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Towards understanding the error analysis thinking of prospective mathematics teachers

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ABSTRACT

High school teachers often encounter incorrect solutions from students, especially when teaching combinatorics. This study investigates the ability of prospective mathematics teachers to assess the correctness of solutions to combinatorial problems and to falsify incorrect ones. 39 second-year prospective teachers participated in the experiment, analysing both correct and incorrect solutions to combinatorial tasks. They were asked not only to judge the correctness but also to identify the cause of procedural or conceptual errors. The results indicate that for incorrect solutions, 44.16% of the error identifications were accurate, and only 36.36% of cases could be successfully falsified. Furthermore, when evaluating correct solutions, 57.1% of the correct solutions were mistakenly judged as incorrect. Overall, 31 out of the 39 participants were unable to accurately determine the correctness of the solutions, despite their advanced mathematical background. These findings highlight the need to foster error-detection skills and thorough reasoning in teacher training programs.

Keywords: error analysis, prospective mathematics teachers, mathematics teaching practices

INTRODUCTION AND MOTIVATION

The field of combinatorics is an important part of both elementary and university studies, especially since it holds significant potential for developing problem-solving skills (National Council of Teachers of Mathematics, 2000). The calculations involved in solving combinatorial problems, in themselves, would not pose a problem. The difficulties that arise are primarily because handling the calculations is linked to weaknesses in other student skills, such as the identification of an implicit combinatorial model (Batanero et al., 1997) or the definition of combinatorial operations (Fischbein & Gazit, 1988).

High school teachers often encounter incorrect solutions from students during their work – especially when teaching combinatorics – as it is generally harder to formulate procedures and methods that students can easily learn and practice in this subject. As a result, students produce many incorrect solutions, which the teacher must, or should, correct. However, many of our prospective mathematics teachers report that during their own high school studies, their teachers judged the correctness of their solutions based on whether the final answer matched their own. That is, they often could not provide a deeper explanation: to uncover the reasons for the error or offer a more detailed justification to aid understanding, that is, to falsify or verify the solution.

The Role of Teaching Combinatorics in (Hungarian) Teacher Training and Public Education

In Hungary, mathematics teacher education places significant emphasis on expanding prospective mathematics teachers' content knowledge: the content knowledge required for teaching high school material is greatly surpassed by the knowledge delivered and assessed during the courses. (This is also the case at our university, where the experiment was conducted.) In the first semester of the program, prospective teachers are required to complete a theoretically focused combinatorics course (Discrete mathematics 1), which consists of lectures and practical sessions. Nearly one-third of the lecture covers the knowledge needed for high school combinatorics topics (permutations, variations and combinations with and without repetition, Pascal's triangle, binomial theorem), as well as a significant portion of proof methods (direct and indirect proofs, case analysis, the complement rule of counting, pigeonhole principle, mathematical induction, inclusion–exclusion principle, invariance principle, double counting).

In the remainder of the semester, prospective teachers are introduced to topics that expand their content knowledge (recursions, guessing game problems, the basics of graph theory, Euler and Hamiltonian paths, graph colouring, matchings, and planarity), providing a thorough introduction to the field of combinatorics from a scientific perspective. During the problem-solving seminar, prospective teachers work independently and in small groups to solve problems related to the lecture topics, most of which go beyond high school material. In fact, the seminar aims to develop prospective teachers' problem-solving skills through solving and understanding difficult and complex problems.

In the second semester, prospective teachers are not required to participate in combinatorics courses, but they are in the third semester, when their acquired combinatorial knowledge is refreshed and reinforced through the Elementary Mathematics 2 problem-solving seminar, with a focus on the didactical processing of the topic. It was during this course that we conducted the experiment with the cohort of prospective mathematics teachers (39 individuals).

Uncovering and Modelling the Error-Detecting and Analysing Thinking of Prospective Mathematics Teachers

It is not an uncommon situation in mathematics – especially in combinatorics – that we want to determine the cardinality of a set. At the end of this process, the need may arise to check our answer (i.e., the result) (Pólya, 1945; Salavatinejad et al., 2021). However, the difficulty of verification can vary greatly depending on the mathematical structure of the problem and its solution. In a combinatorics problem, where we are interested in the cardinality of a set that satisfies certain conditions, it is often the case that while the description of the solution set is simple and clear, neither the counting process that determines the number of elements nor the verification of the relevance of the resulting output set (and number, i.e., the result) is trivial. Moreover, the connections between the solution set and the results of the counting processes are often not easy to grasp.

It is no coincidence that one common characteristic of good problem solvers is that they spend more time reading and understanding the problem's text compared to their less effective peers (Mamona-Downs & Downs, 2005; Schoenfeld, 2014, 2016). This gives them time to develop a variety of representations and models, allowing them to have more "tools" at their disposal when it comes to thinking through the solution. Similarly, a common strategy used by teachers is to recall elements of a known solution when reading a new, unfamiliar one, and they try to relate or compare the new ideas to those familiar elements (Bransford et al., 1986). In doing so, they spend time in the space of the problem and the closely related true statements. Their hope is that this way they can understand the underlying mathematical structures more deeply, and thus grasp and understand the new solution more quickly.

Teachers have a difficult task when they begin reading a solution if the problem is new to them, meaning they have not yet had time to think about it. However, even if they have thought about it but do not know the solution, their task is still not easy. Naturally, by seeing a student’s correct solution, the teacher can also learn the correct answer, and even when reading a student’s incorrect solution, they might be inspired to discover the correct one themselves. It may happen that a student’s incorrect solution leads the teacher to the correct answer, but they may fail to notice that the student’s solution is wrong. More commonly, discovering the correct solution helps the teacher recognize the student’s error as well.

Teachers most often look for mistakes in a solution that they can solve themselves or at least know one possible solution to. However, it is not uncommon for them to have to review a solution that is unfamiliar to them, very different from their own (and possibly incorrect), where their own deep understanding of the solution does not provide essential help.

The process of falsification

„Verifying an answer to a combinatorial problem is a particularly difficult task, because there are no guaranteed ways to ensure the detection of an error. In addition, the detection of an error does not necessarily yield a way to reach a correct solution.” (Mashiach Eizenberg & Zaslavsky, 2004, p. 17.)

Scholars have defined errors and mistakes in a variety of ways, so we use the two terms interchangeably (Borasi 1994; Maharani & Subanji, 2018; Matteucci et al., 2015; Mellone et al., 2015; Radatz 1980; Santagata 2005).

If the error-checking process is successful, meaning the teacher correctly identifies the faulty element of the solution and provides a well-reasoned explanation (for example, by giving a counterexample) of why the solution is incorrect, then it is called falsification.

Focus of the Research

Based on our own experiences and those of our university colleagues error detection is challenging for prospective teachers, as is the process of falsification. The research aimed to quantify and confirm this observation. However, by analysing the prospective teachers' written, detailed responses, we did not only gather quantitative data: in order to uncover the deeper reasons behind the results, we also aim to conduct a qualitative analysis using a new theoretical model.

THEORETICAL BACKGROUND

The theoretical background of our research is presented from two main perspectives: the roles of prospective teachers and the modeling of combinatorial thinking provide important theoretical frameworks for our study. The teachers' understanding and evaluation of errors in students' solutions to combinatorial problems lies at the intersection of these two, and will be detailed at the end of this section.

Prospective Teachers and Problem Solvers – Interpretation Frameworks

The literature on mathematics education has long focused on the relationship between mathematical knowledge and thinking skills. As Pólya (1962) states:

„Our knowledge of any subject consists of *information* and of *know-how*. If you have genuine *bona fide* experience of mathematical work on any level, elementary or advanced, there will be no doubt in your mind that, in mathematics, know-how is

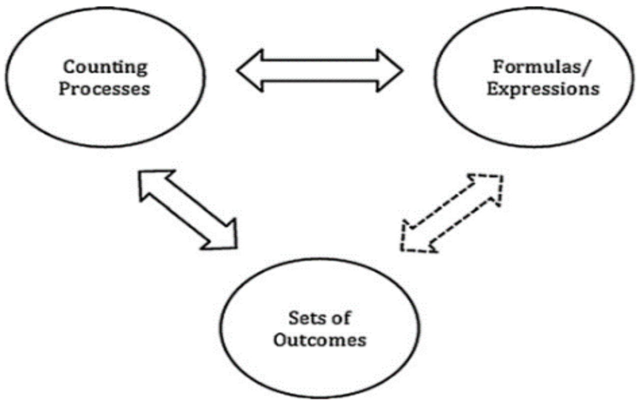

Figure 1. A model of students' combinatorial thinking (Source: Lockwood, 2013)

much more important than mere possession of information. Therefore, in the high school, as on any other level, we should impart, along with a certain amount of information, a certain degree of know-how to the student." (Pólya, 1962, p. 7-8).

Just as mathematical knowledge is obviously necessary in solving mathematical problems, it is equally evident that teachers must possess the ability to apply and impart this knowledge when solving problems encountered during mathematics teaching. Of course, a variety of problem types (social, interpersonal, etc.) can arise during teaching, but this article focuses on mathematical problems, specifically those related to combinatorics. Given that teacher preparation is naturally and inevitably incomplete (Chick, 2011) and cannot cover every problem a teacher may face, successful teachers must be able to apply the knowledge they possess during teaching. The challenges that arise in mathematics teaching are particularly interesting because their solutions are not part of a teacher's routine. Moreover, effectively addressing them is challenging because solutions often need to be constructed immediately, in front of students in the classroom (Lampert, 2001). If the problem focuses on mathematical content, the knowledge to be applied is a mix of mathematical knowledge and a broad understanding of teaching and students. Mason and Davis (2013) write about the importance of spontaneous action and emphasize that greater knowledge – both in mathematics and pedagogy – increases the likelihood of handling new situations. Watson and Barton (2011) explored the use of mathematical questioning in teaching, highlighting the role of prior mathematical experiences, which strongly influence pedagogical decisions.

The prospective mathematics teachers participating in the research therefore play two roles simultaneously: they are both learners and future teachers. These roles accompany and influence them during the experiment, so it is important to distinguish and reflect on them:

1. **Problem Solvers**: They are solving a meta-task, which involves checking if there is an error in the solution.
2. **Teachers**: They aim to reconstruct, interpret, and correct the students' thought processes to help the student improve.

These two interpretation frameworks are not sharply distinct from each other. In fact, much of the prospective teachers' activity can be interpreted from both perspectives. However, given the nature of the experiment, the prospective teachers may feel more like problem solvers rather than teachers, so it is important to differentiate the two frameworks and focus primarily on the problem-solving attitude.

Understanding Students' Combinatorial Thinking

Two models are presented below that describe combinatorial thinking. While the Lockwood model operates with the most fundamental components of the thinking process, the authors of the article by Salavatinejad et al. (2021) use elements of Pólya's model (Pólya, 1945).

The Lockwood model

Lockwood (2013) modeled students' combinatorial thinking, comprising three components: formulas/expressions, counting processes, and sets of outcomes (see **Figure 1**). Formulas/expressions refer to "mathematical expressions that yield some numerical value. The formula could have some inherent combinatorial meaning [...] or it could be some combination of numerical operations [...]." (Lockwood, 2013, pp. 252-253). The term "counting processes" refers to "the enumeration process (or series of processes) in which a counter engages (either mentally or physically) as they solve a counting problem. These processes consist of the steps or procedures the counter does, or imagines doing, in order to complete a combinatorial task." (Lockwood, 2013, p. 253). Finally, the sets of outcomes refer to "the collection of objects being counted – those sets of elements that one can imagine being generated or enumerated by a counting process." (Lockwood, 2013, p. 253).

According to Lockwood, the solution to combinatorial problems involves the use of one or more components, and students' errors often arise from one or more components or the lack of proper connections between them. This model has been used to analyse the process of mathematical proof (Lockwood et al., 2021), and Lockwood (2013) also mentions its role in understanding students' mistakes. However, to our knowledge, no one has yet applied the model to the process of error detection.

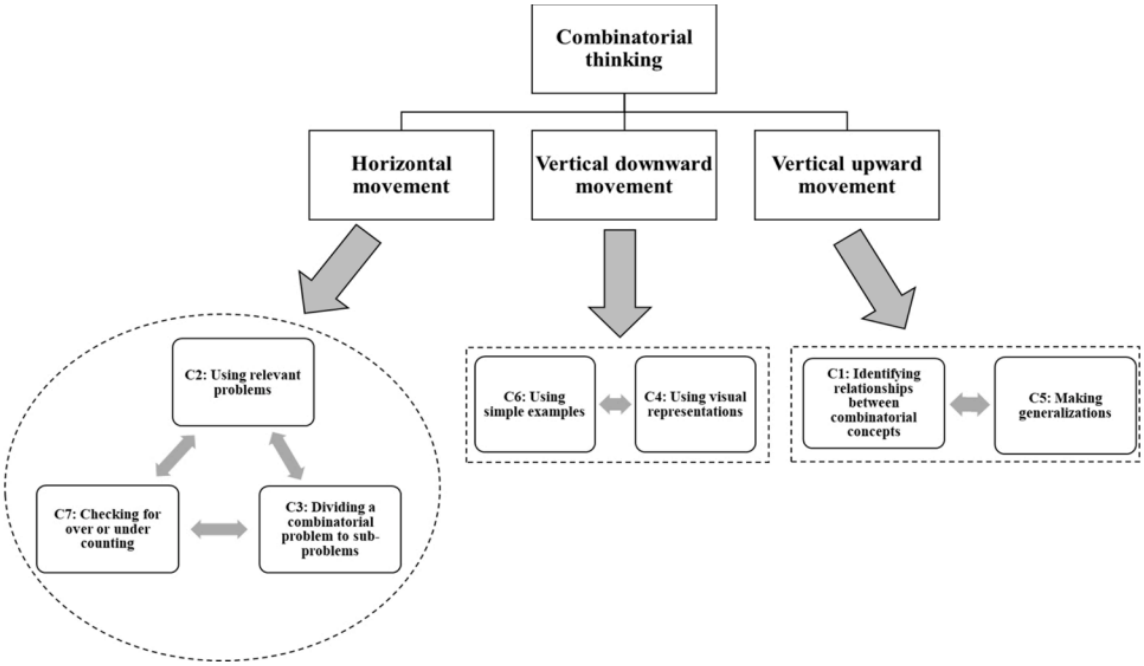

Figure 2. A model of students' combinatorial thinking (Source: Salavatinejad et al., 2021)

The Salavatinejad, Alamolhodaei, and Radmehr Model

In his well-known book, Pólya (1945) divides problem-solving into four main phases: Understanding the problem, making a plan, executing the plan, and reflecting on the solution. Using Pólya's framework, Salavatinejad et al. (2021) applied the "grounded theory" approach (Strauss & Corbin, 1998) to create a model that differs from the Lockwood model.

According to their model (see **Figure 2**), combinatorial thinking can be described as one horizontal and two vertical movements (one upward and one downward). The upward vertical movement involves identifying relationships between combinatorial concepts and generalization, while the downward vertical movement involves using simple examples and visual representations. Horizontal movement includes the use of relevant problems, dividing problems to sub-problems, and checking under- or overcounting. Naturally, elements within the same category work closely together and influence one another.

It is evident that verification, i.e., examining under- or overcounting and correcting it when it occurs, is a crucial part of their model.

The Role of Incorrect Solutions in the Education of Prospective Mathematics Teachers

It is well-established that well-developed examples serve as an effective method for acquiring procedural skills and relevant mathematical knowledge in the early stages of learning (e.g., Adams et al., 2014; Große & Renkl, 2007; Rushton, 2018), often explained through cognitive load theory (Adams et al., 2014; Große & Renkl, 2007). A well-developed example consists of three main parts: the problem statement, the solution steps, and the result. Our focus is specifically on the solution steps and their process.

Incorrectly developed examples differ from correctly developed ones only in that they contain errors in the solution steps. In analysing incorrect examples (briefly referred to as error analysis), students identify and explain the errors (Khasawneh et al., 2023). (They may also correct these errors, which involves justifying their own solutions.) Error analysis not only aids in improving accuracy but also encourages students to develop a deeper understanding of a problem (Rushton, 2018). According to VanLehn's (1999) CASCADE theory, detecting errors and addressing uncertainties is crucial for understanding – especially in the initial stages of acquiring skills, as responses to these (dead-ends) such as self-explanations can foster deeper comprehension.

Well-chosen examples and non-examples, as well as pairs of correctly and incorrectly developed examples, can make critical elements of concepts and problems clearer to learners. Additionally, according to Siegler (2002), these examples facilitate learning, as they increase the likelihood of selecting correct solutions while decreasing the likelihood of choosing incorrect ones. Furthermore, the increase in CK (content knowledge) has an impact on personal mathematics teacher efficacy beliefs, which is an important factor in prospective teacher identity (Austin, 2015; Ball et al., 2008; Fredua-Kwarteng, 2015).

Learning from incorrect solutions poses challenges compared to examples containing only correct solutions. It demands significant working memory resources: students need to represent not only the correct solution steps in their working memory but also find the incorrect step, explain why it is incorrect, and determine possible corrections or changes to the problem statement to make the incorrect step correct. For students with favourable prior knowledge, learning from incorrect solutions can positively impact transfer performance, often more than what can be achieved with structure-focused examples. However, for students with lower prior knowledge, structure-focused examples remain the best choice (Große & Renkl, 2007).

Paulovics et al. (2023) found in their controlled study that regular error analysis during the semester significantly increased prospective teachers' enjoyment of combinatorics and led to significantly better performance on the combinatorics exam at the end of the semester in the experimental group.

Rasiman (2015), based on the levels of critical thinking skills, and As'ari et al. (2017) observed that their prospective mathematics teachers were not yet critical thinkers. Shaughnessy et al. (2021) examined how prospective mathematics teachers attempted to uncover the student's thought process upon seeing a student's incorrect solution: the extent to which they probed the entire process used by the student, the extent to which they questioned the student's understanding, and the extent to which they investigated the student's error, including the reason for the error and the revised process. One of their findings was: "Upon uncovering that the student made a mistake, many of these PTs asked the student to correct their work without any attention to why the mistake was made, which emphasizes getting a correct answer over understanding the student's mathematical reasoning." (Shaughnessy et al., 2021, p. 356). This teaching behaviour not only fails to help the student understand their mistake but also provides the teacher with less information to build upon the student's thought processes.

These findings highlight the challenges faced by prospective mathematics teachers in error analysis and critical thinking. While analysing incorrect solutions can significantly benefit their own understanding and teaching effectiveness, it also requires them to develop deeper critical thinking skills and a more nuanced approach to problem-solving and student interaction.

RESEARCH QUESTIONS

Our research focuses on assessing the accuracy of solutions provided by prospective mathematics teachers to combinatorial problems, as well as examining their ability to falsify incorrect solutions. Our research questions are as follows:

RQ1 Are prospective mathematics teachers able to identify the correctness of solutions to combinatorial problems? Additionally, if they recognize that the solution is incorrect, to what extent are they able to uncover the underlying causes of the error?

RQ2 To what extent are prospective mathematics teachers able to falsify incorrect solutions to combinatorial problems?

METHODS AND DESIGN

In order to answer the research questions, it was needed to design an experiment where prospective mathematics teachers analyse correct and incorrect solutions to combinatorial problems.

Methods

The participants in the experiment were 39 second-year prospective mathematics teachers (ages 19 to 23, 26 female and 13 male) from the Mathematics Teacher Education program at our university, who took part as part of a mandatory course. We created a set of six combinatorial problems. Each prospective teacher was assigned three randomly selected problems from the set, each accompanied by either a correct or incorrect solution. The participants were not informed about the total number of problems in the set or the proportion of correct and incorrect solutions among them.

The participants had 30 minutes to assess whether each solution was correct or contained an error. If they deemed a solution incorrect, they were required to explain in detail which part of the solution was wrong and why. In the case of a correct solution, they needed to argue for the relevance and correctness of the reasoning within the solution, elaborating on it further. They were instructed not to consider potential argumentative gaps as mistakes.

The problems were modifications of high school competition questions or exercises from our university-level practice sessions. These tasks, while not easy, were solvable using basic methods. The incorrect solutions were based on previous experiences from our teaching practice, reflecting real student errors and flawed reasoning pathways.

Design

The most fundamental combinatorial problems can be categorized into three main types: Selection, distribution, and partition (Dubois, 1984). Previous research (Batanero et al., 1997) has shown a close relationship between the implicit combinatorial model of a problem, which serves as a didactic variable, and the difficulty level of the task. Based on this, we designed a set of problems where two tasks were grounded in each of the three models: Selection, distribution, and partition.

A student's solution can be initially classified as either correct or incorrect. Upon further analysis, incorrect solutions can be distinguished as procedurally or conceptually incorrect. According to Mashiach Eizenberg and Zaslavsky (2004), a procedural error occurs when the method employed by the student could lead to a correct solution, but an error occurred during execution. One characteristic of this type of error is that it generally requires relatively little work to correct: by retaining the method but modifying certain elements, a correct solution can be reached.

On the other hand, a solution is considered conceptually incorrect when the method used is incorrect – indicating that the student has misidentified the underlying mathematical model or operations. In this case, the correction process requires a complete or significant reworking of the method used. While we employed this classification system solely for the design of the problem set used in our study, a deeper exploration of the theories related to procedural and conceptual knowledge can be found in Hiebert and Lefevre (2013), and Rittle-Johnson and Alibali (1999).

Table 1. The categories of problems and solutions

	Correct	Incorrect	
		Procedurally	Conceptually
Partition		Mary's Wild Animals	Steve's Cards
Distribution	Bruno and Theresa		Ignatius' Birds
Selection	Median	Heads or Tails	

Using Mashiach Eizenberg and Zaslavsky's (2004) error classification system, the solutions contained either procedural or conceptual errors. **Table 1** illustrates the categories of problems and solutions. A detailed analysis of both the tasks and solutions, as well as the nature of the errors, can be found in the **Appendix**.

Participants' Tasks and Goals

In the case of an incorrect solution, the participants' task was to:

- Identify the incorrect part(s) of the solution,
- Explain the reason for the error in detail,
- Ensure that correct elements of the solution were not wrongly marked as incorrect.

Simply indicating that the solution was wrong was not sufficient. Conversely, when presented with a correct solution, the participants' task was to:

- Avoid marking correct elements as incorrect,
- Verify any parts of the solution they perceived as incomplete.

Therefore, the participants were not required to solve the tasks themselves (although they had already acquired the necessary prior knowledge during their studies).

Data Analysis

Each individual response was evaluated separately according to the following categories:

A: Whether the participant clearly identified the incorrect idea(s),

B: Whether the participant provided a detailed explanation of the identified error,

C: Whether the participant wrongly marked any correct ideas as incorrect,

D: Whether the participant attempted to verify parts of the solution,

E: Whether the participant assessed the solution fully and correctly (according to the criteria outlined above).

For incorrect solutions, categories A, B, C and E were relevant, while for correct solutions, categories C, D and E were considered.

QUANTITATIVE RESULTS

The data are presented first aggregated by participants and then task by task.

Aggregated Results by Participants

The authors aggregated the 39 participants' responses according to Category E, i.e., and determined how many solutions each participant could correctly assess in their entirety. Thus, each participant was assigned a score of 0, 1, 2, or 3. A total of 14 participants (35.9%) scored 0 points, 17 participants (43.6%) scored 1 point, 5 participants (12.8%) scored 2 points, and 3 participants (7.7%) scored 3 points (see **Figure 3**). Therefore, on average, less than one task (0.92) was answered completely correctly out of the three.

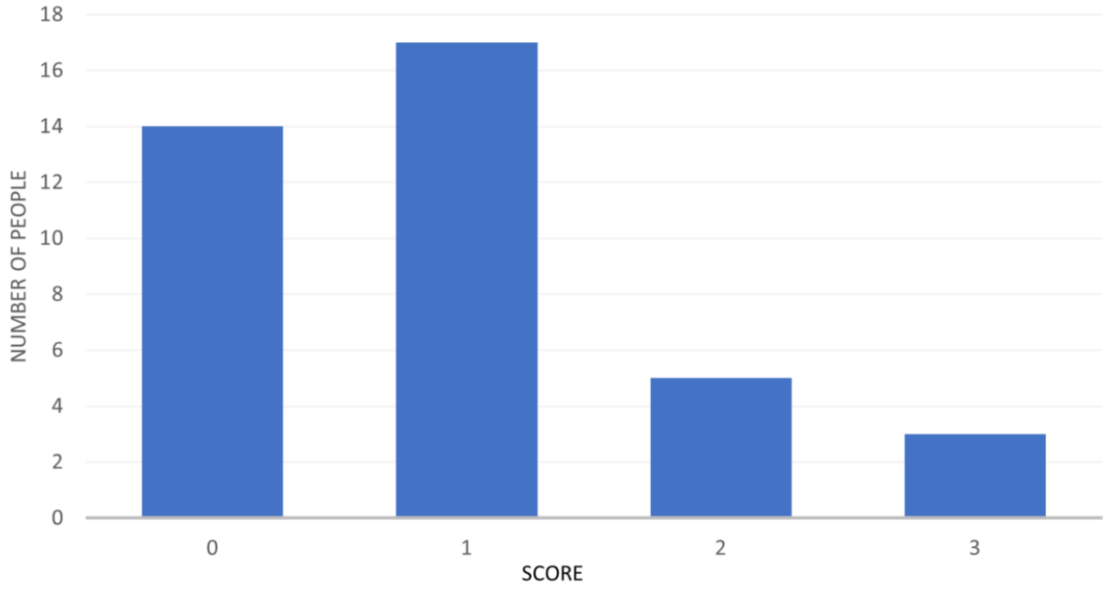

Figure 3. Completely correct answers (Source: Authors' own elaboration)

Table 2. The case of incorrect solutions

Number of people	Task's name	(Category A)	(Category B)	(Category C)	(Category E)
20	Steve's Cards	14	9	12	5
20	Ignatius' Birds	5	5	10	5
19	Heads or Tails	11	10	4	9
18	Mary's Wild Animals	4	4	12	2
77		34	28	38	21

Table 3. The case of correct solutions

Number of people	Task's name	(Category C)	(Category D)	(Category E)
16	Bruno and Theresa	10	9	6
19	Median	10	11	9
35		20	20	15

Aggregated Results Task by Task

The 39 participants submitted a total of 112 evaluable solutions (in 5 cases, they indicated that they did not have enough time to adequately work on the task). There were 77 incorrect and 35 correct solutions.

In the case of incorrect solutions, in 34 out of the 77 cases (44.16%), the participants identified the error (Category A), and in 28 cases (36.36%), they were able to falsify it (Category B). Every response in Category B was also included in Category A, meaning that 82.35% of the participants who identified the error were also able to falsify it. That is, if a student correctly judged which part of the solution was incorrect when reading the incorrect solution, they could usually find the exact reason for the error and explain it in detail. While participants correctly identified the faulty part of the solution in 34 responses, in 38 cases (Category C), they mistakenly considered correct parts as incorrect (of course, both could have occurred within the same response). Therefore, identifying a thought as an error was more often a mistake than accurate (since if a participant deemed multiple correct ideas as wrong within the same solution, it only counted as one instance in Category C). Of the 28 successful falsifications, in 7 cases, the participants also incorrectly identified another thought as wrong, meaning that 21 times (27.27%), they provided a completely correct response (Category E). Every Category E response was also included in Category B, i.e. there were no cases where a student correctly referred to the location and actual cause of the error in their answer, but gave misleading or insufficient detail. In total, there were 11 instances (distributed by category, in order: 0, 5, 2 and 4 pieces) where the entire solution was deemed correct (14.3%). More data are presented in **Table 2**.

In the case of correct solutions, in 20 out of 35 cases (57.1%), some element of the (correct) solution was mistakenly deemed incorrect. If they correctly assessed the solution's accuracy (in the remaining cases), the verification was also successful, meaning their response fell into both Category D and Category E. Naturally, there were also cases where, despite deeming part of the solution incorrect, they verified other parts of the solution. More data are presented in **Table 3**.

DISCUSSION

The results will be evaluated along the research questions using the theoretical models presented in section theoretical background.

RQ1

To answer this question, it is started from the results summarized in section aggregated results by participants. The assessment is based on dividing the participants into two disjoint groups: one group includes those who are "not" or "rather not" able to do this, while the other group includes those for whom we can answer "yes" or "rather yes" to the question.

Considering the 3 tasks, those who were unable to determine the correctness of the solution for any of the tasks fall into the "no" category, and those who gave correct answers for 1 task fall into the "rather no" category. It is important to consider that these prospective teachers will become mathematics teachers, and they will likely need to determine the correctness of many combinatorial problem solutions in the future. The other group includes those who gave correct answers for 2 ("rather yes") or 3 ("yes") tasks.

Thus, overall, 31 out of the 39 participants (79.49%) are (rather) not able to identify the correctness of solutions to combinatorial problems, and only 8 participants (20.51%) can do so.

It is important to emphasize that although the prospective teachers are only in the second year of their five-year program, they have already completed most of their university combinatorics studies, demonstrating knowledge that is far broader than what is required for the tasks in the experiment. They have solved similarly difficult or more challenging problems independently during their exams. Yet, the results show that nearly 80% of them are (rather) not able to identify whether a solution to a combinatorial problem is correct.

In section theoretical background, it has been discussed that the students participating in the experiment take on both the roles of teacher and those of the problem solver. However, in our view, their role as problem solvers is more dominant in the experiment. Assuming this remains the case, one possible reason for the low efficiency could be the lack of an attitude that considers the verification of solutions important. One possible reason for this is a lack of attitude towards the importance of checking solutions. Based on the experience of the authors, it is not typical for Hungarian high school mathematics teachers to explicitly teach the Polya model (Polya, 1945) or even implicitly but consistently incorporate all four phases (Gordon Győri et al., 2020). As a result, the problem-solving culture and competence of prospective teachers applying for the teaching program vary widely, and they generally have insufficient experience with the fourth phase (review). Mashiach Eizenberg and Zaslavsky's (2004) results also indicate that verification is not a practiced and essential part of prospective teachers' mathematical toolkit.

The students, who are experienced in problem-solving, perform weaker in one of the phases of the Pólya model, which describes the problem-solving process. It can be assumed that neither their instructors nor the students themselves placed sufficient emphasis on practicing this phase during their university studies, and consequently, on developing the skills required in this stage. The fact that they have successfully completed university courses with exams containing more complex problems ensures their proficiency in the first two phases. However, the precision required for the flawless execution of the solution plan—the third phase—which is also a crucial component of the verification aspect of the fourth phase, has not been sufficiently developed.

The noticeable absence or underdevelopment of certain components in the model provided by Salavatinejad et al. (2021)-describing combinatorial thinking (C7: Checking for over- and under-counting)—in students' responses suggests that although they are in a problem-solving role, they are faced with a type of problem they are unfamiliar with. Instead of solving a combinatorial problem, they are required to correct an existing solution.

This is further supported by the Lockwood model. Evaluating the students' responses revealed that even the clear presence of multiple elements from the three sets -formulas/expressions, counting processes, and outcomes - does not guarantee the correctness of their cognitive process or the success of error detection. The key to success in combinatorial thinking lies in the strength and transparency of the relationships between components. The transitions between these components create opportunities for mistakes, and avoiding these errors, as well as reviewing them afterward (i.e., correctly executing the thought process and verifying it - see the third and fourth phases of the Pólya model), requires precisely the kind of skills and familiarity with the problem type that are necessary for error detection, which was examined in the experiment.

Whether the causes were interpreted from the perspective of underdeveloped skills necessary for the fourth phase of the Pólya model, from the students' lack of experience with error-detection problems, or from the weak connections between components in the Lockwood model, the explanation for the poor results remains the same: students are not prepared (or have not been prepared) for situations where they must find an error in a solution, understand the cause of the error beyond a superficial level, and explain it. Some aspects of this lack of preparedness are evident based on the above discussion: Insufficient precision, limited experience in error detection, and an unstructured approach to combinatorial thinking. However, additional general characteristics may also define students who performed weaker in the experiment. Unfortunately, the available quantitative data is insufficient to precisely determine these characteristics, but based on our other experiments in the field, we suspect that self-reflection, confidence, the ability to distinguish between incorrect and correct reasoning, a deeper understanding of mathematical concepts and structures, and their approach to problem-solving (from both a combinatorial and methodological perspective) are all important components of effective error detection.

Since the tasks in our study were more complex than those in Mashiach Eizenberg and Zaslavsky (2004), and the prospective teachers may have had more advanced mathematical training and interest, the two results are not comparable. However, looking at our results, the question arises as to what extent our prospective teachers can identify the correctness of their own solutions to combinatorial problems. If to a lesser extent, it is possible that many of their exam results are not reproducible, meaning that their scores may reflect not their actual knowledge but rather decisions made in the moment (such as the implicit selection of a combinatorial model). This could relativize the value of their results and their self-assessment of their knowledge.

Another contributing factor that reinforces the above explanation is the lower level of students' critical or reflective thinking. Several studies examine where the critical thinking level of pre-service teachers can be found (As'ari et al., 2017; Erdoğan, 2020). Although the study did not measure this aspect, our previous (yet unpublished) research also confirmed that the development of critical thinking is an important factor in the cognitive process of error detection. It is planned to conduct our experiment with students in higher academic years as well, allowing us to reflect on the role of increasing self-reflective thinking - associated with more years spent in university (Erdoğan, 2020) - in the process of error detection.

Of course, additional reasons can be identified. It is possible (and hopeful) that in a real situation, the prospective teachers would be much more motivated to help their pupils, and thus – in addition to the resources derived from the problem-solving role – the catalytic effect of the teacher role could also come into play. In this light, it would be interesting for further research to explore how our results regarding prospective teachers change when they are in real-world situations as novice teachers.

RQ2

The above evaluation can be further refined by specifically examining the proportion of cases in which prospective teachers succeed in identifying and justifying errors in incorrect solutions. For this, it is important to rely on the data presented in section Aggregated results task by task. Here, the participants cannot be divided because they received both correct and incorrect solutions, so we perform the analysis at the task level.

The results show that participants were able to identify incorrect solutions in 34 out of 77 cases (44.16%), while they identified 15 out of 35 correct solutions as actually correct (42.86%). Among these participants, the proportion of those who could falsify the error is relatively high (82.35%).

The interpretive framework of the Lockwood model explains this phenomenon by suggesting that some students (either consciously or unconsciously) pay attention to the (perceived) transitions between components. That is, when interpreting a student's solution, if they find a transition between two components insufficiently justified, they take the time to clarify it. Since the lack of clarity in the relationships between components can easily become a source of errors, students participating in the experiment verify potential sources of mistakes.

Examining certain components of combinatorial thinking as described by Salavatinejad et al. (2021) allows for a deeper understanding of the results. The relatively low error identification rate of 44.16% may be explained by the students' inadequate downward vertical and certain horizontal movements: they do not always break down the problem to a sufficient depth to recognize errors (C6: Using simple examples, C3: Dividing a combinatorial problem into subproblems), they do not employ problem interpretation and problem-solving tools or strategies that would effectively support them in executing the task (C4: Using visual representation), and they lack experience in the error-detection process (C7: Checking for over and under counting).

When students correctly identify an error, they move vertically upward, meaning they can place the identified error within a broader mathematical framework and explain it in that context. This suggests that once students become aware of an error, they possess the theoretical background and necessary combinatorial thinking skills to provide an accurate explanation. As a result, they achieved a surprisingly high success rate of 82.35%.

Among students who successfully identified the error, the high falsification rate suggests that when they found a part of the solution unclear or incorrect but could not immediately determine the exact cause of the mistake, they took the time to clarify and investigate it. As a result, falsification occurred in a significant proportion of cases. However, considering all students, the 36.36% successful falsification rate is relatively low, as this implies that only one out of three cases would result in the student receiving complete (though not necessarily entirely correct) assistance.

Thus, a crucial error-detection trait is the persistent clarification of details, as in more than half of the cases (55.84%), prospective teachers did not proceed thoroughly enough: they either deemed the solution correct or considered another correct element incorrect. Striving for thoroughness paid off, as prospective teachers pursued the resolution of unclear conceptual units in the solution.

However, if we look deeper into the responses and only consider those responses as truly correct where a correct conclusion is accompanied by entirely correct reasoning, one can say that the prospective teachers correctly falsified the incorrect solutions in 21 task solutions classified under Category E, which is only 27.27% of all responses. In other words, just over one in four teacher responses would be considered mathematically completely accurate.

CONCLUSION

The research findings highlighted that a significant portion of pre-service teachers is unable to efficiently determine the correctness of solutions to combinatorial problems or, in the case of incorrect solutions, to accurately identify the underlying errors. The experiment revealed that only 20.51% of students could recognize the correctness of solutions, suggesting that the verification aspect of problem-solving is not sufficiently integrated into their mathematical thinking. The low rate of error identification (44.16%) and the even lower occurrence of fully correct responses (27.27%) both indicate that students lack sufficient reflective and critical thinking skills in this area.

These results align with theoretical models of combinatorial thinking, particularly the components defined by Salavatinejad et al. (2021). Deficiencies in these areas hinder students from recognizing correct solutions and providing meaningful justifications for errors.

From the perspective of the Pólya model (1945), which examines students' problem-solving processes, it is evident that they apply the fourth phase - review and verification - at a low level. This is likely due to insufficient experience in developing these skills during their secondary and university education. A lack of familiarity with error-detection problems and the unstructured nature of connections between solution processes contribute to students' frequent inability to mathematically justify their findings in a fully correct manner.

Within the interpretive framework of the Lockwood model (2013), the results indicate that while some students can identify problematic parts of a solution, the lack of clarity in the connections between these parts often leads to incorrect conclusions. Students who took the time to examine the relationships between components were more likely to falsify incorrect solutions. This suggests that persistently clarifying details is crucial for developing sound mathematical reasoning.

The research successfully addressed the posed research questions. While the results are not particularly encouraging, they highlight the importance of teaching error detection in teacher education. Additionally, they raise new questions regarding the effectiveness of teacher training, particularly in developing critical thinking and reflective analysis. Students' mathematical self-reflection, confidence, and ability to distinguish between correct and incorrect reasoning play a decisive role in the success of error detection. Consequently, one goal of future research could be to examine how students' verification skills and critical thinking develop over the course of their university studies. Furthermore, it would be valuable to investigate the extent to which practical experience in a teaching role enhances these skills in real classroom situations.

Improving mathematics teacher education requires multiple, mutually reinforcing pedagogical and methodological changes. Explicitly integrating the teaching of error detection and self-checking strategies into the curriculum could help develop students' reflective and critical thinking. It is essential that future teachers not only master the components of the problem-solving process but also learn to identify errors, uncover their root causes, and correct them. To this end, specialized courses should be designed where students can practice the fourth phase of the Pólya model in various scenarios. Such courses would raise awareness of the error-detection process and provide students with feedback on the accuracy of their reasoning.

Creating a learning environment that fosters self-reflection is essential in teacher education. Structured self-assessment and reflection tasks could help future teachers gain a deeper understanding of their own problem-solving strategies and consciously develop their verification skills. At the same time, it would be beneficial to expand teacher training programs with professional workshops where students can learn error-detection and falsification methods through concrete tasks based on real teaching scenarios.

To improve the professional (methodological) preparedness of university instructors in teacher education, continuous professional development and the integration of research findings into teacher training are necessary. Based on the results of this study, it would be justified to establish short training sessions and workshops specifically focused on developing error-detection skills, reflective thinking, and combinatorial problem-solving. This approach would ensure that future teachers are trained not only in subject content but also in pedagogy and methodology by university instructors who are confident and effective in applying these practices.

A qualitative analysis of students' responses would provide additional insights into the cognitive process of teachers' error detection. However, the theoretical models available to us - and presented at the beginning of this paper - are not sufficient for qualitatively interpreting the obtained data in a meaningful and comprehensive way. The Lockwood model is not suitable for qualitatively analyzing the verification or falsification of students' solutions by teachers: When the task is not to solve a problem but rather to determine the correctness of an existing solution, the model encounters limitations. To address this, a more complex model is needed - an extension of the Lockwood model that develops it into a two-dimensional framework - capable of simultaneously illustrating both the cognitive process of the student's solution and the teacher's evaluation of it. It is planned to develop this model in the near future and publish the results obtained with its help.

It is important to mention that students' attitudes may have been influenced by the fact that they were not evaluating solutions in a real classroom setting. This could have affected their motivation and possibly led them to search for errors even when none were present. Although we explicitly advised them not to have preconceived expectations about the number of incorrect solutions—just as would be expected in a real test-grading scenario—this factor may still have played a role.

Overall, the findings of this study highlight that developing pre-service teachers' skills in verifying combinatorial problems and identifying errors poses a significant challenge. University education should place greater emphasis on the conscious application of verification strategies and the development of critical thinking so that future teachers can effectively evaluate and improve their students' mathematical reasoning.

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APPENDIX

Description and Analysis of Our Tasks

The tasks and their solutions are presented and analysed in the experiment. In addition to the analysis of the mathematical content, the underlying model of the problems and the type of error(s) in the solution are also discussed.

"Steve's cards"

Task: Steve is organizing the cards for his new card game. Each card is unique, and on each card, there are 2 images of cornelian cherry, 3 images of elderberry, and 3 images of rosehip (with the same picture of each type) arranged in some order. The deck contains all possible combinations of these cards. Steve gathers into a group those cards that have two adjacent rosehips. How many cards will end up in this pile?

Solution: "I will calculate the number of cards in the complementary set, that is, the number of cards that do not contain adjacent rosehips, and then subtract this from the total number of cards to get the final result. The total number of cards is $\frac{8!}{2!\cdot 3!\cdot 3!}$. It is not difficult to determine the number of cards without adjacent rosehips: I 'glue' another plant to the right of each rosehip, then arrange the resulting 3 glued plants and the remaining 5 − 3 = 2 elements in $\binom{5}{3}$ ways, and then arrange the non-rosehip plants in $\binom{5}{3}$ ways, and then arrange the non-rosehip plants in $\binom{5}{3}$ different ways. So, the number of these cases is $\binom{5}{3}^2$ Thus, the answer is: $\frac{8!}{2!\cdot 3!\cdot 3!}-\binom{5}{3}^2$."

Analysis: The problem is based on the partition model, as Steve holds all the cards and classifies and organizes them according to a certain criterion: he groups the cards with two adjacent rosehips into a pile. The main principle of the solution – subtracting the number of cards without adjacent rosehips from the total number – is correct. The total number of cases is also correct (permutation with repetitions): $\frac{8!}{2!\cdot 3!\cdot 3!}$. The error lies in determining the number of unfavorable cases, as the solution does not account for cases where there are no adjacent rosehips, but the last plant is a rosehip, such as the card ERERECCR (where C represents cornelian cherry, E represents elderberry, and R represents rosehip). An important observation is that while the reasoning after the error may seem unusual, it is based on fundamentally correct principles, so every other case (where the last image is not a rosehip) is counted exactly once. Therefore, the cards counted do indeed belong to the pile in question, but the number given in the answer is larger than the correct solution. The error is conceptual, as the solution for the subproblem – determining the number of cards without adjacent rosehips – contains a fundamental mistake. A correct solution can be reached by counting the unfavorable cases as follows. Besides the rosehips, each card has five images, which can be arranged in $\frac{5!}{2!\cdot 3!}$ ways. Among these five cards, there are six slots where the rosehips can be placed. If we place no more than one rosehip in each slot, we ensure that no rosehips are adjacent, and all unfavorable cases can be obtained with this method. So, we choose the rosehip positions in $\binom{6}{3}$ways. Each unfavorable case has been counted exactly once, so the solution is: $\frac{8!}{2!\cdot 3!\cdot 3!}-\frac{5!}{2!\cdot 3!}\cdot\binom{6}{3}$.

Understanding the correct solution, as well as the interpretation of the incorrect one, requires knowledge of both permutations with repetition and combinations without repetition.

"Ignatius' birds"

Task: Ignatius is attaching pictures of his 12 favourite bird species (all different) to 4 pieces of furniture in his home. How many different ways can he do this if each piece of furniture must have at least one picture, and the order of the attachments does not matter?

Solution: "Since he wants to attach at least one picture to each piece of furniture, he first chooses four pictures in $\binom{12}{4}$ ways and attaches one to each piece of furniture in 4! ways. Then, for each of the remaining 8 pictures, he can choose any piece of furniture freely. Thus, the solution is $\binom{12}{4}\cdot 4!\cdot 4^8$."

Analysis: The problem is based on the distribution model, as Ignatius holds all the pictures and decides for each one which piece of furniture to attach it to.

The main principle of the solution – that the remaining pictures can be distributed arbitrarily after fulfilling one of the task's conditions – is incorrect. If, in a particular arrangement, both picture A and picture B are attached to the same piece of furniture, this attachment could be achieved by initially assigning either A or B to that piece of furniture. As a result, every valid arrangement has been counted, but some are counted multiple times (and not necessarily the same number of times for different arrangements), so the number given in the answer is larger than the correct solution.

The error is conceptual, as the main principle of the proposed solution is based on incorrect reasoning. A correct solution can be given, for example, by applying the inclusion-exclusion principle (the set of invalid cases is the union of four sets, where the k-th set contains the attachments in which the k-th piece of furniture is left empty).

To understand the incorrect solution, knowledge of combinations without repetition and permutations with repetition is required, and for the correct solution, the inclusion-exclusion principle is also necessary.

"Heads or tails"

Task: We play a heads or tails game as follows: We flip a coin four times, and then we flip it again as many times as the number of heads we got in the first four flips. What is the probability that we will get at least 5 heads in total across all flips?

Solution: "If all four of our initial flips are heads, then in the four additional flips, we will succeed in every case except one (all tails). If we get 3 heads in the first four flips, then we need to get at least 2 heads out of the next three flips. This occurs in 4 cases (1 case for 3 tails; 3 cases for 2 tails). If we get 1 or 2 heads in the first four flips, we will certainly not succeed (i.e., we won't get at least 5 heads in total). Thus, the desired probability is: $\frac{1}{2^4} \cdot \frac{2^4-1}{2^4} + \frac{1}{2^4} \cdot \frac{4}{2^3}$."

Analysis: The problem is based on the selection model, as we are concerned with those coin flip sequences that contain at least five heads. The main principle of the solution – separating cases to create manageable situations, and then solving them essentially through counting – is correct. The solution considers all relevant cases, but there are two minor errors. First, in the "3 heads" case, the parentheses list the wrong cases (it mentions tails instead of heads), but since the number of correct cases equals the number of incorrect cases, the total count is still correct. Second, at the end of the solution, the first factor of the second term is incorrect: it should be $\frac{4}{2^4}$, as there are 4 ways to get 3 heads out of 4 flips. Both errors are procedural, as they can be easily corrected without affecting the overall approach. The correct solution can be obtained by fixing these minor issues.

Understanding the incorrect solution (as well as correcting it to the right one) requires knowledge of case distinction and simple counting techniques.

"Mary's wild animals"

Task: Mary is enthusiastically photographing her favourite animals in a nearby national park: white storks, black storks, bison and otters. She takes pictures of 10 individuals in total. How many ways can there be at most 4 storks and at least 3 otters in the pictures, knowing that she photographed all four species, but no more than 6 of any one species, and the order of the pictures does not matter?

Solution: "The valid cases can be determined by separating them based on the number of storks.

1. case (4 storks): Assume a picture of each species has already been taken, so we only need to decide the remaining 6 pictures. There are 3 ways to choose how many white and black storks to have. Out of the remaining 4 pictures, 2 must be of otters, so there are 2 ways to decide the number of bison. Therefore, we have 2 × 3 = 6 valid cases.
2. case (3 storks): As before, we decide about the remaining 6 pictures. There are 2 ways to choose whether 2 of the storks are white or black. Out of the remaining 5 pictures, 2 must again be otters, so we decide the number of bison pictures in 3 ways. Thus, we have 2 × 3 = 6 valid cases here as well.
3. case (2 storks): As before, we decide about the remaining 6 pictures. Out of these, 2 must be otters, so we only need to decide the number of bison pictures. We can do this in 4 ways. Thus, there are 4 valid cases here.

So the total answer is: 6 + 6 + 4 = 16."

Analysis: The problem is based on the partition model, as the pictures belong to categories defined by the species.

The main principle of the solution – creating manageable cases by separating them and then solving through counting (while using the multiplication rule) – is correct. The solution considers all possible cases, but it makes two minor (and essentially the same) mistakes. In both the 1st and 2nd cases, the solution incorrectly calculates how many ways we can decide the number of additional bison pictures. For 2 (and 3) pictures, the number of bison pictures can be 2, 1, or 0 (or 3, 2, 1, or 0), meaning there are actually 3 (or 4) ways to choose, not just 2 or 3. (One more bison picture remains among the 10 photos, of course.) It's important to note that the 3rd case ends up correct because the upper limit on the number of individuals per species prevents all remaining pictures from depicting otters.

Both errors are procedural and can be easily fixed without affecting the main approach. The correct solution can be obtained by making these minor adjustments.

Understanding the incorrect solution (and correcting it to the right one) requires knowledge of case distinction, the multiplication rule, and basic counting techniques.

"Bruno and Theresa"

Task: We want to distribute 5 different English books, 6 different French books, and 8 different Russian books between Bruno and Theresa (it's possible for one of them to receive no books). How many ways can this be done such that Theresa receives books in at least two different languages?

Solution: "The total number of cases: 2^{19}. It's a bad case if Theresa doesn't receive any books, which is 1 case. Or if she only receives books in one language, which is $2^5 - 1 + 2^6 - 1 + 2^8 - 1$. Therefore, the number of valid cases is: $2^{19} - (1 + 2^5 - 1 + 2^6 - 1 + 2^8 - 1)$."

Analysis: The problem is based on the distribution model, as we are essentially deciding which books go to which person.

The main principle of the solution – subtracting the cases where Theresa receives books from fewer than two languages from the total number of cases – is correct. The determination of the total number of cases is also correct (the number of all subsets of a 19-element set): 2^{19}. The calculation of the invalid cases is also correct: there is indeed 1 case where Theresa doesn't receive any books, and for an nnn-element set, the number of non-empty subsets is $2^n - 1$, and these are all clearly distinct cases.

To understand the solution, one needs knowledge of case distinction and the formula for the number of non-empty subsets of a set with *n* elements.

"Median"

Task: How many three-digit numbers can be formed using the digits 1, 2, 3, 4, 5, and 6, where the middle digit is exactly the median of the three digits? (Here, "median" refers to the middle value when the digits are arranged in order.)

Solution: "We solve the problem by separating the cases based on the number of distinct digits used. If all digits are different, there are $\binom{6}{3}$ possibilities. If two of the digits are the same, there are $\binom{6}{2}$ possibilities. Finally, if all three digits are the same, there are $\binom{6}{1}$ ways to choose the three digits. From each valid set of three digits, two different three-digit numbers can be formed that satisfy the conditions, except in the third case. Therefore, the answer is: $2 \cdot (\binom{6}{3} + \binom{6}{2} \cdot 2 + \binom{6}{1}) - 6$."

Analysis: The task is based on a selection model, as we are interested in those three-digit numbers where the middle digit is the median of the three digits.

The main principle of the solution – dividing the problem into cases based on the number of distinct digits used and solving them by subtraction – is correct. The determination of the number of possibilities in each case is also correct, using the binomial coefficient (selecting some elements from a set without repetition and without regard to order) and ensuring that duplicate digits are not counted twice (or subtracted when they are).

To understand the solution, one needs knowledge of case distinction and combinations without repetition.