginning of class instruction for this unit, PTs completed the iDGi Quadrilaterals Pretests 1 and 2, then after instruction (e.g., iDGi modules, In-Class Activities, class discussions, and homework assignments) for the unit was completed, PTs completed the iDGi Quadrilaterals Posttests 1 and 2 as their end of unit assessment. The iDGi quadrilaterals pre and posttests were identical and were each split into two parts (e.g., Test 1 and Test 2). One possible threat to the internal validity of the pretest/posttest design using the same test is that posttest results might be affected by taking the pretest. However, there are several mitigating factors to consider when analyzing this threat. First, PTs were never told that the pre and posttests were identical, the pre and posttests were locked after completion, and PTs were never shown test answers. Second, in the school-based iDGi project assessment design (Battista, 2019), there was a pretest/posttest control group in which grades 8-10 students—whose beginning and ending shape LP levels are similar to those of PTs (see **Table 4** later in the "Pedagogical Context" section for more details)—studied other parts of their mathematics curriculum while the treatment group did iDGi instruction. The grades 8-10 control group consisted of 250 students (12 classes, 6 teachers). The pre- and post-test time interval of the PTs was 10-12 days, and for the grades 8-10 control group, 8-10 days, so the test-retest effect should be similar to or even less for the PTs versus the grades 8-10 students.

The grades 8-10 control student scores were as follows: Pretest 1: M=55.87, SD=7.18, Pretest 2: M=55.53, SD=8.67, Posttest 1: M=59.22, SD=8.27, Posttest 2: M=60.13, SD=7.25. The pre-posttest relative increases for this control group were, Test 1: 5.99% [(59.22-55.87)/55.87=5.99%], Test 2: 8.29%. Thus, this data suggests that there was only a small test/retest effect in comparison to the pretest/posttest iDGi group gains of Test 1: 70.94%, Test 2: 47.68%.

Test 1 consisted of three sections. In the first and second sections, participants answered a question, then gave a reason for their answer—each scored at 0.5. For a correctness score, participants received .5 if their answer was correct; they received .5 for their reason if it was at the highest LP level possible for that item. For a reasoning level score, LP levels were assigned depending on the correctness of the answer, in combination with the reasoning choice level. Reasons for choices were presented in multiple-choice format, with distractors generated from previous research on the tasks using clinical student interviews (see below). In the third section of Test 1, participants answered several yes or no questions and got 1 point if all were correct—a LP level was assigned based on the configuration of participants' yes or no answers. Test 2 consisted of two sections. In both sections, participants answered a question, then gave a reason for their answer; each part scored at 0.5. For each item, a LP reasoning level was assigned based on the correctness of the answer in combination with the reasoning choice level.



a. Page 1: Student answers (6 possible answers)

Describe exactly how you decide if a shape is a rectangle or not.

R1 ○ I look at the shape to see if it has 4 right angles and its opposite sides are equal and parallel.

R2 ○ I look at the shape to see if it looks like a box or door.

R3 ○ I look at the shape to see if it has 4 right angles.

R4 ○ I look at the shape to see if it has two long sides and two short sides.

R5 ○ I look at the shape to see if it has opposite sides equal.

b. Page 2: Student reasons

Page 1 Answer Total		*Page 2 Reason*	*LP Level*	*Scores: Answer Reason*
<5	and	any	Level 1.1	0 P1 0 P2
≥5	and	R2	Level 1.2	.5 P1 0 P2
≥5	and	R4	Level 2.1	.5 P1 0 P2
≥5	and	R5	Level 2.2	.5 P1 0 P2
≥5	and	R1	Level 2.3	.5 P1 .5 P2
≥5	and	R3	Level 3	.5 P1 .5 P2

c. Answer-reason combination LP Level indicators

Figure 2. Example iDGi reasoning about rectangles assessment task and scoring criteria (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)

iDGi assessment design components: Type and validity of reasoning

Two major components must be considered in analyzing the sophistication of participants' geometric reasoning—their LP levels. The first component is *type of reasoning*, which has several levels and ranges from visual-holistic reasoning to formal proof. The second component is *validity of reasoning*, which involves the accuracy and precision of participants' identifications, descriptions, conceptualizations, explanations, and justifications. Validity is judged relative to level and type of reasoning. For instance, the validity of Level 1 visual-holistic reasoning is judged by accuracy in identifying shapes. In contrast, the validity of Level 4 reasoning is judged by correctness of formal proofs. Pre/post-assessment of participants' learning and within-module branching were accomplished within iDGi *by analyzing the type and validity of their reasoning*. **Figure 2** shows an example of how iDGi includes both components in assessment tasks.

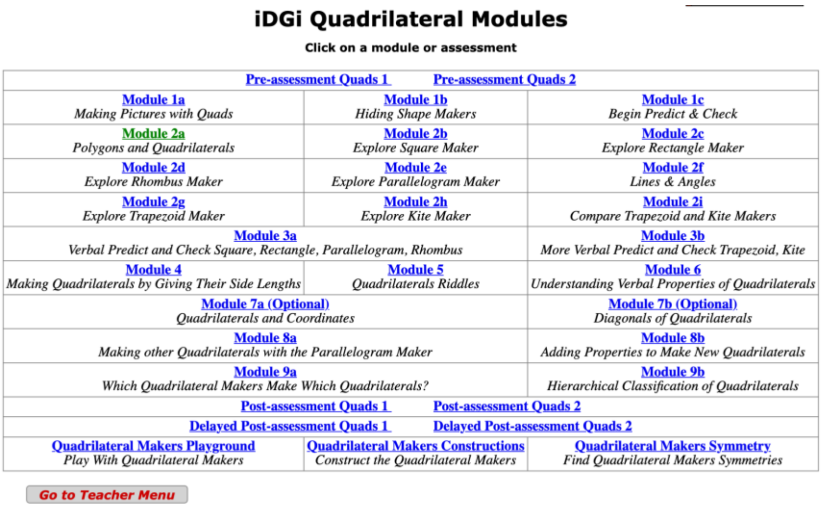


Figure 3. Main menu for the iDGi quadrilaterals modules (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)

Ordered multiple-choice assessment tasks

In large scale studies, individual interviews become impossible, so an alternate way is needed to assess student reasoning. The method used in iDGi is called "Ordered Multiple-Choice [OMC]" by Briggs et al., 2006. OMC items use multiple choice distracters that K-12 students have given in previous interviews and teaching experiments and that correspond to specific LP levels. "Recent evidence suggests that OMC items do a better job than open-ended items of eliciting responses similar to the understanding that students express in think-alouds and interviews (Alonzo & Steedle, 2009, p.702)." Briggs et al., 2006, suggest that using OMC items provides a practical and valid way to assess student reasoning in large scale research. Using OMC items in iDGi seemed especially appropriate because of its large scale, and because of all the extensive previous interview and teaching-experiment research on the topic of special quadrilaterals, both in computer and non-computer contexts, that provides in-depth knowledge of students' reasoning on iDGi instructional topics.

iDGi Modules and Class Instruction

The course was taught using an inquiry-based learning approach (Laursen et al., 2015), which is also known as "ambitious teaching" (Smith, Huinker, Bill, 2017). PTs were always asked to independently complete iDGi modules as iDGi homework assignments prior to the in-class activity/lesson that addressed those module concepts. The instructors wanted the PTs to individually explore the concepts in the iDGi modules prior to the in-class activity to develop a foundation of knowledge about quadrilaterals that could be extended by the subsequent in-class activity and its whole class discussion. Inside and outside of class, each PT used a personal electronic device to complete the iDGi modules and participate in in-class activities. A typical in-class lesson consisted of the following activities: 1) the instructor explained the directions for the class activity, 2) PTs worked collaboratively in small groups on the iDGi activity, 3) the instructor facilitated a whole class discussion related to their work. In-Class activities focused on iDGi module content or supplemental activities from Battista's (2012c) *Shape Makers* curriculum. While PTs worked on in-class activities, the instructor circulated to assess and support PTs and select tasks for whole class discussion. Selected tasks were chosen based on one of three criteria: 1) many PTs struggled or gave incorrect answers, 2) the task extended a concept beyond the modules, or 3) the task aligned with the teacher's learning goals for the activity.

Prior to the whole class discussion, the instructor reminded the class of their talk expectations and norms for whole class discussion, which were adapted from Smith, Huinker, and Bill (2017). During the whole class discussion, the instructor asked multiple PTs to present their answers and strategies and to explain their reasoning for each of the selected tasks. After PTs shared their strategies or answers, the instructor facilitated discussion by encouraging questions, disagreements, and additions from classmates; posing follow-up questions, using turn-and-talks, guiding the class towards consensus; and occasionally summarizing key learning from discussion about quadrilaterals.

In **Figure 3**, is the main menu of iDGi Quadrilaterals Modules 1a through 9b that PTs completed during this unit. All iDGi Quadrilaterals module homework assignments were posted a week ahead of their due date and needed to be completed before class time on a designated day. Recall, to move on to the next module, PTs had to score 80% on the problems in the module. PTs were informed of their score on the last page of the module. iDGi did not reveal which answers PTs got wrong, requiring them to redo the module and reflect on each response to identify mistakes and learn from them. Lastly, PTs were allowed to work ahead on the modules if they wanted to get ahead on their iDGi assignments.

The COVID-19 pandemic lockdown occurred during some data collection, and as a result, six courses were taught online by the same instructor synchronously through Zoom. The instructor of these six courses tried to adapt all the in-class activities to the Zoom platform as similar as possible with the constraints of the videoconferencing software at that time. When PTs worked in small groups, the instructor had Zoom randomly assign PTs to breakout rooms (three PTs per breakout room). Note, when the course was taught on Zoom, the groups were randomly assigned at each meeting whereas when the course was taught face-to-

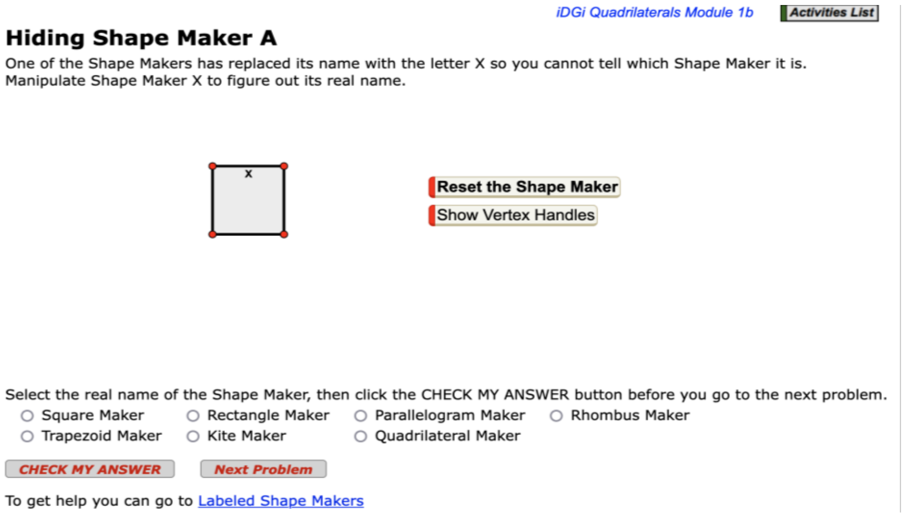


Figure 4. A sample task from Quadrilaterals Module 1b (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)

face, PTs typically worked with the same people who they were seated near each other in the classroom. The instructor uploaded the in-class activity to an online whiteboard – a cloud-based digital canvas that allowed PT groups to work together and write simultaneously in real time. Each group received its own copy of the online whiteboard activity. PTs could download their group's completed activity from their online whiteboard as PDF for future reference. As PTs were working together on the online whiteboard class activity, the instructor would move from breakout room to breakout room to see PTs' progress and reasoning on the in-class activity. Due to the constraint of Zoom only allowing the instructor to be in one breakout room at a time the instructor did not get to as many groups or as fast as when the course was taught face-to-face. This made it difficult for the instructor to implement the same formative assessments techniques as he would in a physical classroom. Whole class discussions were held in the same manner as described in the previous paragraph just in the Zoom meeting with PTs clicking the raise hand button and waiting to be called on by the teacher. During class discussions, the instructor either shared his screen to let PTs annotate their solutions or had PTs share their screens to present their strategies from their online whiteboard.

Below, we describe the implementation of the iDGi Quadrilaterals unit, including the iDGi homework assignments and in-class activities. These descriptions are intended to help researchers understand how the curriculum (i.e., the treatment) was enacted with PTs and to provide sufficient detail for mathematics teacher educators to replicate the unit with their own PTs.

iDGi pretests and iDGi homework assignment 1

Prior to the beginning of class instruction for this unit, PTs were asked to complete the iDGi Quadrilaterals Pretests 1 and 2 (described in the "iDGi Quadrilaterals Tests 1 and 2" section above), then to complete iDGi Quadrilateral Modules 1a through 2f (i.e., iDGi Homework Assignment 1). Modules 1a through 1c had PTs explore the seven basic types of quadrilaterals by providing them with different Shape Makers that could be manipulated only in ways consistent with the properties that are associated with that specific type of quadrilateral (e.g., rhombus). In Module 1a, PTs were shown and encouraged to play with/manipulate the seven types of quadrilateral Shape Makers to create a given image. The module explicitly told PTs that each specific Shape Maker could only make shapes that fit that type of quadrilateral (e.g., the Square Maker can only make shapes that are squares). In Module 1b, PTs were given a series of problems in which they were given an unknown Shape Maker and were asked to manipulate the Shape Maker to determine which type of quadrilateral it represented (see **Figure 4**). After manipulating the unknown Shape Maker, the PTs had to select an answer and then check to see if their answer was correct. iDGi only told the PT if they were correct or incorrect, but not why. If the student was incorrect, they had to reflect on and select a different answer.

iDGi Modules 1c through 2e, and later Modules 2g through 2i, had the PTs explore the following types of quadrilaterals each with their own modules in the following order: quadrilaterals, squares, rectangles, rhombus, parallelogram, trapezoid, and kite. In all these modules the PTs were asked to first predict whether a given blue shape could be made by a specific Shape Maker (e.g., Square Maker) without being able to manipulate the Shape Maker. PTs then checked their initial prediction by manipulating the specific gray Shape Maker (e.g., Square Maker) to determine if it could make the blue shape; they were allowed to change their answer from their initial prediction (see **Figure 5**). Next, PTs selected from a predetermined, research-based, list of four statements the one that best reflected the reason they selected their answer. Lastly, iDGi told PTs whether their answers were correct and let them click a button to play an animation showing the Shape Maker attempting to form the blue shape. These modules challenged the PTs to think about what properties were always true for each type of quadrilateral and how these properties determined the possible spatial configurations of the given Shape Makers. In Module 2f, PTs investigated the angle properties that are associated with parallel and intersecting lines (e.g., alternate interior angles, same side interior angles, vertical angles etc.).

In-class activity 1

In-Class Activity 1 had two parts. In the first part, the instructor led a whole class discussion where PTs shared their answers and reasoning for selected problems from Module 1b (see **Figure 4** for an example problem). The second part of In-Class Activity

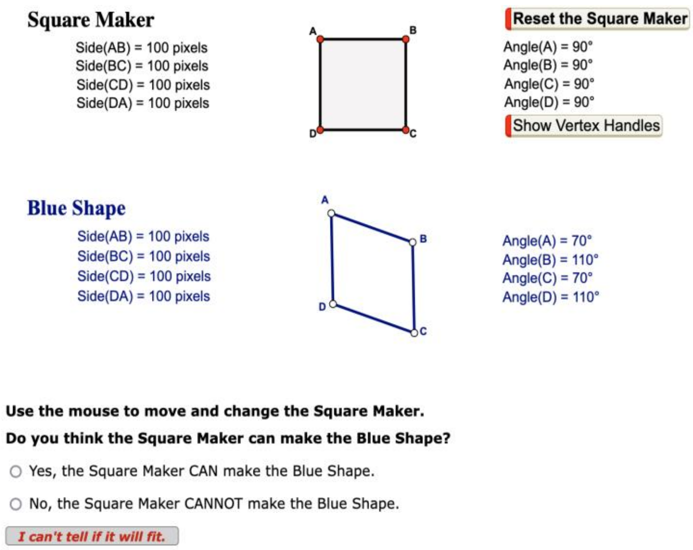


Figure 5. Sample *Predict and Check* activity from Module 1b (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)

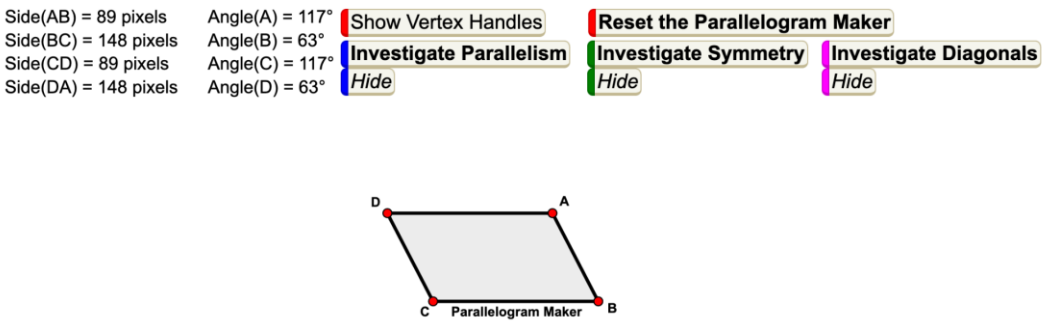


Figure 6. Parallelogram maker (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)

1 was adapted from Battista's (2012c) *Shape Makers* book, the activity entitled "Which Shape Maker Can You Use?" The activity asked PTs to predict if each of the seven types of quadrilaterals could make a quadrilateral fit six different descriptions shown below.

> If a Shape Maker cannot make one of the described quadrilaterals, Explain why or why not.
>
> 1. A shape with all different side lengths.
> 2. A shape with side lengths 120, 120, 180, 180.
> 3. A shape with all different angles.
> 5. A shape with angles 45°, 135°, 45°, 135°,
> 6. A shape with two pairs of sides parallel.
> 7. A shape with at least one line of symmetry.

PTs discussed and had to come to an agreement with their partners, providing an explanation before moving on to the next problem. PTs were asked to make all their predictions for listed shapes 1-6, before using the iDGi Shape Makers to check all their predictions for those shapes with their partners. The iDGi quadrilateral Shape Makers application for this activity included all seven quadrilateral Shape Makers, as shown for example the Parallelogram Maker, in **Figure 6**. In this and subsequent modules, each Shape Maker page had side lengths (in pixels) and angle measures (in degrees), with measurements updating instantaneously as the Shape Maker was manipulated. Each Shape Maker also had buttons that allowed PTs to investigate parallel sides, lines of symmetry, and properties of diagonals.

Battista (2012c) summarizes the learning goals for this activity as:

Riddle 8
I can have 0, 2, or 4 right angles, but never only 1.
Which Shape Maker am I?

Square Maker, Rectangle Maker, Parallelogram Maker, Rhombus Maker, Trapezoid Maker,

Kite Maker, Quadrilateral Maker

Answer:

Informal proof/argument:

Figure 7. Sample riddle from the Quadrilateral Riddles assignment (Module 5) (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)

> We are hoping that students' reasoning is moving away from vague ideas such as 'You can't make different angles with a Parallelogram Maker' towards more property-based reasoning such as 'The Parallelogram Maker always has the angles across from each other [opposite angles] equal.' This activity encourages students to conceptualize shapes in terms of their properties and to develop ways to communicate about these conceptualizations (p. 57).

The activity also started to help PTs notice that some properties were shared among different types of quadrilaterals; for instance, squares, parallelograms, rhombuses, and rectangles all had two pairs of opposite sides parallel. The activity intended to challenge PTs to move toward full attainment of Level 2.3 reasoning which requires them to think about properties as objects, ones that have reached the interiorized level of abstraction so that they can be detached from their original shape contexts and applied to new situations for analyzing shapes (Battista, 2007).

iDGi homework assignment 2

PTs were assigned to complete iDGi Modules 2g through 4 (Modules 2g through 2i were briefly discussed earlier). Modules 3a and 3b continue to have PTs presented with verbal descriptions of quadrilaterals and asked if a specific Shape Maker could make the described shape, similar to the types of problems addressed with the In-Class Activity 1 from *Shape Makers* entitled "Which Shape Maker Can You Use?" (Battista, 2012c). Having PTs complete Modules 3a and 3b directly after In-Class Activity 1 functions as a critical formative step; it allows us to evaluate how effectively PTs can extrapolate core concepts from the collaborative activity and apply them to distinct, unfamiliar problems. Module 4 asked PTs to enter side lengths to make a quadrilateral that is always a specific type of quadrilateral, specifically a rhombus, a parallelogram, or a kite, no matter how they move the vertices. The learning goal for this module was to get PTs to start thinking about the prototypical defining properties involving side lengths that are needed to define a rhombus, a parallelogram, or a kite. This leads into the next in class activity in which PTs reflect on properties that are always true for each type of quadrilateral.

In-class activity 2

For In-Class Activity 2, the instructor wrote each of the names of the seven types of quadrilaterals on a whiteboard and provided the PTs with the following prompt, "For each type of quadrilateral, I want you and your partners to describe the properties that were *always true* for that shape and explain why. You can use the iDGi quadrilateral Shape Makers to investigate the properties that are always true for each shape." First, PTs explored and discussed in small groups which properties they believed were always true for each type of quadrilateral. Then, in a whole class discussion, volunteers shared their group's findings. The instructor asked the class if they agreed or disagreed with each property proposed by PTs and to explain why. When there was disagreement on a proposed property, the instructor let PTs engage in a mathematical debate to try to convince others whether a property was always true. The instructor also asked clarifying and assessing questions to help refine PTs' ideas about the properties they proposed. The learning goal of this activity was, as a class, to get PTs to come to a consensus of what properties were always true for each type of quadrilateral and to clear any lingering misconceptions about the properties. Some PTs would initially state properties of quadrilaterals that were sometimes true (e.g., a parallelogram has a line of symmetry) and did not recognize that the property was not always true until they explored with the Shape Makers or participated in whole class discussion. Note, In-Class Activity 2 often spanned two days. Later in the Findings section we will present a sample vignette of a class discussion from In-Class Activity 2.

iDGi homework assignment 3

PTs were assigned a handout for homework entitled "Quadrilateral Riddles". PTs were assigned ten riddles which gave them clues about a mystery shape's properties and they had to determine which quadrilateral Shape Maker best fit the riddle (see an example of riddle in **Figure 7**). The riddles that were presented in this assignment are the same riddles presented in iDGi Module 5. Once they made their prediction for which one Shape Maker is described by the riddle, they wrote an informal proof or argument explaining why they thought their prediction was correct. PTs were not allowed to check their answers with Shape Makers until after they had submitted their work. The instructor wanted to see how PTs reasoned to initially deduce which Shape Maker was the most appropriate and how they used the verbal properties of types of quadrilaterals to justify their answer. In the directions of the assignment, PTs were told that to "informally prove" meant to write an argument that would convince others that they were correct.

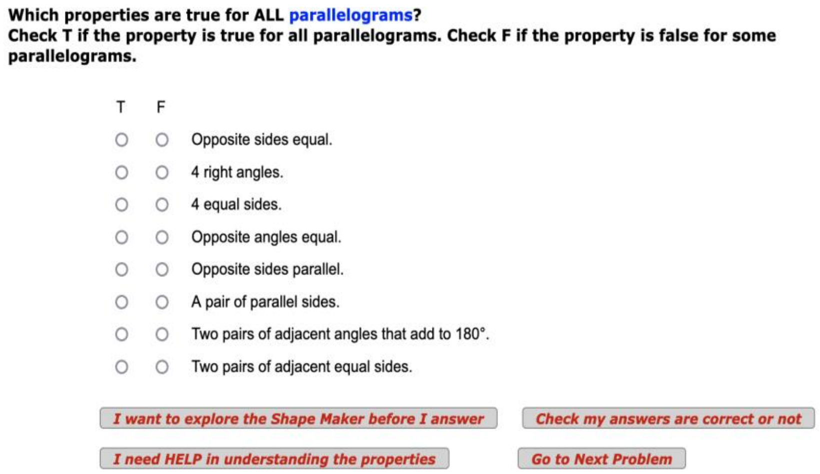


Figure 8. Sample question from Module 6 (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)

In-class activity 3

In-Class Activity 3 was a whole class discussion concerning the quadrilateral riddles assignment the PTs had completed for iDGi Homework Assignment 3 (i.e., iDGi Module 5). PTs were instructed to login into iDGi and open Module 5, the quadrilateral riddles. Each riddle was posted on the projector and read by the teacher. The PTs were asked to turn-and-talk with their partners for a few minutes about how they answered and informally proved their answer was correct. PTs were also allowed to use the Shape Makers during their turn-and-talks. This gave PTs the opportunity to discuss their answers and arguments with classmates, allowing them to further refine their conceptions prior to having a whole class discussion about the riddle.

As part of the role as facilitator, the instructor asked questions that encouraged PTs to critically analyze others' solutions such as: "Is that the only Shape Maker that fits the description? How do you know it is not the Parallelogram maker?" Sometimes the instructor had to act as a naysayer in which he would ask questions to consider other Shape Makers that the PTs did not seem to consider. For example, for Riddle 8 as shown in **Figure 7**, many PTs did not even consider a kite as a possible answer. As a result, the instructor asked PTs "Would a kite work for the riddle? Why or why not? Or I think a kite might work? Tell me why you think a kite would or would not work?" PTs tended to not recognize that a kite can have 0, 2, or 4 right angles and these types of questions forced them to investigate why a kite would not work for this riddle, which led to more productive discussion. After the class came to consensus on an answer, the instructor selected the class answer in iDGi to check if the class was correct. PTs were also encouraged to put in answers into Module 5, which would help complete the Module 5 and be able to move on to Module 6. The instructor facilitated this type of class discussion for as many of the riddles as possible during the class period, most times getting through all the riddles.

iDGi homework assignment 4

After In-Class Activity 3, PTs completed iDGi quadrilateral Modules 6, 7a, 7b, 8a, 8b, 9a, 9b before the next class meeting. Module 6 asked PTs to select the properties that were always true for each of the seven types of quadrilaterals (see **Figure 8**). PTs could either select their answers directly or investigate further using Shape Makers and a help page that explained different properties (e.g., equal opposite sides or angles). When PTs were ready to submit their final answers, they clicked "check if my answers are correct or not" and were either told "some of your answers were incorrect" or "all of your answers were correct". iDGi did not explicitly state to the PT which specific answers were correct or incorrect. It was up to the PT to reflect and determine which answers needed to be changed. The learning goal for this module was for PTs to solidify their understanding of using succinct verbal statements to specify properties of each type of quadrilateral.

In Module 7a, PTs explored how to create various quadrilaterals—rectangles, squares, parallelograms, rhombuses, and trapezoids—by specifying the coordinates of their vertices on the Cartesian plane. Module 7b had PTs investigate the different properties of diagonals for the seven types of quadrilaterals. The learning goal for this module was to introduce PTs to the different properties involving diagonals for special quadrilaterals that are addressed in high school/middle school geometry.

Module 8a asked PTs to investigate how adding properties to a Parallelogram Maker can make other special quadrilaterals. This starts to get PTs to reason about hierarchical classifications of quadrilaterals (i.e., Level 3.4 reasoning). iDGi asked PTs the following questions: "What has to be true about its measurements for a Parallelogram Maker to make a rectangle, a square, or a rhombus?" PTs were given a Parallelogram Maker in which they could enter different side lengths and angle measures to explore the changes that occurred to the Parallelogram Maker in real time. Although Module 8a was not typically directly discussed during in class activities/lessons, its concepts were typically addressed/discussed by PTs in the class activity involving Module 8b.

Module 8b continued to move PTs toward Level 3 reasoning, especially Level 3.3, at which PTs use logical inferences to relate properties and understand minimal definitions. For example, Problem 1 in Module 8b asked "If the Parallelogram Maker makes a shape with a right angle, what is the most accurate name for the shape?" For Module 8b, the instructor assigned PTs to write arguments for each problem in the module that explained why they thought their answer was correct and submit it to Canvas as

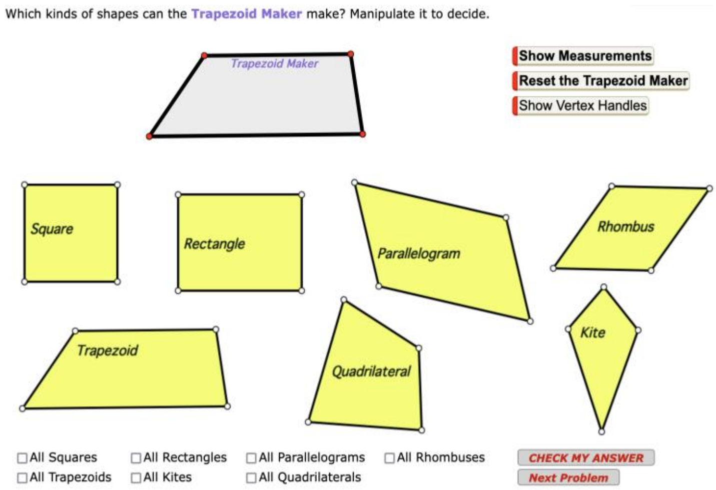


Figure 9. Sample problem from Module 9a (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)

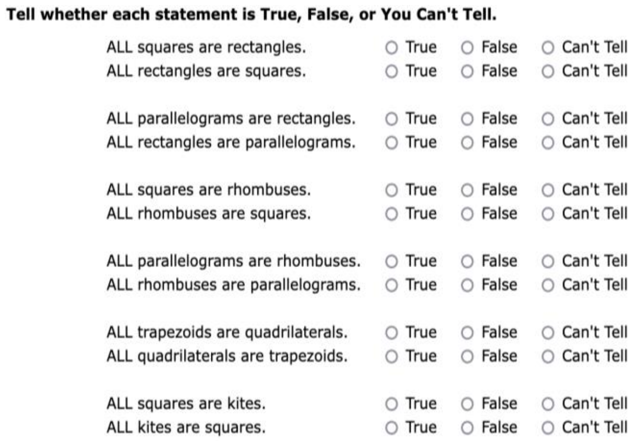


Figure 10. True or false statements asked in Module 9b (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)

a written assignment. Requiring PTs to submit written arguments for each problem served as a diagnostic tool to assess whether their reasoning was approaching this Level 3.3 threshold. At this level, reasoning is characterized as "locally logical," meaning students draw inferences from properties they previously established empirically through the Shape Makers (Battista, 2012b). Consequently, PTs were not expected to employ the organized axiomatic systems typical of Level 4 formal deduction proofs; rather, the objective was to cultivate informal arguments based on geometric properties (e.g., concluding that because opposite angles in a parallelogram are congruent, a single right angle necessitates that its opposite angle is also a right angle). This foundational work directly informs In-Class Activity 4, which is discussed in further detail below.

Module 9a had PTs investigate the interrelationships between the special types of quadrilaterals by asking PTs which other special quadrilaterals a specific Shape Maker can always make. For example, "Which kinds of shapes can the Trapezoid Maker make? Manipulate it to decide." (see **Figure 9**). PTs were able to manipulate the given Shape Maker (e.g., Trapezoid Maker) to help them determine what other special types of quadrilaterals it can always make. This is another step toward understanding hierarchical classifications of quadrilaterals and the interrelationships between them (i.e., Level 3.4 reasoning).

In Module 9b, PTs were asked to predict which of the statements shown in **Figure 10** were true or false. Note, PTs inferred from their prior knowledge that "Can't Tell" was not a valid answer to these questions. As the PTs progressed through the module, each of the "All" statements shown in **Figure 10** were rephrased in a different way to help PTs understand their meaning. For example, "All squares are rectangles," was rephrased to "Every shape made by a Square Maker can also be made by the Rectangle Maker, so all squares are rectangles". Then PTs investigated each of the statements using the provided Shape Makers. This module provides PTs with the groundwork to use the Shape Makers to help understand interrelationships between classes of shapes (e.g., squares and rectangles, parallelograms and rectangles etc.). For example, when PTs notice that a rectangle has all the properties of a parallelogram, but also has right angles, they are seeing a relationship that can enable them to see that a rectangle is a parallelogram with right angles, thus rectangles are parallelograms. PTs can begin to see how one class of shapes is related to another class of shapes, moving them to Level 3.4 reasoning. This module intended to encourage PTs to recognize that a Rectangle Maker can make every shape that a Square Maker can make, but not the other way around, thus, all squares are rectangles. The module continues this type of reasoning and extends it to other pairs of quadrilateral Shape Makers.

In-class activity 4

In-Class Activity 4 involved the instructor engaging the PTs in a whole class discussion about the problems in Module 8b. The instructor presented a problem from Module 8b, then had PTs turn-and-talk with partners to decide on an answer and develop an informal proof/argument. For most PTs, this was their first time discussing Module 8b problems with a classmate, giving them an opportunity to revise their reasoning. Next, the instructor called on multiple volunteers to share their answers and their arguments with the whole class. The instructor did not validate these responses but rather asked for other PTs, if they agreed or disagreed (and why), to add on to the presenter's reasoning, or ask questions of the presenter's reasoning. If key elements of the problem were missed, the instructor asked guiding questions or played the skeptic to deepen student reasoning.

For example, Problem 1 asked, "If the parallelogram makes a shape with *a right angle*, what is the most accurate name for the shape?" PTs often overlooked explaining why one right angle in a parallelogram implies that all angles must be right angles. As a result, the instructor made sure that during discussion the PTs addressed this question. The learning goal for this question was to get PTs to use angle properties of a parallelogram, such as opposite angles equal or all pairs of adjacent angles are supplementary, to explain why one right angle in a parallelogram caused all the angles to be right angles. We will provide a sample vignette of this whole class discussion and its analysis later in the Findings section entitled "Vignette of Class Discussion That Helps Move PTs to a Consistent Level 3 Reasoning". Other concepts addressed during this class activity were how to identify when two lines are parallel if we do not have a "Parallel lines test" button, which indicates the pairs of sides that are parallel on the Shape Maker. The instructor wanted PTs to notice that pairs of parallel lines always had two pairs of adjacent angles supplementary (i.e., Same Side Interior Angles Theorem). Thus, the instructor posed the question in the whole class discussion as a follow up question from problem 7 in Module 8b, which asked "If the Quadrilateral Maker makes a shape with 4 right angles, what is the most accurate name for the shape?" In the context of problem 7, once the PTs came to consensus on the answer of the rectangle, the instructor asked them "how do you know that opposite sides are parallel?" The instructor also pointed out that on the posttest there will be no "Parallel lines test" button to indicate which sides are parallel. The concept – covered in Module 2f – highlighted that if two pairs of adjacent angles in a quadrilateral are supplementary, it forces a pair of parallel sides. The overall goal of the class discussion with the problems from Module 8b was to get PTs to think at Level 3.3 "students use a logical inference to relate properties and understand minimal definitions of shape" (Battista, 2012b, p.9) and to get them started using deductive reasoning to create informal proofs/arguments, which is necessary prior knowledge to be successful at construction formal deductive proofs (i.e., Level 4).

In-class activity 5

In-Class Activity 5 was consistently implemented from Spring 2018 to Fall 2019 and was cut for time when the course was taught on Zoom during the COVID-19 pandemic and was not implemented after face-to-face courses resumed in Fall 2021. For In-Class Activity 5, the instructor used *Polleverywhere* software to create an online poll that had PTs answer each of the questions found in Module 9b (see **Figure 10**) online using their electronic devices at the beginning of class. Using an online poll allowed PTs to honestly answer the problems in Module 9b anonymously, without influence from their peers. This allowed the instructor to get instant feedback and know which problems needed to be discussed as a class based on the poll. For example, if the percentages of true or false questions were split in the poll, then the instructor asked the class to justify their answers in a class discussion. If the class unanimously agreed on a correct answer for a problem, then the instructor only had a brief discussion about the problem with the class. PTs engaged in mathematical debates about what the correct answer was and why, in which the instructor tried to get the class to come to a consensus before moving on to the next question. After PTs debated over a problem, the instructor would do a quick poll of thumbs up for true and thumbs down for false to see if the class could come to a consensus. When possible, the instructor would ask PTs what had changed their answer to explain why they changed their answer. During Fall 2020, the instructor(s) rolled most of the concepts from In-Class Activity 5 into In-Class Activity 6.

Homework assignment 6

Prior to In-Class Activity 6, PTs completed a homework assignment entitled Quadrilateral Relationships assignment. In Module 9b, PTs were asked true or false questions about interrelationships between different Shape Makers, like "All squares are rectangles". The possible answers for Module 9b were only true (i.e., always) or false (i.e., never). For the Quadrilateral Relationships assignment, PTs were asked again about the interrelationships between special types of quadrilaterals, but the answers changed from true or false to always, sometimes, or never. For example, one problem was "A trapezoid is (always, sometimes, or never?) a kite." The PTs had to answer 11 problems like this and had to write convincing arguments that explained why their answers were correct and had to submit them for a grade prior to the class activity (see **Figure 11**). This added complexity to these problems when compared to problems found in Module 9b was the answer of "sometimes" meaning that if one example could be made, then the answer changed from "never or false" to "sometimes". The learning goal was for PTs to understand and justify the interrelationships between the special types of quadrilaterals using prototypical properties and move them to forming hierarchical classifications of the special types of quadrilaterals (i.e., Level 3.4 reasoning).

In-class activity 6

For In-Class Activity 6, the instructor presented a problem from the Quadrilateral Relationships Assignment on the projector and then did a poll in which PTs gave a thumbs up for "always", a horizontal thumb "sometimes or can be", and a thumbs down for "never". After the instructor noted the results, they asked the PTs to turn-and-talk to discuss their answers and present convincing arguments with their partners. Then the instructor had PTs engage in mathematical debates in which they shared their answers and arguments using the properties of the types of quadrilaterals in a whole class discussion. The instructor facilitated discussion by calling on volunteers to share, asking questions, playing a skeptic to have PTs further explain their reasoning, asking

Answer each of the following with either always, sometimes, or never. ***Then write a convincing argument that proves why your answer is correct.*** *If you are stuck, you can use the Quadrilateral Makers Playground on iDGi to explore more with Shape Makers.*

1. A trapezoid is _________ a kite.
2. A kite is ____________ a rectangle.
3. A square is ___________ a trapezoid.
4. A trapezoid is ___________ a rectangle.
5. A rectangle ____________ has only one right angle.
6. A kite is __________ a rhombus.
7. A parallelogram is __________ a trapezoid.
8. A quadrilateral is ___________ a rectangle.
9. A trapezoid _______________ is a parallelogram
10. A kite is _________________ a trapezoid.
11. A parallelogram is ________________ a kite.

Figure 11. Quadrilateral relationships assignment (Adapted from Millsaps, 2013, p. 36)

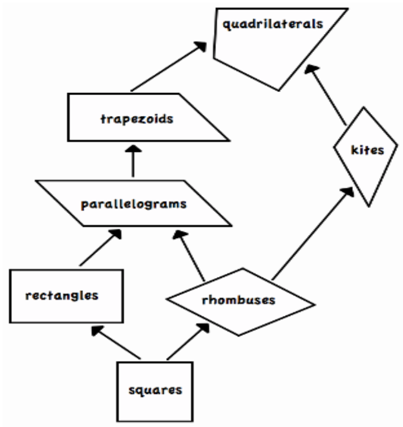


Figure 12. Correct classification chart for the special types of quadrilaterals (Battista, 2012c, p. 91)

PTs to add-on ideas, asking the class if they agree or disagree with a presented argument or answer and why. Sometimes the instructor encouraged PTs to explore their ideas using the Shape Makers when they were unsure of an answer or argument. The instructor would not continue to the next problem until the class came to a consensus and were convinced, which was obtained using the thumbs polling strategy. The instructor typically preselected a small subset of the problems to discuss during the class activity but also allowed PTs to request certain problems to discuss that they struggled with on the homework assignment. The learning goal for the activity was to get PTs to understand that for the relationship to be "always" the first shape (e.g. a square) minimally had to have all the properties of the second shape (e.g., a trapezoid). If the first shape (e.g., a kite) did not have all the properties of the second shape (e.g., rectangle), then the answer was "sometimes" or "never". The answer was "sometimes" if the PT could find an example of the first shape (e.g., use a Rectangle Maker to make a square) that fits all the properties of the second shape (e.g., a kite). The answer "never" would occur when the PTs could justify why there is no example of the first shape that fits the properties of the second shape. As the mathematical debates continued, many PTs started to wonder if there was a way they could create a concept map that showed the hierarchical classifications of the special types of quadrilaterals, which led to In-Class Activity 7.

In-class activity 7

In-Class Activity 7 asked PTs to work collaboratively to create a concept map or classification chart that shows the hierarchy of the special types of quadrilaterals. In-Class Activity 7 was the last class activity completed before the iDGi Quadrilateral posttests 1 and 2. Typically, the instructor gave one day for PTs to complete their charts in small groups and another for a class discussion in which PTs present and explain their classifications. The learning goal for the activity was for PTs to create a chart or concept map that explained the hierarchy of quadrilaterals and the interrelationships between the different types of quadrilaterals. **Figure 12** showed the correct chart that the instructor wanted PTs to obtain by the end of discussion for In-Class Activity 7. The instructor tried to get the class to come to a consensus on which classification chart was correct and why as time permitted.

Analysis Methods for Video Recordings of In-Class Activities

All in-class activities for the iDGi Quadrilaterals unit were video-recorded to capture verbal and non-verbal interactions between the instructor and prospective teachers (PTs). These recordings documented dialogue, writing, drawings, and gestures used during discussions. After data collection, the first author reviewed the footage to identify teacher-facilitated discussions that showed PTs transition from lower to higher levels of the LP. The selection criteria also included identifying PTs' conceptual

Table 1. Summary of the data of pre and posttest scores from all the experimental participants

	Quads 1 pre test scores	Quads 1 post test scores	Quads 1 post-pretest scores (difference)	Quads 2 pre test scores	Quads 2 post test scores	Quads 2 post-pretest scores (difference)
Mean	51.96	88.82	36.89	59.70	88.15	28.47
SD	17.71	11.74	15.50	19.50	12.35	19.25
Median	52.5	92.5	35	60	92.5	27.5
Q1	40	85	26	45	82.5	15
Q3	65	97.5	47.5	76.88	97.5	42.5
min	7.5	25	-7.5	0	25	-22.5
max	92.5	100	90	100	100	92.5
Correlation pre/post.	.507888	.336972		**t-stat**	54.42424	34.36853
				df	542	540
				P(two tail)	<.0001	<.0001
				Result	Significant	Significant
				Cohen's d	2.45	1.74
				C.I. (C=95%)	[35.59, 38.20]	[26.84, 30.10]

difficulties and the role of whole class discussions in resolving them. Consequently, representative video segments of typical daily whole class discussions that met this criteria were identified, selected, and transcribed for analysis. In addition, the three vignettes selected and presented later in the *Findings* section occurred at different points in the unit (early, middle, and towards the end of the unit) and each represented a different transition between the levels of the LP (more details on this later in *Vignette of Class Discussions of iDGi Class Activities* section).

Following transcription, the first author coded the PTs' reasoning—including verbal statements, drawings, and gestures etc.—based on LP levels (for more details on LP levels see Battista, 2012b). Additionally, theoretical memos and commentaries (Charmaz, 2002; Glaser & Strauss, 1967; Strauss & Corbin, 1990) were created to document interpretations, questions, and insights regarding the PTs' mathematical reasoning and instructor's discussion moves. After completing the initial analysis, the first author shared the transcriptions, LP coding, and theoretical memos/commentary with the second author to ensure the interpretations were viable (i.e. valid) and credible (i.e., reliable). The authors then iteratively discussed discrepancies in the analysis and LP coding, refining the results until they reached a full consensus.

FINDINGS

In the following sections, we share our findings and analysis from our statistical tests of PTs' scores on the iDGi quadrilateral tests. We also analyze three sample vignettes from whole class discussions with PTs that illustrate how iDGi-related class activities and discussions were used to enhance PT learning in the iDGi modules, particularly in helping them transition to higher levels of reasoning and resolving their conceptual difficulties.

Analysis of Mean iDGi Test Scores

We did a statistical analysis of pre and posttest scores to see if the iDGi Quadrilaterals unit and class instruction are associated with the increased PTs' achievement. We removed PTs who were repeating the course as well as PTs who did not complete one of the two pre or posttests. Recall that the iDGi quadrilaterals pre and posttests were identical and were each split into two parts. PTs were never told that the pre and posttests were identical, the pre and posttests were locked after completion, and PTs were never shown test answers (see earlier section entitled iDGi Quadrilateral Tests 1 and 2 for more details).

Quadrilaterals Test 1 measured attainment of Level 2.3. Quadrilaterals Test 2 measured attainment of Levels 3.2-3.4. A summary of the data from pre and posttest scores is shown in **Table 1**.

There is a correlation of 0.51 between the pre and posttest scores for quadrilaterals 1 that shows a moderate positive linear association. There is a correlation of 0.34 between the pre and posttest scores for quadrilaterals 2 that shows a moderately weak positive linear association. PTs with higher pre-test scores tended to have higher post test scores. Note that due to the cutoff of 100 (the scores of the tests were from 0 to 100 and no PTs could score above 100), therefore linearity fails for high scores (i.e., if a PT scored something close to 100 in the pretest, there is not much room for improvement. This means that the difference of a posttest score minus a pretest score would be artificially small and it would suggest a smaller impact of the iDGi curriculum than it truly has). This ceiling effect did not seem to cause any problems in our experiments as only 0.2% had scored 90% or better at the Pretest 1 and only 3.5% on the Pretest 2. Those percentages went up to 50.1% and 51.1% respectively for the posttests and the percentages of students who had the maximum possible grade went from 0 and 0.09% for Pretests 1 and 2 to 24.5% and 15.1% respectively, showing large improvement.

Having looked at the distributions of the data and their correlations led us to perform two matched-pairs hypothesis t-tests, one for each of the pre-posttest pairs, Quadrilaterals Test part 1 and Quadrilaterals Test part 2, to test whether there was a significant difference between pre and posttest scores. We used a significance level of $\alpha=0.05$ to determine whether the differences in the scores were statistically significant and whether we could reject that there is no difference between scores before and after

Table 2. Summary of two-sample hypothesis t-tests

	Online	In-person	10-week	8-week	Dr. W	Other teachers
Quads 1 df	95	446	501	40	461	80
Quads 1 t-score		1.27		0.51		-0.01
Quads 1 P-value		0.21		0.61		0.99
Quads 2 df	94	445	499	40	459	80
Quads 2 t-score		-0.49		-0.18		0.89
Quads 2 P-value		0.63		0.86		0.38

using the iDGi system. Despite the cutoff of 100, we saw that there is significant improvement in scores with a high level of effect size.

The results for both pre-posttest pair differences were statistically significant ($p < .0001$), and we concluded that the posttest scores were significantly higher than the pretest scores. Additional analysis found no evidence to support statistically significant different score improvement based on teacher, quarter, or modality. The mean scores of pretests 1 and 2 were much lower than the mean scores of posttests 1 and 2, and the standard deviations of the pretest scores were higher than the standard deviations of the posttest scores. Overall, 99.6% of PTs had posttest scores greater than pretest scores for test 1 and 94% for test 2.

The results for both tests showed that the data were statistically significant and that we can reject the null hypothesis of no difference and conclude that there is a difference between the pretest and posttest scores. We decided to use the two-sided alternative hypothesis as a more conservative approach. The one-sided alternative hypothesis (Ha: $\mu>0$) would give a P-value that is half as much as the one of the two-sided alternative and therefore would produce even more significant results for the same significance level.

Since the majority of our data were from PTs who took the course with a particular teacher, during a 10-week quarter (Fall, Winter, or Spring), and with in-person instruction but there were also PTs who took the course with different teachers, during an 8-week quarter (Summer) or during COVID when the instruction was via zoom, we wanted to see if that also had an effect on the results. We performed three different two-sample hypothesis tests per part of the test of the quadrilaterals for a total of six tests. **Table 2** shows the results of the two-sample hypothesis tests. All the P-values are greater than the significance level $\alpha=0.05$ and therefore the results are not statistically significant, indicating no differences between the paired groups.

In addition, we looked at a control group (same teacher, equivalent activities) that did not use iDGi to see if there was a difference in understanding of quadrilaterals at the end of the term. Control group PTs ($n=39$) took the same posttest as treatment PTs (Control: Posttest 1: $M=83.01$, $SD=14.57$, Posttest 2: $M=81.54$, $SD=16.07$, Treatment: Posttest 1: $M=88.57$, $SD= 12.11$, Posttest 2: $M=88.06$, $SD=12.43$). Our statistical analysis shows that iDGi treatment PTs' posttest scores were statistically higher (with a medium level effect size) than those of control group PTs (Test 1: $p=.024$, Cohen's $d =0.49$, 95% C.I.=[1.04, 10.57]) (Test 2: $p=.017$, Cohen's $d=0.52$, 95% C.I.=[1.36, 11.86]), suggesting that iDGi-based instruction helps PTs understand quadrilateral properties better than instruction that does not use iDGi.

Vignettes of Class Discussions of iDGi Class Activities

In this section, we address our second research question by presenting three sample vignettes of whole-class discussions that took place at different points in the unit. These vignettes illustrate how teacher-facilitated discussions supported PTs in progressing from lower to higher levels of reasoning within the LP. They also show how such discussions helped PTs resolve conceptual difficulties related to quadrilaterals and their properties. Dr. W. served as the instructor for all three vignettes, having taught the majority of the courses utilizing the iDGi curriculum. These vignettes accurately reflect the standard whole class interactions and discussions that took place between the PTs and the instructor during the Quadrilateral unit.

The first vignette, from In-Class Activity 1 on the first day of instruction, shows PTs struggling to understand two properties of kites. Through discussion, they begin transitioning from informal or imprecise descriptions of the kite properties (Levels 2.1–2.2) to more formal ones (Level 2.3). The second vignette, from In-Class Activity 2 (typically on days 3 and 4), captures PTs discussing an angle property of trapezoids. With the instructor's guidance, they begin to transition from describing properties formally (Level 2.3) to interrelating properties and shape categories (Level 3). The final vignette, from In-Class Activity 5 near the end of the unit, illustrates PTs explicitly addressing the interrelationships among the properties of the quadrilaterals (Level 3). Here, the instructor facilitates discussions that help PTs make logical inferences and construct arguments about the interrelationships between parallelograms and rectangles. We also present quantitative data from this problem, comparing PTs' reasoning on the pretest, written responses from iDGi Homework Assignment 4, and the posttest, to show how PTs' reasoning evolved and progressed throughout the unit.

Vignette 1: Class discussion that helps transition PTs from level 2.1/2.2 to level 2.3

Recall, in Module 1b, PTs were given an unknown Shape Maker and were asked to determine which type of quadrilateral it represented (see **Figure 13**). After manipulating the unknown Shape Maker, the PTs had to select the quadrilateral it represented and then check to see if their answer was correct. iDGi only told the PTs if they were correct or incorrect, but not why. If the PT was incorrect, they had to reflect and select a different answer. The learning goal was to help move PTs from Levels 1, 2.1, or 2.2 reasoning (depending on where they were individually reasoning in the LP) to more consistent Level 2.3 reasoning. This activity happens early in the unit (day 1), with PTs who are operating at lower LP levels using language about the visual characteristics/properties that are informal or imprecise (e.g., straight sides to mean parallel sides). However, some PTs drew

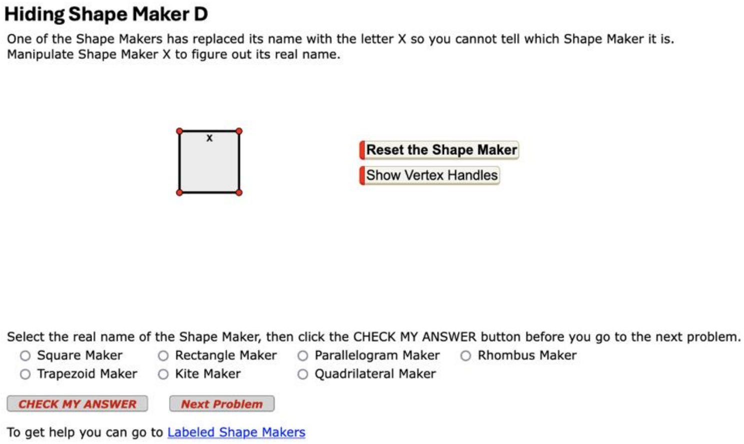
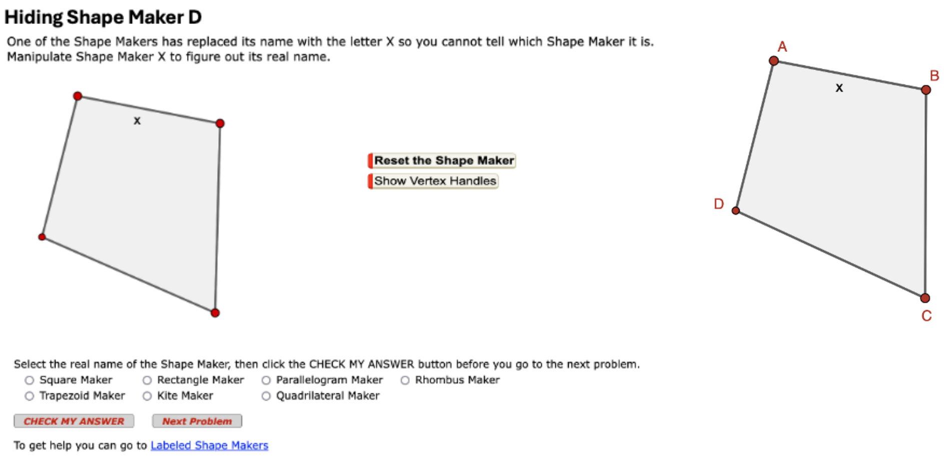


Figure 13. Problem D from Module 1b (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)



(a) (b)

Figure 14. Problem D's Shape Maker after it has been manipulated by the instructor (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)

upon their prior knowledge from geometry courses as well as ideas they learned from completing Modules 2a-2e to use more formal language to describe these visual characteristics (e.g., parallel, perpendicular, adjacent sides etc.) or shape properties.

It was observed that on the LP Pretest, only 26% of the PTs used Level 2.3 reasoning 2/3 of the time for the four most common quadrilaterals. Another 18% used it 2/3 of the time for the three most common quadrilaterals, and 56% of the PTs used levels of reasoning below Level 2.3. So, the most reasonable instructional goal for the PTs at the beginning of the quarter was for the instructor to encourage and help PTs move to consistent and general Level 2.3 reasoning. Further discussion and details about PTs' Level 2.3 reasoning will occur later in the Discussion section in the subsection entitled "Pedagogical Context".

In this episode, the PTs are working on problem D shown in **Figure 13**, which happens to be a Kite Maker, for which the PTs must determine its identity and explain their reasoning. The instructor displays the problem on the projector and manipulates the Shape Maker in various ways, eventually stopping at the configuration shown in **Figure 14a**. PTs then discuss with their partners what shape they think the gray Shape Maker represents and why. The transcript begins at the start of the whole class discussion of the problem. In the transcript, IN stands for instructor and all student names are pseudonyms.

IN: What do you guys think the answer is and why do you think that?

Joanne: We said it was the Kite Maker because it has adjacent sides are equal.

Liz: We said the same thing except there is always one pair opposite equal angles.

IN: Which ones would they be in this case?

Liz: So, these two [points to angles labeled B and D in **Figure 14b**].

IN: So, these two [points to angles labeled B and D in **Figure 14b**]. Okay. Do you guys buy that? [Long pause]. Anything else? [Instructor writes "Kite" on the whiteboard]. What are the things we know again about kites?

Billy: Two adjacent sides congruent. [Instructor writes “Two adjacent sides congruent” on the whiteboard under the kite heading].

IN: Which ones would that be in this instance [points to **Figure 14b**]?

Billy: The bottom one and the one going straight up.

IN: These two [points to sides CD and BC in **Figure 14b**].

Billy: Yes. Adjacent means side-by-side.

IN: So, just these two [points side CD and BC in **Figure 14b**]. Is everyone okay with that? Those are the two adjacent sides that are equal. Anything else?

Ellie: Are you saying that it is only those two [sides]?

IN: Good question, I will go back to you [Billy]. Is it just these two [points side CD and BC in **Figure 14b**]?

Billy: No, when I said two adjacent sides, I meant that one and the other one too.

IN: Which other one? So let me draw it out with letters on it [Draws Kite ABCD from **Figure 14b** on the whiteboard].

Billy: AD and AB are congruent and then these two [points to sides BC and CD. The instructor marks the two pairs of sides congruent on the diagram on the whiteboard. See **Figure 15**].

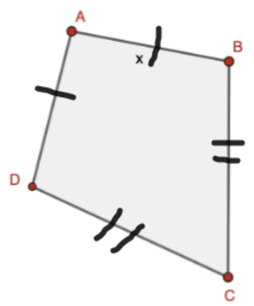


Figure 15. Instructor’s markings of two opposite pairs of adjacent sides congruent on his kite drawing (Source: Authors’ own elaboration)

IN: Do we agree with that? [Long pause]. How do we need to adjust what we said [on the whiteboard]? Or do we need to adjust what we said?

Alexis: Do we need to say two pairs of adjacent sides equal?

IN: I don’t know. That is what I am asking you. Are you asking that or saying that?

Alexis: I am saying that [as a statement].

IN: What do you guys think? Do we need to say two pairs of adjacent sides congruent?

[Many PTs say yes and nod their heads in agreement].

Tyrone: What about all adjacent sides congruent?

IN: That is a good question. What about all adjacent sides congruent? Does that work too?

[Many PTs, including Lily, say “no”].

Lily: Because AB and BC are adjacent, but not congruent.

IN: So, it is not all of them.

Evie: So, the opposite sides are congruent.

IN: Are they opposite sides?

Evie: It is opposite adjacent sides congruent.

IN: Is opposite adjacent sides congruent? What does opposite adjacent sides mean?

Evie: I don't know.

IN: No worries. I am just trying to clarify. Liz.

Liz: There are just two pairs of adjacent sides congruent.

IN: So, you want to say that there are just two pairs of adjacent sides that are congruent. What do you guys think? I am not going to validate here. I want you guys to make the decision. Maybe we need to step out for a second [and change what we are looking at]. [The instructor opens the Kite Maker on the projection screen]. So, this is a Kite Maker and it only makes kites. So, the answer is kite for the problem. But this [the Kite Maker] shows measurements of angles and sides (see **Figure 16**). So, as I move things around [manipulates the Kite Maker] what do you notice? Do we agree with what Liz is claiming? Liz is claiming that two pairs of adjacent sides are equal. Is that actually what we want to say? [The instructor continues to manipulate the Kite Maker in many ways on the projector screen]. What do you guys think? Are we satisfied? Or do we want to say something else? Is it two adjacent sides congruent?

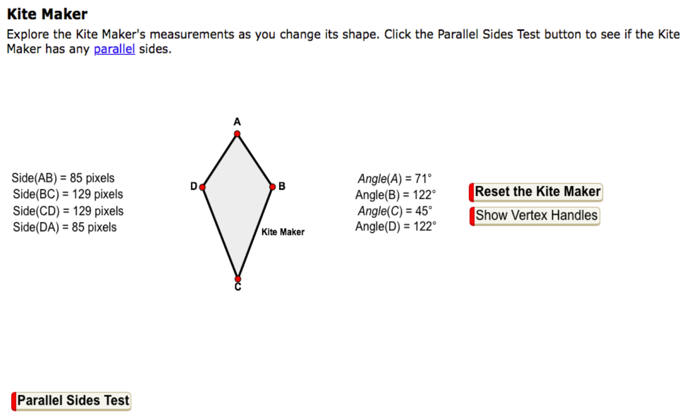


Figure 16. Kite Maker with measurements (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)

Joe: What two sides are you talking about?

IN: I don't know.

Joe: Because like DC and CB are the same. And those are adjacent sides. DC and DA are not the same.

IN: […After some student discussion of opposite angles congruent] I want to go back to this one [points to the "two adjacent sides congruent" written on the whiteboard]. Is it pairs or is this okay [points to the "two adjacent sides congruent" on the whiteboard]? [Most PTs say "pairs"]. So, two pairs. [The instructor writes "two pairs of adjacent sides congruent" on the whiteboard].

The discussion began with Joanne claiming that the Shape Maker was a kite due to it having "the adjacent sides equal". It appears that Joanne is basing this claim on her visual perception of the equality of sides of the Shape Maker (i.e., the sides look equal therefore they are equal); however, it is also possible that she is making it based on her prior knowledge of the property of kites or from her work with the assigned Modules 2a-2f. Joanne's claim is ambiguous in that it might mean one pair of adjacent sides, or all adjacent sides, equal, when in fact, a kite has at least two opposite pairs of adjacent sides equal or congruent[4]. So, Joanne is reasoning at Level 2.2; in which she is either incorrect or insufficient in her formal description of this kite property. The instructor was aware of Joanne's struggle, but recognized that it was early in the unit, so he chose not to validate or invalidate her claim. Instead, the instructor wanted the class to discuss each claim and validate or invalidate it based on their current understandings, while encouraging the class to try to reason at Level 2.3 (i.e., using correct formal language).

Liz agreed with Joanne's claim but added on to it by stating that there was always one pair of opposite angles equal. To get Liz to be more explicit with her claim, the instructor asked Liz to explicitly state which pair of opposite angles were equal, which Liz indicated by pointing to angles B and D on the kite shown in **Figure 14b**. Liz's claim is interesting because the Shape Maker provided with the problem did not show angle measures.

To get the class to think more critically about kites, and to assess their current understanding, the instructor asked the class what they knew about the properties of a kite and wrote what they stated on the whiteboard. Billy stated that two adjacent sides

[4] Note that iDGi does not discriminate between congruence and equals as it is used interchangeably throughout the unit as its original audience were elementary and middle school students. The difference between congruence and equals is addressed at Level 4 reasoning which is not addressed until formal geometry (i.e., high school geometry).

were congruent and stated, "The bottom one and the one going straight up" (indicating sides CD and BC on **Figure 14b**). Billy's claim of "two adjacent sides congruent" is different from Joanne's claim that "the adjacent sides are equal."

Ellie questioned Billy's claim and asked if it is only those two adjacent sides congruent. Billy clarified his statement to mean two pairs of adjacent sides equal indicating that sides BC and CD is one pair adjacent sides equal and sides AD and AB as other pairs of adjacent sides equal as shown in **Figure 15**. Billy's clarification reveals that his initial claim was not a misconception, but rather a conceptual struggle to use precise formal language to convey his reasoning. This illustrates that Billy's initial claim was Level 2.2 because his description of the "two adjacent sides equal" for kite was insufficient and informal because it lacked the precise language of "two pairs" to be a formal description. However, it is Ellie's questioning of Billy's claim in discussion that makes Billy reflect and revise his claim thus initiating his transition to more formal and precise language of the property and helping him to start transitioning into Level 2.3 reasoning.

Notice that when Billy was clarifying his claim to be "two pairs of adjacent sides equal" the instructor recognized the informal language of "the bottom one and the one going straight up" and the use of pointing to sides might be difficult for other PTs to follow in discussion. To help Billy be more explicit with which specific parts of the kite that he was referring to, the instructor labeled vertices with letters. This instructor helped PTs move from gestures to more general, communicable, and precise, diagram-based language.

Alexis stated that they needed to revise the kite property on the whiteboard to "Two pairs of adjacent sides are equal". In response, Tyrone asked whether the property should say "All adjacent sides equal" and revisited Joanne's earlier claim. This shows evidence that Tyrone was trying to make sense of whether the kite property was more accurately stated with Billy's/Alexis' claim or Joanne's claim. Therefore, it is likely that Tyrone is reasoning at a Level 2.1 or 2.2 because he does not recognize which description of the property, Alexis' or Joanne's, is the correct and formal one. Next, many PTs answered "no" to Tyrone's question. Lily explained that in quadrilateral ABCD (see **Figure 15**), sides AB and BC are adjacent but not congruent, assuming the figure represents a kite—an interpretation the other PTs seemed to share. Her visual counterexample showed that in a kite, not all adjacent sides must be congruent.

Ellie's and Tyrone's questions, and Lily's explanation illustrate how class discussions can help PTs work through conceptual difficulties by collaboratively asking and answering each other's questions. So, not only the instructor, but the PTs, as well, seem to be encouraging each other to move from Level 2.2 to Level 2.3 reasoning.

Evie stated that a kite has its opposite sides congruent. The instructor thought that Evie misspoke and said, "opposite sides" instead of "adjacent sides", therefore resulting in asking Evie for clarification. Evie revised her statement and replied with "It is opposite adjacent sides are congruent", another ambiguous, and therefore Level 2.1 or 2.2 type reasoning. Recognizing this, the instructor asked Evie what she meant by opposite adjacent sides. Evie responded with "I don't know' which suggests that she was most likely either having some kind of conceptual difficulty about the two types of side descriptors or difficulty finding the precise formal language to explain her reasoning for her statement, most likely Level 2.2.

Liz stated that the side property of a kite was "two pairs of adjacent sides congruent", which is not quite formally correct (more on this in a moment). To have the class evaluate this reasoning, the instructor used the Kite Maker tool, which shows real-time changes in side lengths and angle measurements (see **Figure 16**). After the demonstration, the instructor asked whether the class agreed with Liz's statement or still believed that only two adjacent sides are equal. Note: PTs were allowed to use the Kite Maker Tool on their own laptops at this time. This question was intended to gauge the PTs' confidence in correctly describing this side property of a kite.

Joe responded by asking, "What two sides are you talking about?" Joe further elaborated that he noticed that in the Kite Maker, sides DC and CB are adjacent and congruent, while DC and DA are not congruent. This observation suggests that Joe recognized from the Kite Maker example that all pairs of adjacent sides were not congruent, but still seemed puzzled by whether it was two adjacent sides or (but still not correct) two pairs of adjacent sides equal. Joe's question suggests that, despite the demonstration and earlier discussion (There are just two pairs of adjacent angles congruent), most PTs were still unclear/unsure about how to correctly describe this property, or they did not understand the property yet. This suggests that Joe was still transitioning his reasoning from Level 2.2 into Level 2.3. This confusion is understandable, as the activity takes place early in the quadrilaterals unit, when many PTs are still developing their understanding of kite properties.

The reason why the instructor showed the class the Kite Maker with dynamic measurements was to allow PTs to empirically investigate the adjacent sides property. The instructor was encouraging PTs transition from Level 2.1 or 2.2 reasoning addressed in Module 1b toward Level 2.3 reasoning, which were addressed in the Modules 2a-2i.

At this point, the class period was coming to an end and the instructor wanted to try to resolve the adjacent sides' equal property discussion. The instructor asked the class if the side property of the kite was two pairs of adjacent sides equal or two adjacent sides equal? The consensus of the class appeared to be that it was "two pairs of adjacent sides equal". As a result, the discussion seemed to help PTs move toward resolving their confusion with the wording of this kite property.

This vignette shows that PTs seemed to develop understanding of a kite property informally and empirically from their observations of Kite Maker manipulations. They struggled, however, with using precise formal language to describe one of the kite's characteristics or properties. Indeed, one sufficient condition for a quadrilateral to be a kite is that it has at least two opposite pairs of adjacent congruent sides. We see evidence that PTs seemed to understand this condition informally with gestures and pointing at a specific image—it is the precise verbal formalization of the general prototypical defining property that was difficult for them. One example is when Billy gestured to the correct two opposite pairs of adjacent equal sides on the pictured Shape Maker. But he, like all his classmates in this vignette, was unable to describe such a property in correct, precise, non-gesture supported language. Another example is when Evie said, "It is opposite adjacent sides congruent", but was unable to clarify what

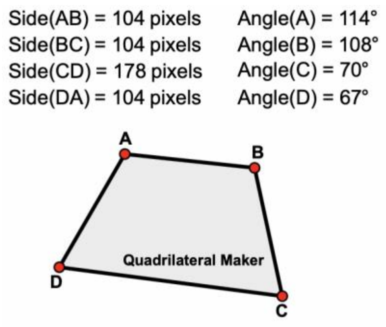


Figure 17. Counterexample to student descriptions of Kite side-congruence property (Source: This figure/image is part of the iDGi system created by Battista under NSF grant 1119034)

she meant by this statement when the instructor asked, possibly because her statement was based on visual inspection of the instructor's diagram. In fact, Billy's and Evie's contributions illustrate well the struggle students often have moving from Level 2.2 to 2.3 reasoning for quadrilaterals whose properties are more difficult to specify than those of squares, rectangles, rhombuses, and parallelograms. Note that it is only when a quadrilateral configuration such as that in **Figure 17** is shown that the need for more precise language might be recognized by PTs. Indeed, in **Figure 17**, AB and BC is one pair of congruent sides, and AB and AD is another pair of congruent sides. But because they are not opposite pairs, the shape is NOT a kite. It may be that it is not until students move from full attainment of Level 2.3 into beginning attainment of Levels 3.3-3.4 that this kind of inference-based reasoning becomes readily accessible to students. Note finally that understanding the verbal characterization of a kite's prototypical defining property of "two opposite pairs of adjacent equal/congruent sides" is difficult because it requires precise understanding of several terms—"opposite," "pair," adjacent sides," and "equal." And for these PTs, understanding these terms can probably be developed only by pointing and describing labeled vertices and sides.

This episode illustrates how class discussion was used to help PTs move toward resolving their conceptual struggles and promote thinking about the properties of a kite. Allowing PTs to question each other (e.g., Ellie and Tyrone) allowed them to challenge each other to reflect on and clarify their claims and reasoning, which encourages deeper understanding and makes PTs be more explicit with their ideas. The class discussion allowed for multiple PTs to compare their different interpretations of a side property of kites (e.g., one pair versus two pairs of adjacent sides congruent) allowing them to reflect and reevaluate their understanding of this property and begin to move to a consensus. Whole class discussion also allowed PTs to revisit each other's ideas and provide additional justification to support their peer's claims. Moreover, the instructor's decision to introduce the Kite Maker with dynamic measurements to help PTs visually confirmed some kite properties and begin to transition from reasoning using the shape's visual appearance (Level 1) to reasoning about their geometric properties using empirical evidence (Level 2, more specifically Level 2.3). During discussion, the instructor made a conscious decision not to validate or correct PTs' statements, which fostered productive struggle for PTs and allowed for misconceptions to be revealed naturally.

Lastly, to be able to facilitate a meaningful class discussion, the instructor needed to create a safe environment in which PTs were comfortable and willing to expose their ideas to the whole class, even when they were incorrect. This involves the instructor being strategic when to follow up on PTs' reasoning (e.g., Billy) and when not to (e.g., Evie), due to reading the social cues being exhibited by the PT. The instructor would typically follow up with Evie to determine her definitions of adjacent sides and opposite sides or ask the class what their meanings for these two terms. However, the instructor noticed from Evie's body language that she was frustrated and upset with herself that he decided not to push the matter further and move on in the discussion. He recognized that it was early in the unit and there was time to correct this difficulty, and it was not worth making Evie any more uncomfortable than she already was at that moment so that she would still actively participate in future discussions.

This vignette shows how collaborative whole class discussion, peer questioning, and strategic teacher facilitation help PTs begin to transition from informal or incorrect ideas about properties of a kite to a more formal and precise reasoning. Through the process, PTs not only learned about two properties of kites (two opposite pairs of adjacent sides equal; one pair of opposite angles congruent) but also gained experience in beginning to use more precise mathematical language and engaging in constructive mathematical discourse—key practices for both learning and future teaching.

Vignette 2: Another class discussion that helps PTs transition from level 2 to level 3 reasoning, and towards level 3 reasoning

This episode of the class discussion illustrates how PTs struggled with the property of "at least two pairs of adjacent angles are supplementary in a trapezoid" and how through whole class discussion PTs build on each other's ideas to clarify their understanding of this trapezoid property. It also shows how class discussion of the properties of quadrilaterals can help PTs transition from Level 2 toward Level 3 reasoning.

This vignette occurred on the fourth day of the iDGi quadrilateral unit during In-Class Activity 2. Prior to In-Class Activity 2, PTs individually completed iDGi Modules 1-4, in which they explored quadrilateral properties by manipulating Shape Makers and solved a sequence of problems about properties, without being directly told the properties. In-Class Activity 2 was split into two days. On day 1, PTs were asked to work in pairs to list all the properties that always applied to each type of quadrilateral, based on their exploration with iDGi modules and In-Class Activity 1. After 15 minutes of group work, the instructor led a whole class discussion in which PTs shared their ideas about properties for each type of quadrilateral. The class then debated, agreeing or disagreeing with the proposed properties, and justifying their reasoning. This whole class discussion would continue for a second day of class. The goal of the class discussion was to foster mathematical debate to help PTs reach consensus on the correct descriptions of shape properties, while addressing any misconceptions or gaps in understanding of the properties of quadrilaterals. The goal was through discussion to get the class to be consistently reasoning at Level 2.3, "Student formally described shapes completely and correctly" (Battista, 2012b, p. 9) and begin transitioning PTs to Level 3 reasoning (interrelating properties and categories of shapes).

By the end of day 1, the class had discussed the properties that were always true for quadrilaterals and squares and had begun exploring those of parallelograms. The episode below is taken from day 2 during the whole class discussion. Earlier in the discussion from day 1 and day 2, PTs agreed on the properties of quadrilaterals, squares, parallelograms, and rhombuses, which were written on the whiteboard. The episode begins with the instructor asking the class to list properties of trapezoids that are *always true*. The first property identified was that a trapezoid has at least one pair of parallel sides. The transcript begins shortly after this is written on the board. Note that in iDGi, adjacent angles in a quadrilateral are defined as angles that share a common side.

> *IN*: What else for trapezoids?...
>
> *Marilyn*: Adjacent angles add up to 180.
>
> *IN*: Is it all adjacent angles? Or is it a specific set? Or is there a certain number?
>
> *Marilyn*: I am just thinking. [long pause]. If not all the sides are parallel, then not all of them will. So, I would say at least one.
>
> *IN*: At least one. [Writes "at least one pair of adjacent angles are supplementary" under the trapezoid category on the white board.]. [The instructor noticed there were a few PTs' hands raised. He calls on Jessica to share her thoughts].
>
> *Jessica*: I am going to argue that there are at least two pairs. I know that when there are two sets of parallel lines you can make up to four sets of adjacent angles supplementary. [The instructor puts on the iDGi trapezoid maker on the projector screen]. So, theoretically, if there is one pair of parallel lines, then there are two sets of adjacent angles that are going to be supplementary.
>
> *IN*: What do we think about that? Did everyone hear her? [Some PTs shake their heads "no"]. So, she thinks that it is at least two because the lines are parallel, you would have to have both sets [of angles] that are going against those two parallel lines. Did you want to add anything Emma?
>
> *Emma*: It has to add up to 360 altogether. So that would be another thought. Well, you don't need to add that [to the whiteboard], but just to help them with that fact.
>
> *IN*: Oh, you mean in terms of the argument. Can you explain a little more about why that fact helps with the argument?
>
> *Emma*: If one pair [of adjacent angles] is 180, then we can assume that hopefully the other [two] angles will also add to 180.
>
> *IN*: So, what you are saying if all four angles add up to 360 because we know that from what we did earlier.
>
> *Emma*: Right.
>
> *IN*: So, if we had one pair that is adjacent angles that are supplementary, then the other pair of angles would have to add up 180 so you could get 360. So, it makes sense that you would have to have two [pairs of adjacent angles supplementary]. Does everyone agree? [A few PTs say "yes". The instructor then changes the property written on the board to "at least two pairs of adjacent angles supplementary"]. You could say "at least one" and it would still be correct. Think about how it was stated. It is that "at least" thing. It was said that it was "at least" meaning it could be more. It could be two. But at least one is true if you have two.
>
> *Mallory*: Would it still say at least two or is it going to just be two?
>
> *IN*: That is a great question. I would like to hear what other people think about that question.
>
> *Mallory:* I mean at least one makes sense, because you can still have two. But would it still be at least one?
>
> *IN*: I see, if we write at least two, does it cause issues with at least one?
>
> *Mallory*: Or does it mean that it can have more?
>
> *IN*: That is a great question.
>
> *Nicky*: But a trapezoid can turn into a parallelogram. So, you can have more.

IN: So, if you can turn it [trapezoid] into a parallelogram, you can have more. Does that answer your question, yes or no?

Mallory: Yeah, well a trapezoid is always a parallelogram, isn't it?

IN: That is a good question. What do you guys think? Is a trapezoid always a parallelogram? [Some PTs in the class say "no".] Let's hear some thoughts on this. …

Helena: To answer your question about is a trapezoid always a parallelogram. I would say that it is not. It is always a quadrilateral, but to be a parallelogram it would have to have opposite sides parallel. And [with] a trapezoid they are not always parallel.

IN: First, does everyone agree with Helena? Then we will need to go back to Danae's question to make sense of all that in a moment. Whatever the idea is, we will need to clear that up [as well]? Yes, Joanna, would you like to add something?

Joanna: Yes, I do want to add something. If we are talking about that. Parallelograms are always trapezoids, but it is not the other way around. Because the definition of trapezoid is at least one set of parallel sides. So, from the list, we can go all the way down to say a rectangle, and it will have four sets of [adjacent] angles supplementary. There can be four angles that are supplementary.

IN: Four angles or four pairs?

Joanna: Four sets of angles. Can I get up and show the class?

IN: Yes, of course. [Joanna goes up to the white board and begins writing on it].

Joanna: Let's say we have a rectangle or a square [draws a rectangle ABDC on the whiteboard (see **Figure 18**)]. We are going to say that these are 90 [degrees]. …

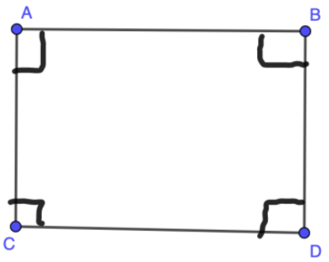


Figure 18. Joanna's drawing on the board (Source: Authors' own elaboration)

Joanna: We know this [points to angle A] is 90 degrees. So, we know that this angle [points to angle A] and this angle [C] add up to 180. We know that this angle [A] and this angle [points to angle B] add up to 180. So already that is two [pairs of angles]. There are [angles] C and D and those would add up to 180. And then [angles] B and D would also add up 180. So, in total we know there are going to be four pairs or sets of adjacent angles that add up 180.

IN: So, you are saying that there can be more than two [pairs of adjacent angles]?

Joanna: There can be more than two, but no more than four for adjacent angles.

IN: So, no more than four pairs of adjacent angles supplementary. Does everyone get what she is saying?

Joanna: But then we have to go through the hierarchy because we know that parallelograms are trapezoids.

IN: Yes, we will get to that in a second. I want to be sure everything is written out first, before we talk about that. I want to go over the properties of the other ones [quadrilaterals] first. [Instructor calls on Addy, who with their hand raised].

Addy: I heard someone saying that is for a square, but it also works for a parallelogram because it would be the same concept that surrounding angles would be supplementary to [angle] A.

IN: Yeah, I think you are right because we said this [points to the written statement "All pairs of adjacent angles supplementary" on the board that was listed as a property of rhombus and parallelogram]. All pairs of adjacent angles are supplementary. There are actually four pairs. Any additional thoughts? Emma.

Emma: Well, are we still thinking about at least two pairs [of adjacent angles supplementary] for a trapezoid, correct?

IN: Yes.

> *Emma*: So, for all the time because a trapezoid is not always a square. So, I am just going back to the thought of every single time, for whatever the trapezoid looks like.
>
> *IN*: So, we are looking at if it is an "at least" situation or "only" situation?
>
> *Emma*: So, at least two would apply. So, then there could be four. My brain just had to get it [cleared up]. …
>
> *IN*: To clear it up just to make sure we are all on the same page. At least two pairs [of adjacent angles supplementary] will work. In reality at least one can work too because we can deduce that at least two can come from that. Because of parallel lines. When we have a set of parallel lines and have to connect it would be two [pairs] but at least one would work. Does that make sense? Because it can have four as we pointed out because it can be four if we can make it a rectangle or a parallelogram. It can have four, so it can have more than two.

The episode began with Marilyn stating that trapezoids have "adjacent angles add up to 180 [degrees]". Due to the ambiguity of Marilyn's statement, the instructor asked her if it was all pairs of adjacent angles or a specific set of adjacent angles that are supplementary in a trapezoid. The instructor asked this question because earlier for parallelograms and rhombuses the class agreed that "all pairs of adjacent angles are supplementary" and the instructor wanted to see if Marilyn argued the same for trapezoid or claimed that some subset of pairs of adjacent angles was supplementary. This prompt exemplifies the types of scaffolding questions the instructor consistently used throughout whole class discussions to try to advance the PTs to higher levels of geometric reasoning. Marilyn then stated, "If it is not all sides parallel, then not all of them will. So, I would say at least one,". This suggests that Marilyn has some understanding of the relationship that if a quadrilateral has a pair of parallel sides, then it forces at least one pair of adjacent angles to be supplementary. Thus, she seems to be transitioning into Level 3 reasoning. What is not explicit from Marilyn's statement is which pair(s) of adjacent angles would be supplementary in a trapezoid (i.e., adjacent angles formed by the parallel sides and a transversal to be supplementary) or why she said at least one pair.

To get other PTs to reflect on and evaluate Marilyn's initial conjecture, the instructor wrote "At least one pair of adjacent angles supplementary" on the whiteboard. In response, Jessica argued that trapezoids had to have at least two pairs of adjacent angles supplementary. She justified this claim by arguing that if there are two pairs of parallel sides, then all four pairs of adjacent angles are supplementary. Therefore, if there is one pair of parallel sides, there should be two pairs of adjacent angles supplementary (which she does not justify logically—but perhaps is thinking empirically via her explorations of the Trapezoid Maker, or even strictly numerically, as in ratio[5]). So, Jessica refines Marilyn's conjecture of "at least one pair of adjacent angles supplementary" to "at least two pairs of adjacent angles are supplementary in a trapezoid".

Emma then analytically/logically refines Jessica's argument by saying that if two adjacent angles add up to 180 degrees, then since the sum of interior angles of quadrilateral adds up to 360 degrees, then the remaining two angles must also add to 180 degrees (implicitly claiming that if there is one pair of adjacent angles supplementary in a quadrilateral, there must be two). Emma's statements add a supporting logical justification to Jessica's claim that there must be at least two pairs of adjacent angles supplementary in the Trapezoid Maker using another known property that the sum of the interior angles of a quadrilateral is 360 degrees.

The discussion between these three PTs shows that they are moving into the beginning sublevels of Level 3 of the LP, which states "Student interrelates properties and categories of shapes," (Battista, 2012b, p.9) and students explain their reasoning using informal justifications (Battista, 2007). Marilyn started by arguing that a trapezoid had to have at least one pair of adjacent angles supplementary because it was a consequence of the trapezoid's prototypical property of "at least one pair of parallel lines". Thus, Marilyn was interrelating property 1 "at least one pair of parallel lines" with property 2 "at least one pair of adjacent angles supplementary" by suggesting that when property 1 is true, it implies property 2 must also be true, thus exhibiting Level 3 reasoning. Jessica either empirically from her explorations with a Trapezoid Maker or numerically as she thinks about an equal ratio, built on this interrelationship further by asserting that if all four pairs of adjacent angles are supplementary when there are two pairs of parallel lines (a prototypical property of a parallelogram), then, when there is one pair of parallel sides (a prototypical property of trapezoid) it must have two pairs of adjacent angles supplementary. Emma provided an analytic/logical interrelationship to further help justify Jessica's assertion in which she argued that if a trapezoid has at least one pair of adjacent angles supplementary (Marilyn's conjecture), then using the quadrilateral property that the sum of interior angles is 360 degrees it can be logically deduced that other remaining pair of adjacent angles of the quadrilateral must also be supplementary (Jessica's assertion). Emma's assertion demonstrates a transition into Level 3.3 reasoning. To analyze this shift, we utilize the coding framework established by Battista (2012b), which defines this level as being 'locally logical' because participants 'string together logical deductions based on "assumed-true" propositions' (p. 40). Emma's dialogue matches this analytical baseline, as she accepts propositions to be true based on her own experience and intuition. Because consistent Level 3 reasoning serves as a critical prerequisite for the formal geometric proofs expected in high school (Senk, 1989), the Level 3.3 reasoning observed here demonstrates that these PTs are successfully developing the specific mathematical foundation necessary to guide their future K-8 students toward that benchmark.

Mallory asked whether the trapezoid property is only two or at least two pairs of adjacent angles supplementary. She stated the connection that "at least one pair of adjacent angles supplementary" would also allow the trapezoid to have two pairs of adjacent angles supplementary. However, she seemed to be wondering whether a trapezoid could have more than two pairs of

[5] Numerical—equivalent fraction ratios, in geometric context: (**2** pairs of parallel sides) is to (**4** pairs of adjacent angles supplementary) as (**one** pair of parallel sides) is to (**two** pairs of adjacent angles supplementary.

adjacent angles supplementary. As a response to Mallory's question, Nicky stated that a trapezoid can be changed into a parallelogram, so there can be more than two pairs of adjacent angles supplementary. Mallory extended on this reasoning by asking "Well a trapezoid is always a parallelogram, isn't it?". This shows how whole class discussion can be used to help reveal PTs' misconceptions about the properties of shapes (i.e. trapezoid is always a parallelogram). It also provides evidence that through discussion the class is transitioning from a property of a trapezoid (i.e., at least two pairs of adjacent angles supplementary) to the incorrect, but hierarchical classification of shape classes (i.e., is a trapezoid always a parallelogram?). Recall the goal of the class discussion was to get PTs to correctly understand the properties of each type of quadrilateral (Level 2.3) and to start transitioning them to implication (Level 3.1-3.3), and later the interrelationships between types of quadrilaterals (Level 3.4). In this instance, Mallory's question is beginning to address the hierarchical classifications between shapes (Level 3.4), something that the instructor planned to address in a later activity.

Another PT, Helena, argued that a trapezoid is not always a parallelogram because a trapezoid does not always have both pairs of opposite sides parallel. Joanna then further elaborated on Helena's claim by arguing that "parallelograms are always trapezoids, but it is not the other way around" because a trapezoid has at least one set of parallel sides. Joanna recognized from their list of properties on the whiteboard so far that they had agreed that a rectangle has four pairs of adjacent angles supplementary. She further elaborated this idea by drawing a rectangle/square (see **Figure 18**), then arguing that since a rectangle or a square has four right angles it will have four pairs of adjacent angles supplementary. This means that rectangles and squares meet the trapezoid's minimum requirement of at least two pairs of adjacent angles supplementary, thus a rectangle or a square will always be a trapezoid. Joanna is using Level 3.4 reasoning in which she is forming hierarchical classifications between trapezoids, parallelograms, rectangles, and squares using properties they have in common (opposite sides parallel and adjacent angles supplementary). Joanna also explicitly mentioned the need "to go through the hierarchy because we know that parallelograms are trapezoids" showing further evidence of her Level 3.4 reasoning. Joanna's argument shows how the class discussion of the properties of quadrilaterals, a Level 2 activity, can help PTs start transitioning to a Level 3 reasoning. It is also noteworthy that Joanna drew a specific example of a rectangle on the whiteboard to help convince her classmates of her claim that rectangles and squares are always trapezoids. This suggests that some PTs still need specific examples to support their logical-analytical reasoning, and that they are still transitioning to being able make inferences like this using logic alone.

Facilitating whole class discussion provided opportunities for PTs to articulate, reflect on, and refine their ideas, through which Mallory's misconception was revealed. Rather than correcting them, the instructor encouraged peers like Helena and Joanna to respond, fostering student-led mathematical discussion. Helena and Joanna's responses introduced alternate arguments, prompting the class to evaluate their claims against those of Mallory (i.e., a perturbation), thus engaging in deeper sense-making.

Building on Joanna's argument, Addy stated that "four pairs of adjacent angles supplementary" not only works for squares and rectangles, but also parallelograms. The instructor agreed with Addy's claim and pointed out that earlier in discussion, the class agreed that all pairs of adjacent angles are supplementary for parallelograms. To get some clarification/validation, Emma asked whether a trapezoid always had at least two pairs or only two pairs of adjacent angles supplementary. The instructor then asked Emma how she would answer her own question. Emma replied that it would be at least two pairs of adjacent angles because it could have four pairs in which the trapezoid is a square (Level 2.3 moving toward Level 3). Note at Level 2.3, PTs have interiorized properties themselves and can see them as applying to multiple shapes, thus starting the move to Level 3.3, in which students start interrelating shapes and their properties. This exchange indicates that the class is approaching consensus with Helena's and Joanna's reasoning – that a trapezoid possesses at least two pairs of adjacent supplementary angles, and that while parallelograms and squares are trapezoids, the converse does not hold.

The episode concluded with the instructor summarizing the class's consensus: a trapezoid has at least one pair of adjacent angles supplementary angles, which implies two – and potentially four if it is a parallelogram (or square or rectangle). This clarification was meant to ensure a shared understanding by the class before transitioning to a new property in the discussion.

Vignette 3: A class discussion that helps PTs move to consistent level 3 reasoning

Recall, In-Class Activity 4 involved the instructor engaging the PTs in a whole class discussion about the problems in Module 8b and PTs had to submit their answers and written arguments for these problems prior to this activity. Below is a vignette from a whole class discussion of Problem 1 from Module 8b. It is followed by our analysis of the episode, along with statistical data on PTs' responses and explanations for Problem 1 at three different points in the unit, highlighting how their answers and reasoning evolved. We selected this vignette because it shows an example of how iDGi class activities, particularly whole-class discussions, helped PTs overcome conceptual difficulties as well as getting them to reason at a more consistent Level 3.3 in the LP.

The transcript begins at the start of the whole class discussion. The instructor has already presented Problem 1,"If the Parallelogram Maker makes a shape with a right angle, what is the most accurate name for the shape?" and asked PTs to turn-and-talk with a partner.

> *IN*: What do you think and why?
>
> *Chloe*: I said a rectangle because both a parallelogram and a rectangle have opposite sides equal. And the parallelogram has at least one right angle; it would make it shift down to a rectangle.
>
> *IN*: Please explain why it makes it a rectangle. What is it doing?

Chloe: Usually when it is a parallelogram the angles are like a little bit over 90. So, when the parallelogram gets 90 degrees or makes a right angle, then all the edges are then straight and straight horizontal. So, it is slightly at an angle. So, 90-degree makes it straight.

[Instructor calls on Holly who has her hand raised].

Holly: Like adding on to that is the reason that they are straight is because like with the last problem where we were talking about how the Parallelogram Maker has to have two pairs of supplementary angles. Because it has two pairs of parallel sides all of the time. If you make one right angle both of the adjacent angles are next to it would have to be 90 then. Then that leaves the last one [angle] to be 90. So, it automatically all becomes right angles.

IN: Why is the last one [angle] 90?

Holly: Because it [parallelogram] has to equal 360.

IN: Do all of you buy that? Can anyone explain in their own words what Holly was saying?

Kristen: I kind of thought of it using the Parallelogram Shape Maker … like if you move the bottom corner to be a right angle then the adjacent one would have to be right as well.

IN: Which adjacent angle would be right, top or over? [The instructor draws a right angle on the whiteboard].

Kristen: The top would be easier.

[The instructor draws **Figure 19** on the board].

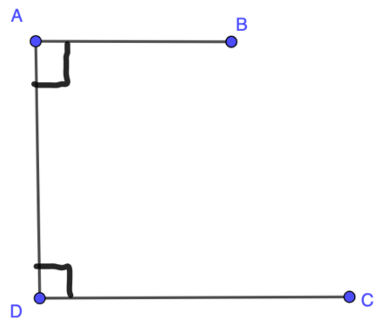


Figure 19. The teacher's initial drawing of Kristen's idea (Source: Authors' own elaboration)

Kristen: So, then the opposite angles are also congruent, so it automatically gets you four. 90 degrees [angles]. [The instructor draws a rectangle shown in **Figure 20** on the whiteboard].

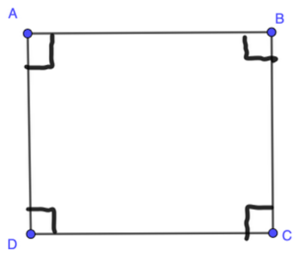


Figure 20. The teacher's completed drawing of Kristen's idea (Source: Authors' own elaboration)

IN: So, once you do this one [points to the right angle D on rectangle shown in **Figure 20**] then you are saying this one [points to angle A on the rectangle] is adjacent and supplementary to [points to angle D]. So, that is 90 and that is 90 [points to angles A and D, respectively] so that is 180.

Kristen: Because they are parallel, so they have to be 180 [degrees]. [The instructor then marks that side AB is parallel to side DC as well as side AD is parallel to side BC on rectangle ABCD].

In many classroom discussions for In-Class Activity 4 like this one, many PTs—like Chloe—initially overlooked explaining *why* one right angle in a parallelogram logically implies that *all* angles must be right angles. Chloe offered a spatial transformation argument, visualizing a parallelogram and suggesting that if one angle were made right, the rest would follow. Chloe's reasoning for this conclusion seems based on all examples she has seen or imagined of a parallelogram with one right angle that forces all the angles to be right angles, thus trying to transition into Level 3.1 reasoning. It is the use of the term "straight", which is informal language (Level 2.1) that suggests that Chloe's reasoning does not quite make Level 3.1 reasoning. As PTs attempt to move to Level

3 implication, their reasoning might drop down to informal. Therefore, PTs' description levels might dip a bit in the transition from Level 2.3 to Level 3 or it is possible that students can use more than one level on a problem (Siegler, 2005; Siegler & Chen, 2002) (more discussion on this later in "Vignette Analysis Elaborations of Current LP Theory" section). So, with this latter possibility, Chloe's reasoning showed characteristics of both Level 2.1 and Level 3.1. However, the learning goal for this problem was for PTs to use the *angle properties* of parallelograms—such as opposite angles being equal and adjacent angles being supplementary—to construct a logical argument.

In the vignette, the instructor did not validate Chloe's explanation, but instead encouraged further discussion by inviting others, such as Holly and Kristen, to share their reasoning. In other implementations of In-Class Activity 4, in which no one volunteered, the instructor would prompt the class with, "How do we know that all four angles would be right angles if a parallelogram has just one right angle?"

Holly claimed since parallelograms must have two pairs of supplementary angles, if one angle is 90°, its two adjacent angles must also be 90°. However, her claim that *two* pairs of supplementary angles implies that both adjacent angles to the 90° angle are supplementary is incorrect. The property she needs for this inference is that *four* pairs of adjacent angles are supplementary. So again, as students attempted Level 2.3 reasoning for more quadrilaterals needing more complex property descriptions, they seem to have to attempt Level 3.3-3.4 reasoning, which they were struggling with in this vignette.

Holly also stated that the final angle must also be 90°, but she did not initially justify this claim. Only after the instructor prompted her did she explain that the interior angles of a quadrilateral sum to 360°, which implies the fourth angle must be 90° (i.e., $360° - 90° - 90° - 90° = 90°$). While this explanation is attempting Level 3.3 reasoning, Holly's argument was not fully explicit or logically complete on her own. She was NOT able to completely, logically justify why all the angles in the parallelogram had to be a right angle if one angle was. Often, as in this episode, PTs' initial attempts at higher level reasoning had gaps or mistakes, but they began to sharpen their reasoning with prompts and questions from the instructor, or better, other PTs (more on this using Kristen's argument below).

To check for understanding, the instructor asked someone to revoice Holly's argument. Instead, Kristen offered an alternative explanation. She described visualizing the use of a Parallelogram Maker: if angle D is 90°, then adjacent angle A must also be 90° (see **Figure 19**). At this point, Kristen's argument is at Level 3.1 because she recognized empirically that if angle D is 90°, then adjacent angle A must also be 90°, but she did not logically justify this conclusion. Kristen then pointed out that opposite angles are congruent, concluding that all four angles must be 90°. In this case, she did draw a correct logical inference in which she deduced that if angles D and A are 90° and since opposite angles are congruent in a parallelogram, then angles B and D have to be 90° as well. This shows that she was in transition from Level 3.1 to Level 3.3 reasoning. The instructor then revoiced Kristen's argument inadvertently adding on the using Holly's previous idea that adjacent angles were supplementary, which is why if angle D is 90°, then angle A had to be 90°. However, Kristen justified this by noting that in a parallelogram, opposite sides are parallel, so adjacent angles must be supplementary. Therefore, Kristen's reasoning with the teacher's prompt of adjacent angles being supplementary, moved into a Level 3.3 reasoning.

As a result of this discussion, two logical arguments emerged for why one right angle in a parallelogram implies all angles are right. First, Holly's argument—which was logically incomplete—that in a parallelogram, adjacent angles to the given right angle must be supplementary (90°), and the final angle must also be 90° because the interior angles of a quadrilateral sum to 360°. Second, Kristen's argument in which, in a parallelogram, adjacent angles are supplementary (so, an adjacent angle to the given 90° angle must be 90°) opposite angles are congruent, implying all four angles are 90°. This PT discussion, with instructor prompting, illustrates how such discussion helped move PTs toward more formal, logical reasoning, or Level 3.3 reasoning. To see more evidence of this claim, we need to look at statistical results and how this class of PTs answered this problem on three different assessments.

We collected data on PTs' answers and reasoning for Problem 1 from Module 8b across three assessments—pretest, iDGi Assignment 4, and posttest—administered at different points during the iDGi quadrilaterals unit. This allowed us to track how PTs reasoned about the problem before, during, and after the implementation of the iDGi curriculum and instruction.

In the pre- and posttests, PTs selected a type of quadrilateral and then chose from a list of predetermined arguments (determined by previous clinical student interviews). There were five arguments for why the correct answer was a rectangle, with two arguments representing Level 3.3 reasoning or the most sophisticated reasoning on this problem. In contrast, for iDGi Homework Assignment 4, PTs were required to generate their own written justifications for each problem without a provided list of reasoning options. For this assignment, PTs submitted answers and explanations for all Module 8b problems prior to In-Class Activity 4 electronically to the Canvas platform.

Table 3 shows how the PTs in this class responded to Problem 1 from Module 8b across three assessments. On the pretest, over half of the PTs did *not* identify the correct answer (rectangle), suggesting difficulties understanding quadrilateral properties. PTs who answered incorrectly were reasoning at best at Level 2.2 (using informal or partially correct shape descriptions) and at worst at Level 1 (identifying shapes based on visual appearance alone). On the Pretest, only about 31% (8/26) of PTs chose both the correct answer and provided a valid reason, and 15.38% (4/26) selected the correct answer with insufficient reasoning. This means that one-third (4 out of 12) of those who answered correctly struggled to logically derive from parallelogram properties why the shape must be a rectangle.

Table 3. Results from pre and posttests and written assignment from Module 8b for Problem 1

Assignment or assessment name	# of PTs	% of PTs with correct answer (rectangle)	% of PTs with incorrect answer	% of PTs with correct answer and reasoning	% of PTs with correct answer and incorrect reasoning
Pretest	26	46.15% (12/26)	53.85% (14/26)	30.77% (8/26)	15.38% (4/26)
iDGi HW Assignment 4	22	100.00% (22/22)	0.00% (0/22)	27.27% (6/22)	72.23% (16/22)
Posttest	26	96.15% (25/26)	3.85% (1/26)	80.77% (21/26)	15.38% (4/26)

In iDGi Assignment 4**,** 100% of PTs correctly identified the rectangle, but only about 27% provided a valid written argument. This indicates improvement in identifying properties of rectangles and parallelograms, but many PTs still struggled to logically derive *why* a rectangle was the correct answer. Most PTs with incorrect reasoning failed to explain why one right angle in a parallelogram implies that all angles must be right angles using logical inferences. Some PTs made a spatial transformation argument, like Chloe, in which they visualized a parallelogram with one right angle, and it made a rectangle, but did not logically using geometric properties to explain why all the all the angles of the Parallelogram Maker must be right (i.e., Level 3.1).

On the posttest, 96.15% of PTs selected the correct answer, "Rectangle" (only 1 PT incorrectly chose "Square"). Although this was a slight drop from iDGi Assignment 4, where 100% answered correctly, 84% of PTs (21/25) who selected a rectangle on the posttest selected a Level 3.3 justification—the most sophisticated level of reasoning on this problem. In contrast, only 27.27% of PTs demonstrated Level 3.3 reasoning in their written arguments for iDGi Homework Assignment 4.

Pedagogical Context: PTs Compared to K-12 Students

In the original iDGi study (Battista, 2019), a large-scale field-test of iDGi Quadrilaterals was conducted with 46 teachers and 2100 students in grades 5, 8, 9, and 10. Student learning was measured not just by mean test scores but by where students were located in terms of levels in the LP. **Table 4** shows, by grade level, means of teacher means of percent of students[6] who were located at the Level 2.3 Two-thirds Benchmark (LP 2.3 {2/3}), along with the percentages for PTs for a sample of two PT classes. Achieving Level 2.3 {2/3} indicates that students used Level 2.3 or 3 reasoning for at least 2/3 of LP test tasks for 3 of 4 of the shapes—squares, rectangles, parallelograms, and rhombuses. LP 2.3 {2/3} is the most reasonable indicator of initial attainment of Level 2.3 reasoning because it indicates that most of the time, for a majority of the most common quadrilaterals, students used Level 2.3 reasoning.

Table 4. Grade level teacher means for quadrilaterals 1 tests using LP 2.3 {2/3} benchmark

	Grade 5 (35 teachers, n=1524)	Grade 8 (3 teachers, n=126)	Grades 9-10 (8 teachers, n=450)	PTs (n=54)
Pre- LP 2.3B	6%	65%	31%	44%
Post- LP 2.3B	35%	100%	90%	96%

Grade 5: Students PTs would typically teach. The mean teacher percent of Grade 5 students who achieved the LP 2.3 {2/3} on Quadrilaterals Test 1 increased from 6% on the pretest to 35% on the posttest. So, although Grade 5 students made significant progress toward attaining Level 2.3 reasoning, they still had much to learn.

Grade 8 (Advanced): Students PTs might teach. The mean teacher percent of advanced Grade 8 students who achieved the LP 2.3 {2/3} increased from 65% to 100%. Even more impressive, the percent of Grade 8 students reasoning at Level 2.3 on ALL tasks for 3 of 4 basic shapes increased from 28% to 93%.

Grade 9-10: Reasonable comparison group to PTs. The mean teacher percent of Grade 9-10 students who achieved the LP 2.3 {2/3} benchmark increased from 31% to 90%. However, the percent of students' reasoning at Level 2.3 on ALL tasks for 3 of 4 basic shapes increased from 7% to only 56%, evidence of the difficulty that even typical high school students have using Level 2.3 reasoning consistently.

PTs. The mean percent of PTs who achieved the LP 2.3 {2/3} benchmark increased from 44% to 96%. The 44% of PTs is far below the post-percent for Grade 9-10 students—suggesting that often, PTs who have not had university geometry instruction understand quadrilaterals less well than iDGi-instructed high school students. Although, the percent of Grade 9-10 students' reasoning at Level 2.3 on ALL tasks for 3 of 4 basic shapes increased to only 56%, the percent of PTs' reasoning at Level 2.3 on ALL tasks for 3 of 4 basic shapes increased from 26% to 87%, further evidence of the effectiveness of iDGi in moving PTs to Level 2.3 reasoning.

Note that PTs' pretest scores were only slightly better than Grade 5 students' iDGi Posttest scores, and lower than Grade 8 students' pretest scores, suggesting that without iDGi instruction or its equivalent, PTs are not at the level they need to be to teach these students effectively. It requires full attainment of Level 2.3, and at least partial attainment of Level 3 to teach Grade 5 students, and full attainment of Levels 2.3 and 3 to teach Grade 8 students conceptually.

[6] For each teacher, the percent of that teacher's students who had reached the Level 2.3 Two-thirds Benchmark was calculated. Then, the mean of these teachers' percentages was calculated.

DISCUSSION

Elaborations of Current LP Theory Derived from Vignette Analysis

Levels and informal justification

As the vignette evidence indicates, during class discussions, the PTs were often using and sharpening both their Level 2.2-2.3 reasoning and their forms of informal justification that occur in Level 3. For instance, students often used locally logical reasoning (occurring within Level 3.3), which is a form of informal justification/argumentation. In locally logical reasoning, students make logical deductions based on propositions they assume to be true, that is on, "propositions that they accept as true based on their experience, intuition, or authority;" they use logic without questioning the starting point of their logical reasoning (Battista, 2007, p. 853). Also, within locally logical reasoning, usually the logical strings are very limited—often consisting of just a single step. That informal justification occurs at Level 3 is supported by Fuys, Geddes, and Tischler (1988) and Clements and Battista (1992)[7]. As Fuys et al. claim, at their level 2 , which is our LP's Level 3, students might provide an informal deductive argument to justify an area rule, for instance, explaining how two congruent right triangles can be arranged to form a rectangle so the area of the right triangles is one-half the area of the rectangle, which is the product of its base and height. More generally, for Fuys et al., in "Level 2 [or LP's Level 3]: The student logically interrelates previously discovered properties/rules *by giving or following informal arguments*" p. 5. And, as Clements and Battista (1992) explain, at our LP's Level 3, students can form precise definitions, consider necessary and sufficient sets of conditions for being a shape, and understand and sometimes provide logical geometric arguments. They can also classify figures hierarchically, giving informal arguments to justify their classifications; for example, they can consider a square as a rhombus because you can think of it as a rhombus with some extra properties. At Level 3, students "can discover properties of classes of figures by informal deduction. For example, they might deduce that in any quadrilateral the sum of the angles must be 360° because any quadrilateral can be decomposed into two triangles, each of whose angles sum to 180°" (Clements & Battista, 1992, p. 427).

Level progress elaboration

To better understand the configuration of levels of reasoning used by PTs during the vignettes and iDGi treatment, more generally, we refer to Siegler's overlapping waves model of learning, which has been applied in various areas of mathematics (Clements & Battista, 1996; Siegler, 2005; Siegler & Chen, 2002). "The overlapping waves model's basic assumptions are that individual children typically know and use multiple strategies; that the strategies usually coexist and compete with each other over prolonged periods of time; and that with age and experience, children increase their use of the relatively advanced approaches and also discover new strategies" (Siegler & Chen, 2002, p. 452). This theory is consistent with our observation that PTs often used elements of Levels 2 and 3 at the same time. In addition, even though "the overall trend of learning is toward greater use of more advanced approaches … many regressions occur along the way" (Siegler, 2005, p. 772). In particular, as students encounter new, more challenging, tasks that require higher levels of reasoning, for parts of the tasks, they may temporarily use a less sophisticated strategy that they had seemingly abandoned previously (see an example of level variation in the geometric measurement context in Battista (2011)).

Complementary iDGi Assessments

Correct/incorrect quantitative iDGi assessment scoring is based on correctness of answer AND LP level exhibited [see **Figure 2** for an example of how the quantitative assessment scores included LP levels]. LP scoring describes the percent of students/PTs reaching an LP level, specifically, LP 2.3 {2/3}. So, the two assessments are related, but not directly comparable in an illuminating way. On the one hand, the analysis of iDGi quantitative assessment scores showed large pre to posttest effect sizes, suggesting a significant overall quantitative increase in geometric reasoning about quadrilaterals. On the other hand, the LP analysis is directly connected to students' progression through iDGi LP levels, providing an important theory-based instructional context for the results. Thus, the two assessment measures—quantitative score and LP analysis—are complementary components of the analysis of a critical educational goal in mathematics learning: "Instructional programs from prekindergarten through grade 12 should enable all students to—Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships" (NCTM 1999). That is, a professionally recommended major goal for school geometry instruction is for students to develop fluency in meaningfully using the property-based conceptual system that supports reasoning about shape and space. For instance, when students study quadrilaterals, it is essential that they understand and learn to use the property-based conceptual system that analyzes angle and length measurements, and perpendicular-parallelism, to specify spatial relationships within and among various types of quadrilaterals. The iDGi LP results indicate positive movement toward this instructional goal.

CONCLUSIONS AND FURTHER DISCUSSION

Research suggests that over a variety of curricula in a variety of countries and using a variety of assessments, a reasonable estimate for the percent of students who achieve (on posttests) Level 2 or higher reasoning in the van Hiele LP in any of grades 5-9 is about 36% (Battista, 2012b; Clements & Battista, 1992). Furthermore, only about 60% of high school students achieve Level 2

[7] The level descriptors in the LP (Battista, 2011) are distillations of research descriptions that were, through extensive research with practicing teachers, deemed appropriate and useful for that audience.

reasoning by the end of proof-oriented high school geometry (Senk, 1989). We can compare the 36% Level 2 achievement reported above to the posttest means for iDGi Quadrilaterals LP 2.3 {2/3}, which are 35% for Grade 5, 100% for bright Grade 8 students, 90% for iDGi grades 9-10 students, and 96% for PTs. Overall, this shows strong evidence suggesting that iDGi-based instruction helped PTs move significantly toward attaining Level 2.3 reasoning. One *might* argue that K-8 PTs who used iDGi have sufficient attainment of LP 2.3 to teach properties of quadrilaterals to Grade 5 students, who, at best, only start transitioning to Level 2.3 reasoning. However, PTs might struggle to keep up with gifted Grade 8 students in learning interrelationships between quadrilaterals.

Overall, iDGi-based classroom instruction seems to have a positive impact on PTs' understanding of the properties of quadrilaterals, as reflected in their significantly improved posttest scores after iDGi and related classroom instruction. Neither the mode of instruction, quarter of enrollment, or university instructor seemed to have any impact on the results, suggesting their generality.

We argue that effective mathematics instruction requires more than just a solid grasp of the subject matter itself. Studies show that teachers must also understand how their students think about mathematics. Specifically, they need to recognize the typical stages/levels of reasoning students progress through when learning specific content, along with how they reason at each stage/level (e.g., Carpenter & Fennema, 1991; Fennema et al., 1996; Steffe & D'Ambrosia, 1995). In practice, this means utilizing detailed, research-backed LPs to guide their teaching (Battista, 2019; Carpenter & Fennema, 1991; Cobb et al., 1990; Fennema et al., 1996; Steffe & D'Ambrosia, 1995). So, PTs' learning within iDGi deepens their geometric reasoning by encouraging their progress through the LP levels, which in turn, can affect their students' learning. Indeed, Lumbre et al., (2023) found that the achievement of ninth grade students whose teachers were operating at level 5 (our LP's Level 4) in the van Hiele LP was significantly higher than that of students whose teachers were operating at level 2. In addition, PTs' learning within iDGi familiarizes them, through their own learning, with the different LP levels of reasoning, helping them understand the general stages that their students pass through in acquiring geometric reasoning about shapes.

Although the iDGi online system could be used by university instructors to have PTs independently complete the modules and take the post-assessments in isolation, that was not the basic design goal of our instructional approach. Indeed, the goal of the iDGi online system was not to replace the instructor, but rather to give them a powerful instructional tool to help them teach their students/PTs geometry topics with conceptual understanding. We argue that to make iDGi online system an even more effective learning tool, iDGi should be linked to related classroom discourse activities that facilitate meaningful small group and whole class discussions to help students and PTs apply, make sense of, and refine what they learned from the iDGi modules. Smith, Steele, and Raith (2017) argued that:

> Mathematical discourse should build on and honor student thinking, provide students with opportunities to share ideas, clarify understandings, develop convincing arguments, and advance the mathematical learning of the entire class. When teachers and students engage in mathematical discourse, teachers can better understand students' thinking and support them in clarifying, refining, and connecting their explanations. The opportunities that teachers offer for students to engage in mathematical discourse also have powerful implications for student identities as mathematical knowers and doers. (p. 123).

Class activities that support meaningful mathematical discourse support PTs' learning by helping them clarify misconceptions, refine ideas, and construct new shared knowledge (Chapin et al., 2013). These class activities should also be linked to level progression in a research-based LP.

The three sample vignettes illustrate how thoughtfully facilitated whole class discussions, when integrated with targeted instructional tools and curriculum components like iDGi, can significantly enhance PTs' geometric reasoning. Vignette 1 demonstrates that the PTs' recorded struggles not only moved them toward the upper bounds of Level 2 reasoning, but actively laid the foundation for Level 3.3 reasoning, specifically regarding the identification of minimal definitions. As Smith et al. (2017) and Warshauer (2015) emphasize, engaging learners in productive struggle allows them to grapple with mathematical structure and conceptual conflicts, laying the foundation for deeper conceptual understanding rather than procedural recall. By navigating these cognitive conflicts within an iDGi-based environment, these specific PTs advanced their own geometric proficiency while simultaneously experiencing the exact learning progressions necessary to understand and address similar conceptual difficulties in their future pupils. This transition from analyzing individual shape properties toward understanding logical inclusions and minimal definitions aligns with research showing that preservice teachers often require guided, collaborative opportunities to move beyond descriptive reasoning (Battista, 2007; Kuzniak & Rauscher, 2007). Furthermore, when instructors elicit and strategically build upon evidence of student thinking during these critical moments of struggle, they start to guide learners toward formal mathematical connections without removing the cognitive demand of the task (Smith et al., 2017; Stein et al., 1996).

Vignette 2 shows the importance of the role that whole class discussion has on PTs' reasoning about shape properties in the iDGi curriculum. Having PTs complete the modules prior to related in-class activities allowed them to individually develop their own empirically based conceptions (Harel & Sowder, 2007) about quadrilaterals prior to the class activities and discussions. It is during the class activities and subsequent whole class discussion that PTs are challenged with new ideas (sometimes conflicting ideas) from their peers about quadrilaterals. Unlike working individually through the iDGi modules, PTs construct and validate knowledge collaboratively through class activities and discussion. Many researchers have argued that effective discourse encourages students to share, clarify, and justify their mathematical thinking, transforming classrooms into environments where collective understanding is built (Chapin et al., 2013; Smith et al., 2017), The instructor facilitated a meaningful whole-class discussion, which allowed PTs to reflect on and question peers' ideas, justify their reasoning, clarify misconceptions, ask questions for understanding, and deepen their knowledge of quadrilaterals based on what they learned from the iDGi modules. By strategically orchestrating student contributions and posing targeted questions (Chapin & O'Connor, 2007; Chapin et al., 2013; Smith et al., 2017), the instructor helped make mathematical reasoning explicit for the entire group**.** This vignette shows how class discussion can evoke precise and correct verbal descriptions of shape properties—such as the difference between "at least," "only," or "always"—as well as help PTs transition from Level 2 reasoning (properties of shapes) toward Level 3 reasoning (interrelationships between shape properties).

In Vignette 3, we analyzed a whole class discussion, pre/posttest results, and students' work from iDGi Assignment 4, Problem 1, in Module 8a, finding substantial growth in PTs' sophisticated geometric reasoning rather than merely accurate identification. Although accuracy remained high across both measures (100% on iDGi Assignment 4 compared to 96.15% on the posttest), the percentage of PTs providing Level 3.3 justifications increased sharply from 27.27% to 84%, demonstrating clear growth in their ability to construct complex mathematical arguments. Furthermore, when comparing pretest and posttest scores for Problem 1, we observed a 50% increase in the number of PTs who answered correctly and a 50% increase in those who both answered correctly and selected the highest iDGi level of reasoning. Thus, these results indicate that PTs demonstrated substantial progress in their Level 3.3 reasoning by the end of the iDGi quadrilaterals unit.

These findings suggest that the whole class discussion from In-Class Activity 4, alongside subsequent iDGi assignments, effectively supported PTs in developing stronger reasoning skills and constructing logical arguments. As Vignette 3 illustrated, advancing classroom discourse within a math-talk community shifts PTs' focus from merely reporting correct answers toward defending mathematical claims and articulating more precise explanations (Hufferd-Ackles et al., 2004). This shift—moving beyond procedural correctness toward structural, deductive justifications—reflects a critical transition in PTs' proof schemes from empirical verification toward formal mathematical deduction (Harel & Sowder, 2007; Stylianides et al., 2016). By anticipating, eliciting, and connecting PTs' thinking during whole class discussion, the instructor provided the necessary scaffolding for PTs to refine their informal ideas into formal, precise mathematical reasoning (Smith & Stein, 2018; Stein et al., 2008).

For teacher educators, the vignettes offer important insights into how to structure whole class discussions that support PTs in transitioning from informal or visual understandings (i.e., LP Levels 1 and 2) to more formal, empirically based, and logical reasoning (i.e., LP 2.3 and 3) (Battista, 2007). In addition, these vignettes highlight the value of designing classroom environments where PTs not only acquire knowledge but also negotiate meaning through mathematical discourse within their classroom community (Cobb et al., 1992). Moreover, these vignettes show examples of how the instructor created a safe environment in which PTs felt comfortable sharing and examining both their correct and incorrect ideas—a foundational practice for establishing sociomathematical norms and fostering authentic mathematical discourse (Borasi, 1996; Ing et al., 2015). Lastly, from the third vignette, the posttest data showed a substantial increase in the number of PTs demonstrating Level 3.3 reasoning, suggesting that structured discussions, paired with opportunities for justification and critique, can effectively support the development of more sophisticated geometric reasoning (Stein et al., 2008; Stylianides et al., 2016).

Together, these vignettes illustrate how providing future teachers with experiences in whole class discussion and iDGi-based instruction mirrors the ambitious, inquiry-based practices they will be expected to facilitate in their own classrooms (Laursen et al., 2016; Smith et al., 2017). As Vignette 1 demonstrates, engaging PTs in productive struggle around geometric tasks allows them to experience firsthand how cognitive perturbation fosters deeper structural understanding rather than surface-level procedural recall (Warshauer, 2015), laying the foundation for Level 3.3 reasoning with minimal definitions. Because most PTs enter content courses having experienced primarily direct instruction during their own K–12 education (Szydlik et al., 2003), they often default to viewing direct instruction as effective teaching. The first author observed this exact trend among participating PTs. Experiencing inquiry-based environments firsthand allows PTs to build the pedagogical content knowledge required to anticipate pupil difficulties, while making them significantly more likely to adopt ambitious, discourse-rich teaching practices in their future classrooms (Ball et al., 2008; Battista, 2007; Szydlik et al., 2003). For mathematics teacher educators, this underscores the necessity of modeling strategic discussion facilitation and immersing PTs in both individual and collective inquiry (Stein et al., 2008).

This research adds to the discussion of how teacher educators should design mathematics content courses for PTs and how curricula based on LP support PTs' learning of K-8 geometry. These findings encourage other researchers to develop similar LP-based curricula for PTs for non-geometric topics. More research is needed on creating and using LP tests to measure PT's content knowledge for other topics and on how these LP tests can be used to inform instruction in PTs' content courses. Finally, research should investigate how teachers taught with iDGi affect their pupils' learning of geometry. Related to this, in the original iDGi study, over 80% of teachers using iDGi to teach thought that it helped them better understand their students' reasoning. This is an important outcome because research shows that students of teachers who understand and focus on student thinking have greater achievement (Battista, 2019; Carpenter & Fennema, 1991; Cobb et al., 1990; Fennema et al, 1996; Steffe & D'Ambrosia, 1995).

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