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# The nature of reasoning-and-proving items in textbooks: The cases of Türkiye, Norway, and Slovakia

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**ABSTRACT**

In this cross-national study, we explore the different ways of reasoning-and-proving (RP) presented in three 8$^{th}$ grade textbooks, one from each country: Turkey, Norway, and Slovakia. While the analysis revealed that all three textbooks contain similar numbers of problems involving some form of RP, differences exist in terms of the dominating ways of reasoning. In particular, while empirical reasoning dominates in the Turkish textbook, deductive reasoning is the dominant way of reasoning in the Slovak and Norwegian textbooks, with the Norwegian textbook having a significantly larger proportion of one-step deductions than its Slovak counterpart. This study is limited to only three countries, with only one textbook selected in each country. We discuss the findings and implications of our findings for textbook developers and teachers. Recommendations for further research, considering the limitations of the study, are also given.

**Keywords:** reasoning and proving, textbook analysis, cross-cultural study

## INTRODUCTION

Reasoning is a foundation for doing mathematics because it goes beyond knowing a set of rules and principles; instead, it requires understanding, analysis, connection, verification, justification, and proof (Stacey & Vincent, 2009). Stacey and Vincent (2009) stated that “understanding that mathematics is built on a foundation of reasoning and is not just a collection of arbitrary rules is an important message to convey to students” (2009, p. 271). Additionally, Stylianides (2009) highlighted that reasoning is a broad term that includes processes that are not always related to proof. To exclude those and place emphasis on the type of reasoning that is connected to proving, he used the term “reasoning-and-proving” (RP), meaning that mathematical activity involves “identifying patterns, making conjectures, providing non-proof arguments and providing proofs” (p. 259). Bieda et al. (2014) used the same term and defined RP tasks at elementary levels in the same fashion to distinguish between the processes connected to reasoning and those connected to proving: “reasoning involves engaging in processes to generalize mathematical phenomena and/or conjecturing about mathematical relationships, whereas proving involves justifying a mathematical claim to be true for the domain to which the claim applies, using logically valid reasoning” (p. 72).

Ways of reasoning have caught researchers’ attention in mathematics education research (Kollosche, 2021; Stacey & Vincent, 2009) since RP are accepted as essential parts of mathematics instruction at all levels (Stacey & Vincent, 2009). Despite its importance and the belief that it should be given a place in mathematics classes, research demonstrates that, in reality, students have difficulties in RP at all grade levels (Harel & Sowder, 2007; Healy & Hoyles, 2000; Knuth et al., 2009; Stylianides, 2009). Many reasons can be responsible for this weakness. Bieda et al. (2014) stated that a lack of solid experiences during K-8 education related to RP might be one of the reasons. The trends in international mathematics and science study (TIMSS) video study (Hiebert et al., 2003) indicated that most countries’ math instruction does not include RP activities. Hiebert et al. (2003) further analyzed the level of sophistication of the problems addressed in mathematics lessons, the way teachers and students find solutions, the connections made between mathematics topics, and the use of resources and materials such as technology, worksheets, and textbooks. As Hiebert et al.’s (2003) TIMSS video study reported, the level of mathematical reasoning and connections was low in 8$^{th}$ countries’ mathematics lessons, and textbooks were the leading resource of mathematics lessons in all these countries.

Textbooks are a crucial learning resource for teachers and students since they translate the abstract curriculum into a more concrete style that students and teachers can use (Sievert et al., 2021a, 2021b). They play an essential role in deciding which tasks will be used and how they will be used in the classroom (Oates, 2014; Stylianides, 2009; Tarr et al., 2006). In addition, textbooks are familiar sources that teachers use to enable students to master given mathematics problems (Depaepe et al., 2009; Hiebert et al., 2003). Furthermore, Johansson (2006) claimed that teachers' reliance on the textbook while deciding on the content coverage and emphasis in teaching is even greater in mathematics than in other school subjects. In this respect, textbooks are essential to gain an idea of mathematics teachers' practice in the classroom.

Since textbooks constitute the leading resource (e.g., Davis, 2012; Mullis et al., 2012; Oates, 2014; Sievert et al., 2021a, 2021b; Stacey & Vincent, 2009; Stylianides, 2009; Vicente et al., 2022) to enhance mathematics learning, a balanced distribution of problems in the textbook should be considered (Siegler & Oppenzato, 2021). For instance, students' achievements are surprisingly poor in simple arithmetic problems using fractions and decimals that are underrepresented in their textbooks (Siegler & Oppenzato, 2021). Similarly, Törnroos (2005) has stressed the positive relationship between topics in the textbook and student achievement on that topic. Textbooks and instructional strategies also play a critical role in the learning environments designed by teachers (Sievert et al., 2021a, 2021b). Keeping these ideas in mind, we consider the opportunities given to students in their textbooks that can influence their RP abilities. Thus, this research aims to compare and contrast how reasoning items are presented in the 8th grade mathematics textbooks of three countries that differ in structure and pedagogical orientation. The rationale for selecting those countries is given in the following section.

## REASONING AND PROVING

In the related literature, several studies focus on the characteristics of different modes of proving and the ways of reasoning (e.g., Harel & Sowder, 2007; Kollosche, 2021), particularly how they are given a place in textbooks (Stacey & Vincent, 2009; Stylianides, 2009; Vicente et al., 2020, 2022). Harel and Sowder (2007) used the term "proof schemes" and defined it as "what constitutes ascertaining and persuading for that person (or community)" (p. 809). According to Harel and Sowder (2007), the taxonomy of proof schemes includes three dimensions: "external convictions including authoritarian, ritual, and non-referential symbolic schemes; empirical class including inductive and perceptual schemes; and deductive class, including transformational and axiomatic systems" (p. 809).

Stylianides (2009, p. 262) investigated RP opportunities available in mathematics textbooks, specifically looking at RP items under two related dimensions. The first dimension, "making mathematical generalizations," consists of "identifying a pattern" and "making a conjecture." The second dimension supports mathematical claims consisting of "providing a proof" and "providing a non-proof argument".

In their analysis of several Australian mathematics textbooks, Stacey and Vincent (2009) generated seven categories of the mode of reasoning: Appeal to authority, qualitative analogy, concordance of a rule with a model, experimental demonstration, deduction using a model, deduction using a specific case, and deduction using a general case. Deduction using a general case, a particular case, and a model corresponds to Harel and Sowder's (2007) deductive proof schemes, concordance of a rule with a model and experimental demonstration aligns with empirical proof, and appeal to authority corresponds to external conviction in Harel and Sowder's (2007) proof scheme class. The last category, the qualitative analogy, is new and has no equivalent in Harel and Sowder's (2007) proof scheme classes. The most frequent modes of reasoning are deductive and empirical, which cover approximately two-thirds of the topics in textbooks. The researchers further explored deductive reasoning in terms of two subcategories, deduction using a general case and deduction using a specific case; however, they did not observe a significant difference between these frequencies. Hence, literature does not present one set of RP methods; rather, each study contributed to the field by exploring new ways of reasoning.

In the related literature, although limited, there are research studies on RP regarding the items in textbooks. Bieda et al. (2014) stated that, in the common core state standards for mathematics, RP received less attention than the procedural and conceptual understanding of mathematical concepts. In their analysis of seven fifth-grade textbooks used in the U.S., they found that 3.7% of the tasks in the textbooks were related to RP, and these mostly involved making and justifying claims empirically. In a similar vein, Stylianides' (2009) exploration of RP opportunities in school mathematics textbooks revealed fewer opportunities for students to make conjectures and provide empirical argumentation. On the other hand, quite a high percentage of tasks ask for rationales, which are considered an essential aspect of RP. More specifically, in his exploration of the American mathematics textbook series, he analyzed the tasks in the CMP book series for different grade levels and content areas. Among the 4,855 analyzed tasks, about 40% were categorized as offering at least one opportunity for RP tasks, and more than 50% were classified as not giving opportunities to engage in RP tasks. Acknowledging the role of textbooks in teachers' lesson plans and teaching practices, Davis (2012) examined three secondary school mathematics textbooks to identify RP opportunities, particularly in polynomial functions. The textbooks that he concentrated on were different in terms of the organization of the units, which were divided into a conventional curriculum unit, a hybrid curriculum unit, and a reform-oriented curriculum unit. The findings of the study indicate that the reform-oriented curriculum unit mostly contained the ways of RP mentioned by Stylianides (2009). However, there were few opportunities for students' conjecture testing in the textbook tasks.

Fujita and Jones (2014) chose to analyze the geometry component of a commonly used 8th grade textbook since it is during geometry lessons at this level that Japanese students are introduced to the idea of mathematical proof. Their analysis divided each geometry lesson into smaller structures, such as narratives, solved examples, exercise sets, figures, and activities. The content of these "blocks", as a unit of analysis in this study, was further analyzed, revealing that 35.5% of 299 "blocks" were related to RP. By contrast, only 7.7% of all "blocks" were found to be related to performing routine procedures, which indicates the emphasis given to argumentation in 8th grade geometry teaching. Another finding supporting this conclusion is that RP-related blocks are identified in almost all (94%) of geometry lessons included in the textbook. Although various aspects of RP are provided, especially in non-exercise blocks, there was a clear emphasis on direct proving. Fujita and Jones (2014) point out the over-emphasis on direct proofs and the limited opportunities for non-proof arguments as possible reasons why many 14-to-15-year-old Japanese students do not fully understand the need for formal proof in mathematics.

In another research study, Sears and Chávez (2014) investigated the differences in the features and cognitive demand of proving tasks. In this study, the researchers examined two high school geometry textbooks used in the USA and explored how the textbooks chosen by the teacher affected mathematical difficulty and students' opportunities to engage with proof during the lesson. It was found that teachers' actions (influenced by their beliefs, students' disposition, and external factors) and the textbook impact on the level of cognitive demands of proving tasks used during enacted lessons, as well as the extent of students' engagement with these tasks. In a similar vein, Vicente et al. (2020) compared problem solving approaches in Spanish and Singaporean mathematics textbooks and observed that Singaporean textbook supported reasoning more while less focused on step-by-step procedural solutions, which the researchers claimed as a potential reason for Singapore's success in international exams.

In addition to investigating textbooks in nation-based studies, cross-cultural studies on textbook analysis are also popular in literature. In their investigation of sixth-grade Turkish, Singaporean, and the US textbooks, Erbas et al. (2012) stated that different countries' textbooks present different design features regarding visual elements, the balance of text densities, the presentation of important notes for students, and the identification of different types of tasks, such as solved problems, activities, and exercises. Erbas et al. (2012) found that a high density of visual elements with a low text density characterized Singaporean textbooks. By comparison, US textbooks use a high density of text, and more topics are covered. Turkish textbooks were considered to fall in the middle of these two countries' textbooks in terms of the use of visual elements and text, and one of the distinctive features of Turkish textbooks was found to be tasks that explicitly connect mathematics and the real world. Similarly, Vicente et al. (2022) focused on illustrations in their comparative textbook analysis study and found that Singaporean textbook included a larger number of illustrations that support reasoning in arithmetic word problem solving compared to the case in Spanish textbook. Vicente et al. (2022) also compared the characteristics of arithmetic word problems in addition to illustrations. Their analysis revealed that the number of arithmetic word problems in Singaporean textbooks is higher than in Spanish textbooks, but both books have a similar problem variety. However, Singaporean textbooks consist of organizational illustrations that help clarify a problem's structure without relying on simple strategies. In other words, the Spanish textbook's illustrations are primarily figurative and do not address student problem-solving, compared to the Singaporean textbook.

In another cross-cultural study, Mayer et al. (1995) compared three Japanese and four the US seventh-grade textbooks on the addition and subtraction of integers. Based on the analysis, many conclusions and statements were raised, for example, that the instructional parts of the textbooks are much longer in Japanese books than in the US books, but the exercise set is about the same length in both nations' textbooks. Worked-out examples, concrete analogies, and relevant illustrations are more common in Japanese textbooks, whereas irrelevant illustrations are more common in the US textbooks. Moreover, Japanese textbooks use more space for explanations, while US textbooks use more space for unsolved exercises and interest-grabbing illustrations. Mayer et al. (1995) further elaborate that meaningful instructional methods, emphasizing the coordination between different representations, are more common in Japanese than in the US textbooks. Although in the US textbooks, drilling and practice, where space is devoted to unsolved exercises involving symbol manipulation, are given a place, in Japanese textbooks, cognitive modeling, where space is dedicated to presenting and connecting multiple representations through worked-out examples, is highlighted.

To gain insights into how curriculum or textbook features may have influenced students' performance, Xin (2007) examined word problem distribution across various types in one the US and one Chinese mathematics textbook series and its relation to students' success rates in terms of solving multiple problem types. Furthermore, he conducted a cross-cultural analysis of how a sample of the US and Chinese students performed differently in solving multiplication and division word problems. The results indicated different patterns of word problem distribution in the US and Chinese textbooks and that the ability of the US participants to solve specific problem types better than other problem types may be directly related to the design of the US textbooks.

As can be deduced from the above studies, cross-cultural studies on the comparison of RP items in textbooks are limited. However, since textbooks play a crucial role as a learning resource for schoolteachers and students (Oates, 2014), it is worth investigating the kinds of opportunities for reasoning students are offered by textbooks. Thus, in this research study, we refined and extended previous schemes and compared and contrasted the number and nature of RP tasks in three countries' textbooks.

## METHODOLOGY

This study was conducted as a part of a larger European project, MaTeK, involving five partners: Slovakia, Czech Republic, Italy, Norway, and Turkey, aiming to explore and improve pre-service mathematics teachers' knowledge for supporting reasoning and proof. As a need analysis for the larger project, we carried out textbook analysis with pairs and/or groups of different partners and explored what opportunities for reasoning and proof are offered in commonly used textbooks in the above-mentioned countries. While in this study we only focused on the textbooks from three MaTeK consortium countries due to space limitation, comparison of other countries, and each country's sole investigation of their textbook will be the outcomes of the project. The selected countries' textbooks differ in structure and pedagogical orientation, and the details are given below. Hence, this study investigated the nature of solved examples (i.e., problems with explained solutions) of RP tasks provided by Turkish, Slovak, and Norwegian 8$^{th}$ grade textbooks in terms of the presented ways of reasoning. Following the ethical considerations of the larger project, we

**Table 1.** Comparison of content of three countries' mathematics textbooks

| Turkish textbook | Slovak textbook | Norwegian textbook |
|---|---|---|
| **Unit 1:**<br>*Section 1:* Factorization<br>*Section 2:* Exponentials<br>**Unit 2:**<br>*Section 1:* Square roots<br>*Section 2:* Data analysis<br>**Unit 3:**<br>*Section 1:* Probability<br>*Section 2:* Algebraic expressions and identities<br>**Unit 4:**<br>*Section 1:* Linear equations<br>*Section 2:* Inequalities<br>**Unit 5:**<br>*Section 1:* Triangles<br>*Section 2:* Congruency and similarity<br>**Unit 6:**<br>*Section 1:* Transformation of geometry<br>*Section 2:* Geometric solids | **Unit 1:** Rational numbers, percentages, ratio, direct and inverse proportionality, congruence of triangles, prism, its volume and surface area, basic geometric constructions<br>**Unit 2:** Powers with integer and zero exponent, square root, powers of 10<br>**Unit 3:** Pythagorean theorem<br>**Unit 4:** Arithmetic expressions, monomials and polynomials, formulas $(a \pm b)^2, a^2 - b^2, (a \pm b)^3$<br>**Unit 5:** Circle, its circumference and area, chords, arcs, the relative position of a straight line/circle and a circle, incircle and circumscribed circle, Thales' circle<br>**Unit 6:** Linear equations and linear inequalities<br>**Unit 7:** Use of loci, axial and central symmetry in geometric constructions<br>**Unit 8:** Graph of direct and inverse proportionality, functional dependence between quantities<br>**Unit 9:** Random trials, relative frequency, probability, and their calculation<br>**Unit 10:** Topographic surveying in the field<br>**Unit 11:** Basics of logic<br>**Unit 12:** Solving tasks using basics from graph theory<br>**Unit 13:** Review exercises | **Unit 1:** Numbers and number calculations<br>*Section 1:* Calculation strategies<br>*Section 2:* Factorization and fractions calculations,<br>*Section 3:* Exponents and square roots<br>**Unit 2:** Algebra<br>*Section 1:* Exploring patterns,<br>*Section 2:* Algebraic expressions<br>*Section 3:* Exploring algorithms<br>**Unit 3:** Functions<br>*Section 1:* Coordinate systems<br>*Section 2:* Linear functions, straight lines<br>*Section 3:* Proportionality and inverse proportionality<br>**Unit 4:** Equations and formulas<br>*Section 1:* From text to equations and from equations to words<br>*Section 2:* Strategies for solving equations<br>*Section 3:* Formulas,<br>*Section 4:* Composite units |

conducted a content analysis where we employed both a qualitative coding procedure and a descriptive quantitative analysis. Below, we present an overview of the features of each textbook and the data analysis process in detail.

### Data Sets

Our dataset involved one 8th grade textbook from each of the three countries of Turkiye, Slovakia, and Norway. We focused on 8th grade textbooks because we agreed in our meetings that a great variety of mathematical content, particularly different forms of argumentation and proof, did not occur in the early years of middle school. Furthermore, 8th grade was considered as an important level in transitioning to more advanced mathematical thinking in high school. **Table 1** presents an overview of the content of the three countries' textbooks.

#### *Turkish textbook*

The Turkish 8th grade mathematics textbook selected for this study is published by the Ministry of National Education (MoNE, 2021) in Turkey and distributed to all middle schools. Although some teachers use it as the main resource for mathematics instruction, some use it as a supplementary book; nevertheless, all teachers have access to the book, as it is available online on the digital education platform. The textbook consists of six units, and each unit consists of two sections. Each section starts with an introductory task (e.g., an interesting question or a real-world problem) followed by an activity that demonstrates the procedure, often involving the use of materials, and asking the students to reason about the situation. Some of the solved examples (between two and 14 in each section, i.e., the problem and the explanation of the solution process) are followed by note-boxes, including a formal mathematical description of the situation, an introduction of mathematical terminology, or the symbolic form of the concepts and procedures. Following the solved examples, each section has some unsolved problems that are structurally similar to the solved examples to allow students to practice what they learned from the solved examples. Lastly, at the end of each section, there is a set of questions for students to practice more and test their knowledge; the answers to those questions are given at the end of the textbook.

#### *Slovak textbook*

In Slovakia, as in the other two countries, schools may choose from several mathematics textbooks. The Ministry of Education provides schools with funding to purchase selected mathematics textbooks included in the list of approved textbooks. As part of the project and based on our own experiences and interviews with experienced teachers, we analyzed older textbooks, which have a long tradition of being used and are still among the most used textbooks even today. These textbooks are not included in the list but have been available to schools since the past; innovative versions for individual years are gradually being developed. The 8th grade mathematics textbook selected for this study has 13 units (unit 10 to unit 12 cover an expanded curriculum, and the last unit is purely repetitive). For the most part, each unit begins with a solved example. This is followed, like in the Turkish textbook, by note-boxes that students should remember. In the next part, there are other solved examples and problems, which are enriched by different tasks to help students practice what they have learned, based on the solved examples and problems. Various note-boxes with summaries may also be found in this section. At the end of each section, there are other unsolved exercises for students to practice on. The number of solved tasks in unit 2 to unit 12 (unit 1 and unit 13 do not contain these types of tasks) varies from 5 to 46. At the end of the textbooks, solutions are found to all the examples, problems, tasks, and exercises that are not solved directly in the text.

**Table 2.** The integrated framework used in the analysis of solved examples (Sevinc et al., 2022, p. 2085)

| Way of reasoning/specification | R* | T* | RT** | None | Description |
|---|---|---|---|---|---|
| 1. Appeal to authority | | | | | "In *appeal to authority*, the warrant (in Toulmin's sense) given to justify an assertion is that a figure of authority (e.g., Euclid and a textbook) says it is so. From a mathematical point of view, this is no explanation or reasoning at all: perhaps it might be called a 'null- explanation" (1. p. 278). |
| 2. Simple (1-step) deductive reasoning | | | | | Simple (1-step) deductive reasoning is a single deduction from one or more premises (cf. 6, p. 235). |
| 3. Mathematising | | | | | In our context, under mathematising we understand the explanation/justification of transformation/decontextualization of a word problem/a problem defined in the real world, to a strictly mathematical form (cf. 7, p. 81). |
| 4. Reasoning by analogy | | | | | "Reasoning by analogy involves making a conjecture based on similarities between two cases, one well known (the source) and another, usually less well understood (the target)" (9, p. 110). |
| 5. Reasoning with empirical arguments/specific cases: (a) making claims and generalizing and (b) justification of a claim (extra note if experimental demonstration is used) | | | | | Reasoning begins with specific cases and produces a generalization from these cases [cf. 9, p. 88]; and testing claims using "evidence from examples (sometimes just one example) of direct measurements of quantities, substitutions of specific numbers in algebraic expressions, and so forth" (10, p. 809). |
| 6. Developing conclusions/ justifying/refuting through deductive reasoning: a generic example, a counterexample, a systematic enumeration, and other | | | | | "Deductive reasoning... is the process of inferring conclusions from known information (premises) based on formal logic rules, where conclusions are necessarily derived from the given information and there is no need to validate them by experiments" (11, pp. 235-236). |
| 7. Other (abductive reasoning, etc.) | | | | | "Abductive reasoning... [is] the search for a general rule from which a specific case would follow" (Eco's description in [9. p. 1011]). |

Note. *R using at least 2 different representations: Graphical (G), Symbolic (S), Verbal (V), Real-world situations (R), Manipulatives (M); *T using (digital) technology (e.g., calculator, GeoGebra, math apps, etc.); & **RT using both technology and at least two different representations together

### *Norwegian textbook*

In Norway, schools can freely choose among all available textbooks found on the market. The selection of the textbook used for the present study was based on two reasons. First, it is a commonly used textbook in the 8th grade. The second reason is that when this study started, the Norwegian Ministry of Education had just published a new national curriculum which RP were emphasized (Utdanningsdirektoratet, 2019, p. 3), and the selected book was one of the books that were updated accordingly. The book is divided into four main units, which are further divided into sections, the content of which is presented in **Table 1**. Each section begins with a box containing a list of the learning goals the students are expected to meet. Then, the sections are divided into narratives, worked-out examples, exercise sets, and activities, although the order of these is not consistent across the various sections. The purpose of the narratives is to explain concepts, terminology, and procedures. The sections all end with a reflection question to initiate classroom discussion. Often, the narrative argues a certain way of thinking or a method, with the worked-out examples that follow offering only the procedure without explanation. Many of the exercises are meant to be solved and discussed in the classroom and, although not part of this study as these are not solved RP tasks, it is worth mentioning that there is a clear focus on argumentation, as in most of the exercises the students are asked to explain their solutions to their fellow students. The various activities are exploratory practical activities, where the students are expected to cooperate and be creative. Toward the end of each unit, there is a set of worked-out examples connected to the goals that were set at the start of the unit's sections. The units end with a set of varied exercises, many of which represent real-life interdisciplinary problems. The items analyzed in this study come from the narrative, worked-out examples appearing in the main body of the units, and the worked-out examples included at the end of the units.

## Data Analysis Procedure

The data analysis of ways of reasoning in RP included in the textbooks mainly involved qualitative content analysis and descriptive quantitative analysis, which are explained below.

### *Data analysis framework*

To conduct a content analysis of solved examples, we used a framework that was integrated from multiple studies and compiled a set of various ways of reasoning (Sevinc et al., 2022). **Table 2** presents the list of reasoning and representational specifications we used in our coding system.

As can be seen, we focused mainly on six different types of reasoning but also included the "other" code, indicating that we are open to other types of reasoning emerging from the solved examples in different countries' textbooks. Furthermore, in our analysis, we aimed to capture different representations accompanying different ways of reasoning. Therefore, we also coded the solved examples, whether they included multiple representations and use of technology or not.

### *Data analysis process*

For qualitative content analysis of solved examples, we followed the steps listed below.

1. Examining all solved examples and identifying RP tasks: RP tasks involve a mathematical claim that can be in the form of an answer to a (real-life) word mathematics problem, a result of a "mathematization" such as an equation or graph, or a

**Table 3.** Percentage of RP tasks in the textbooks

| | Turkish textbook | Norwegian textbook | Slovak textbook |
|---|---|---|---|
| Number of RP tasks | 118 | 98 | 153 |
| Total number of solved tasks | 204 | 228 | 262 |
| Percentage of RP tasks | 57.84% | 42.98% | 58.40% |

(general) mathematical statement. In addition, it involves argumentation that supports the claim, not just a step-by-step solution to the problem using a standard/given algorithm (Sevinc et al., 2022, p. 2083).

2. Coding the ways of reasoning for the solved examples that were identified in the previous step.
3. Selecting representative tasks for each way of reasoning and translating these into English.
4. Having pairs of teams (Norwegian-Turkish, Slovak-Turkish, and Slovak-Norwegian) to look at the tasks and the codes assigned to these tasks.
5. Having both pair-team meetings and whole group meetings to discuss the discrepancies between codes and/or examining the cases identified by the national team.
6. Completing the data analysis tables.

We want to note here that, for the qualitative content analysis, it is important to identify the unit of analysis, especially if the study involves a comparison of cases (Miles et al., 2014). The unit of analysis for coding was each solved example if the task had only one solution. For the solved examples had two or three ways of solving the task, and each solution was coded separately. Hence, the solution of the task was the determinant of the unit analysis.

After analyzing the solved examples and completing the analysis table (see **Table 2**), we combined each country's analysis tables into one (see **Table 3**), which constitutes a transition step from the qualitative content analysis to the descriptive quantitative analysis that was used to make comparisons among the textbooks. Hence, our comparative analysis is three-fold. First, using the overview table of all content analysis results of all three textbooks, we visualized the distribution of ways of reasoning in each textbook separately on a pie chart and compared those pie charts with one another. Second, we used a mixed-methods data analysis tool called MAXQDA (VERBI Software, 2021) and created a multiple-case code distribution map. This map visualizes the constant comparison of different ways of reasoning in the Turkish, Slovak, and Norwegian textbooks, including the percentages of each reasoning type observed in each textbook. Third, we created bar graphs showing ways of reasoning using multiple representations and technology. We first constructed those bar graphs separately for each textbook and then examined the similarities and differences among the graphs.

The reliability check of the data analysis process started earlier when the analysis framework was selected and improved to meet the needs of the study (see Sevinc et al., 2022) for the details of collaborative coding and testing the reliability). During the entire data analysis process with the integrated framework, each national team involved more than one coder to triangulate the codes of the entire textbook. That is, multiple coders code each textbook and met several times to reach a consensus on the codes for the whole of the solved RP items. After the full agreement was met within the national teams, we had pairs of national teams work together to triangulate selected codes to ensure reliability (Miles et al., 2014). In this process, we selected multiple items for each type of reasoning and translated them into English to examine together with the partner team. More specifically, we formed three teams Norwegian-Turkish, Slovak-Turkish, and Slovak-Norwegian, and each team explored each other's selected examples and discussed the codes. Hence, we continued our triangulation meetings until we reached full agreement with partner teams.

## FINDINGS

This study aimed to compare and contrast the number and the nature of solved RP items in 8th grade Turkish, Norwegian, and Slovak textbooks. Based on the research questions, the findings are presented as the comparison of the number of items addressing different ways of reasoning in the three textbooks based on each subcategory of reasoning.

### Comparison of the Number of Reasoning-and-Proving Tasks

According to the numbers presented in **Table 3**, we can deduce that 118 items among the 204 solved tasks (57.84%) in the Turkish textbook, 98 items among the 228 solved tasks (42.98%) in the Norwegian textbook, and 153 items among the 262 solved tasks (58.40%) in Slovak textbook can be identified as RP tasks. Thus, a few more than half of the solved tasks in the Turkish and Slovak textbooks and close to half of Norwegian textbook are classified as RP items.

Further analysis of the textbooks revealed that most of RP tasks in the Turkish textbook are categorized as reasoning with empirical arguments/specific cases. The dominant category in the Norwegian textbook is "simple deductive reasoning," and in the Slovak textbook, it is developing conclusions, justifying, and refuting through deductive reasoning. Hence, each textbook is mainly characterized by a different way of reasoning, although they all involve RP in almost half of the solved tasks. The distribution of the number of items according to the methods of reasoning is presented in **Table 4**.

As can be seen in **Table 4**, in all three textbooks, there exists an imbalance in the distribution of the different ways of reasoning. This imbalance is more significant in the Slovak and Norwegian textbooks, where the most and least favorable categories comprise about half and 1–2% of all RP tasks, respectively. While this imbalance is smaller in the Turkish textbook, the least favorable way of reasoning is quite underrepresented compared to the two most favorable ones. A complete overview of the distribution of the ways of reasoning is presented in **Figure 1**.

**Table 4.** Distribution of items according to the ways of reasoning

| Way of reasoning | R | | | | | | T | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | TT | | NT | | ST | | TT | | NT | | ST | |
| | Yes | No | Yes | No | Yes | No | Yes | No | Yes | No | Yes | No |
| 1. Appeal to authority | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2. Simple (1-step) deduction | 35 | 1 | 17 | 33 | 7 | 43 | 1 | 0 | 2 | 0 | 0 | 0 |
| 3. Mathematising | 23 | 0 | 5 | 0 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4. Reasoning by analogy | 0 | 0 | 1 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Reasoning with empirical arguments/specific cases | 43 | 1 | 10 | 3 | 6 | 12 | 0 | 0 | 0 | 0 | 0 | 0 |
| (a) making claims and generalizing | 36 | 1 | 5 | 1 | 4 | 11 | 0 | 0 | 0 | 0 | 0 | 0 |
| (b) justification of a claim | 7 | 0 | 5 | 2 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Developing conclusions/justifying/refuting through deductive reasoning | 14 | 0 | 7 | 20 | 57 | 17 | 0 | 0 | 0 | 0 | 0 | 0 |
| (a) generic example | 0 | 0 | 1 | 2 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| (b) counterexample | 1 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (c) systematic enumeration | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (d) other | 13 | 0 | 4 | 18 | 55 | 16 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7. None of the previous | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Total number of fitems | 115 | 2 | 40 | 56 | 81 | 72 | 1 | 0 | 2 | 0 | 0 | 0 |

Note. R: Using at least 3 different representation; T: Using digital technology; TT: Turkish textbook; NT: Norwegian textbook; & ST: Slovak textbook

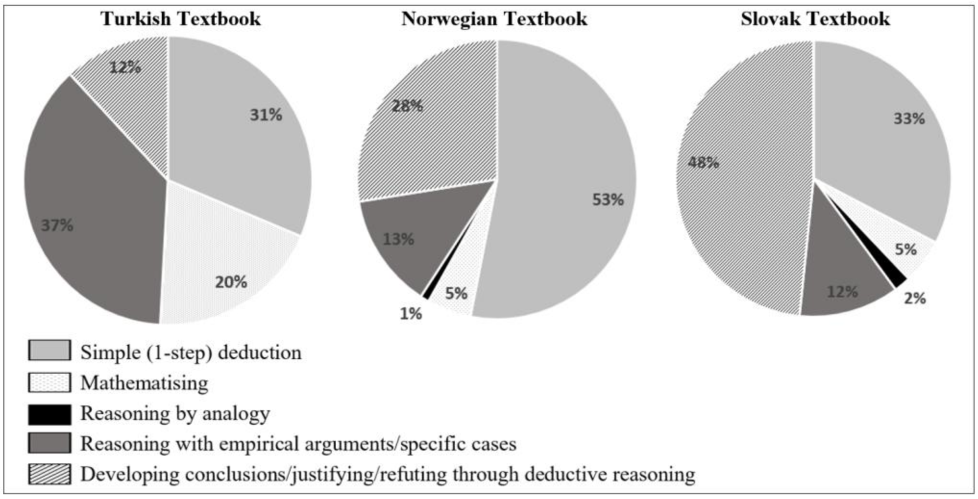



**Figure 1.** Distribution of ways of reasoning in the three textbooks (Source: Authors' own elaboration)

The above distribution presented various ways of reasoning addressed by solved tasks in each textbook. We also performed a comparative analysis of the ways of reasoning across textbooks and developed a co-occurrence model, as illustrated in **Figure 2**.

As this model indicated, simple (one-step) deduction is one of the common ways of reasoning that covers a considerable number of the solved RP tasks in the three textbooks. Specifically, the Turkish and Slovak textbooks are similar in terms of simple deductive reasoning tasks. By comparison, the Norwegian textbook mainly uses this way of reasoning in solved RP tasks.

Another important finding that this model illustrates is developing conclusions/justifying/refuting through deductive reasoning, an advanced form of simple (one-step) deduction. Among the three countries' textbooks, only the Slovak 8$^{th}$ grade textbook has more RP tasks (48%) compared to simple deduction (33%). In fact, developing conclusions/justifying/refuting through deductive reasoning appears to be a way of reasoning characteristic of Slovak RP tasks. The Turkish and Norwegian textbooks include more simple deduction tasks (ST: 31% and SN: 53%) and relatively fewer developing conclusions/justifying/refuting through deductive reasoning tasks (DT: 12% and DN: 28%).

In addition, while Norwegian and Slovak textbooks are similar in reasoning with empirical arguments/specific cases (ES: 12% and EN: 13%), this way of reasoning comprises the highest percentage in the Turkish textbook (37%). Indeed, reasoning with empirical arguments/specific cases characterizes the ways of reasoning in solved RP tasks in the Turkish textbook.

Another parallel finding was observed in the tasks addressing mathematization. In the Slovak and Norwegian textbooks, 5% of the solved RP tasks indicate mathematization, whereas 20% of the solved RP tasks in the Turkish textbook indicate mathematization. The third commonality between Norwegian and Slovak textbook is reasoning by analogy. Although there were a few (S: 2% and N: 1%), that is, at least one to two instances of such tasks, this way of reasoning was not found in the solved RP tasks in the Turkish textbook. Lastly, the comparison model demonstrated that appeal to authority was not observed in any textbooks.

This comparative analysis of ways of reasoning in Turkish, Slovak, and Norwegian textbooks indicates that each textbook has a different characteristic way of reasoning:

- Turkish textbook → Reasoning with empirical arguments/specific cases

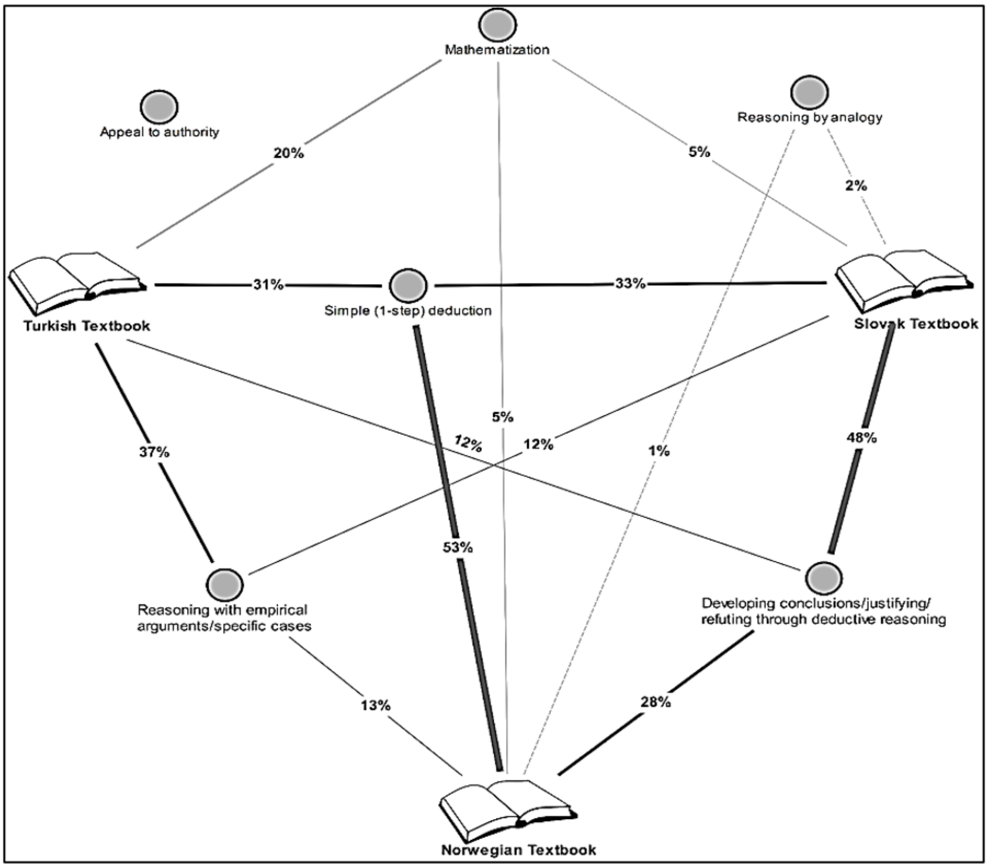



**Figure 2.** Comparative analysis of ways of reasoning in Turkish, Slovak, and Norwegian textbooks (Source: Authors' own elaboration, using MAXQDA data analysis software)

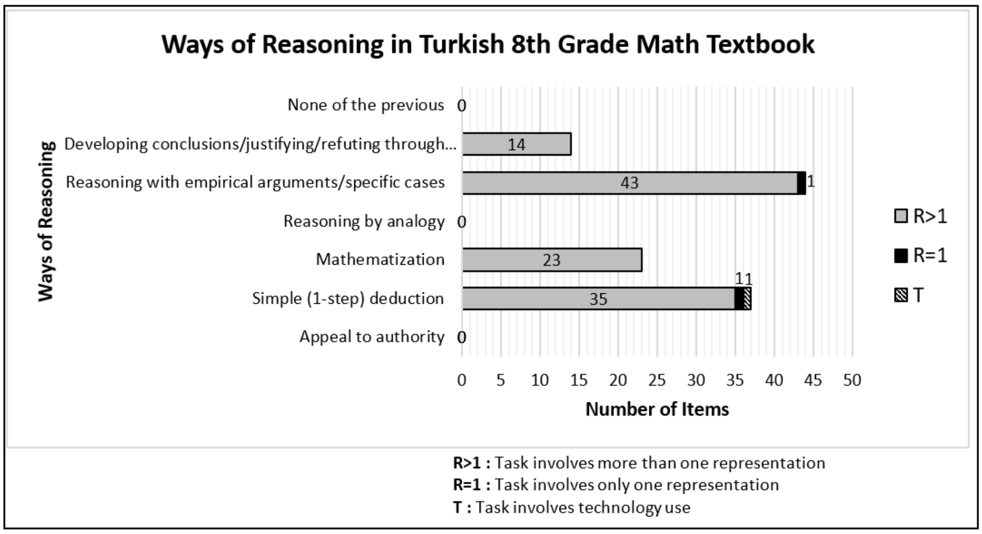



**Figure 3.** Distribution of RP items in the Turkish textbook according to the representations (Source: Authors' own elaboration)

- Slovak textbook → Developing conclusions/justifying/refuting through deductive reasoning
- Norwegian textbook → Simple (one-step) deduction

There were some commonalities between Slovak and Turkish textbooks and between Slovak and Norwegian textbooks. The distribution of ways of reasoning within each textbook did not indicate a similar pattern between Turkish and Norwegian textbooks regarding the solved RP tasks.

### Ways of Reasoning in Textbooks Regarding the Use of Representation/Technology

Further analysis focused on the ways of reasoning in relation to the use of representation/technology. We categorized the number of solved RP tasks that involve representation (R = 1), more than one representation (R > 1), and technology (T), and visualized the distribution of RP items in each textbook. **Figure 3** illustrates the distribution in the Turkish textbook.

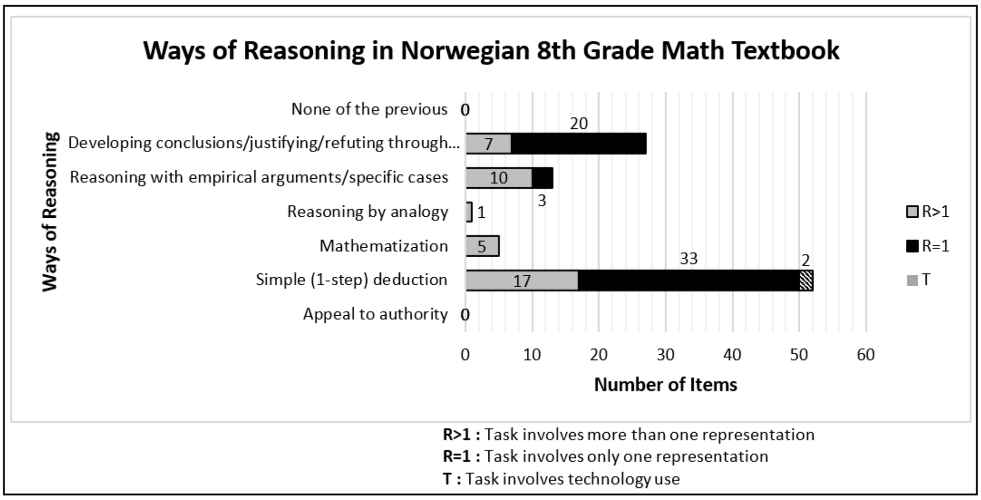



**Figure 4.** Distribution of RP items in the Norwegian textbook according to the representations (Source: Authors' own elaboration)

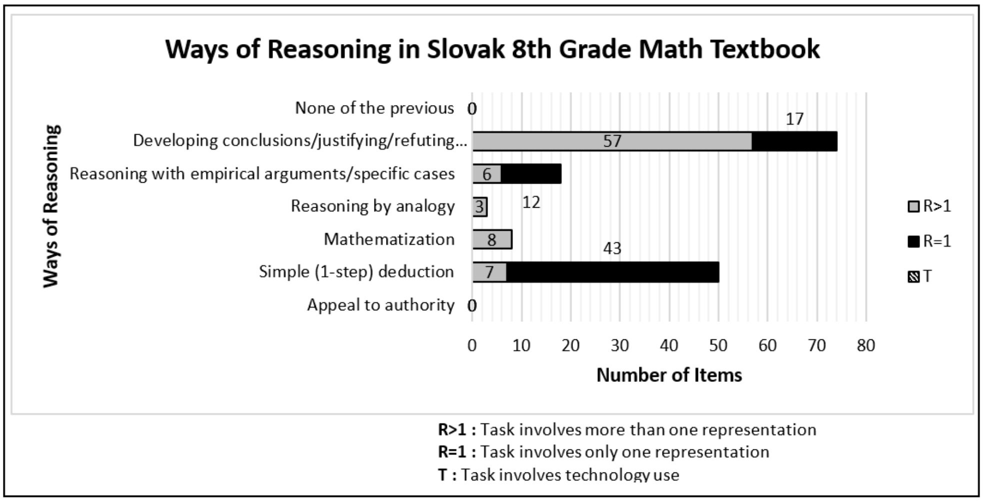



**Figure 5.** Distribution of RP items in the Slovak textbook according to the representations (Source: Authors' own elaboration)

A close examination of RP tasks in the textbooks indicates that, except for three items, all RP items include more than one representation. In addition, only one item involves technology use in the Turkish textbook. This situation is different in the Norwegian and Slovak textbooks. In the Norwegian textbook, more than half of the items (56 out of 98) consist of RP tasks that involve only one representation (see **Figure 4**). Similar to the Turkish textbook, the number of items involving technology use is also limited in the Norwegian textbook; that is, only two RP tasks involve technology use. Interestingly, in both the Turkish and Norwegian textbooks, RP tasks that involve technology use simple (one-step) deduction.

Compared to the Turkish and Norwegian textbooks, the situation is different in the Slovak textbook. Less than half of RP tasks (72 out of 153) involve only one representation. In addition, there is no task involving technology use in the Slovak textbook (**Figure 5**).

### Reasoning-and-Proving Examples in the Three Textbooks

Having compared the number of RP items in the three textbooks, we give examples of RP tasks from each category in this section. According to the analysis, none of the textbooks contained RP items falling under the category "appeal to authority." The other subcategory that received little attention is "reasoning by analogy." The Norwegian textbook contained only one example, the Slovak textbook three, and the Turkish textbook had no RP items related to reasoning by analogy. The example from the Norwegian textbook is presented below (see **Figure 6**).

To identify a given task as reasoning by analogy, there should be a transfer of information from one system to another (see also **Table 2**). In the example, the equation is matched with the balance model and solved by maintaining equivalency on both sides.

The simple deduction category in the given framework receives considerable attention in all three textbooks: 35 items of the 114 RP items in the Turkish textbook, 52 items of the 98 in the Norwegian textbook, and 50 items of the 155 items in the Slovak textbook are coded under this category. Items in the Turkish textbook mainly consist of at least two different representations (34 out of 35). However, simple deduction items in the Norwegian and Slovak textbooks focus particularly on a single representation.

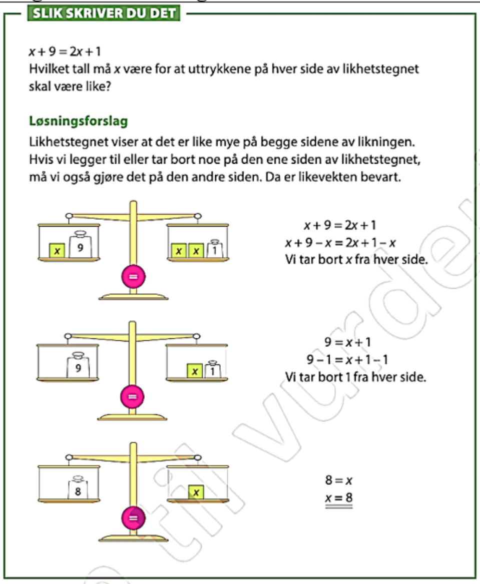
| Original Task: Norwegian Textbook | Translated Version |
|---|---|
| <br>SLIK SKRIVER DU DET<br>$x+9=2x+1$<br>Hvilket tall må $x$ være for at uttrykkene på hver side av likhetstegnet skal være like?<br>**Løsningsforslag**<br>Likhetstegnet viser at det er like mye på begge sidene av likningen. Hvis vi legger til eller tar bort noe på den ene siden av likhetstegnet, må vi også gjøre det på den andre siden. Da er likevekten bevart.<br>$x+9=2x+1$<br>$x+9-x=2x+1-x$<br>Vi tar bort $x$ fra hver side.<br>$9=x+1$<br>$9-1=x+1-1$<br>Vi tar bort 1 fra hver side.<br>$8=x$<br>$x=8$ | $x+9=2x+1$<br>Which number does x have to be so that the expressions on each side of the equals sign will be equal?<br><br>**Suggested solution**<br>The equals sign shows that it is just as much on both sides of the equation. If we add or remove something on one side of the equal sign, we have to do the same to the other side. Then, equality is preserved.<br><br>[Image of a scale with corresponding weights on each side]<br>$x+9=2x+1$<br>$x+9-x=2x+1-x$<br>We take away $x$ from both sides<br><br>[Image of the scale where weight equal to has been removed from each side]<br>$9-1=x+1-1$<br>We take away 1 from both sides.<br><br>[Image of the scale where weight equal to 1 $is$ removed from both sides]<br>$8=x$<br>$x=8$ |

**Figure 6.** Reasoning of analogy example in the Norwegian textbook (Tofteberg et al., 2020, p. 225)

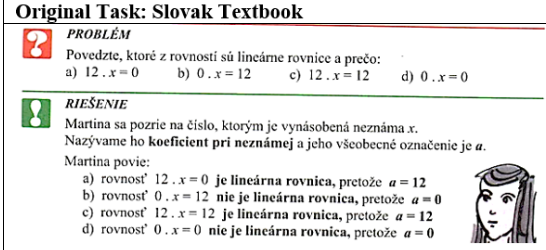
| Original Task: Slovak Textbook | Translated Version |
|---|---|
| <br>*PROBLÉM*<br>Povedzte, ktoré z rovností sú lineárne rovnice a prečo:<br>a) $12 . x = 0$ b) $0 . x = 12$ c) $12 . x = 12$ d) $0 . x = 0$<br>*RIEŠENIE*<br>Martina sa pozrie na číslo, ktorým je vynásobená neznáma $x$.<br>Nazývame ho **koeficient pri neznámej** a jeho všeobecné označenie je $a$.<br>Martina povie:<br>a) rovnosť $12 . x = 0$ **je lineárna rovnica,** pretože $a = 12$<br>b) rovnosť $0 . x = 12$ **nie je lineárna rovnica,** pretože $a = 0$<br>c) rovnosť $12 . x = 12$ **je lineárna rovnica,** pretože $a = 12$<br>d) rovnosť $0 . x = 0$ **nie je lineárna rovnica,** pretože $a = 0$ | **PROBLEM**<br>Say which of the equalities are linear equations and why.<br>a) $12 \cdot x = 0$, b) $0 \cdot x = 12$, c) $12 \cdot x = 12$, d) $0 \cdot x = 0$.<br><br>**SOLUTION**<br>Martina looks at the number by which the unknown $x$ is multiplied.<br>...<br>Martina says:<br>a) equality $12 \cdot x = 0$ is a linear equation because $a = 12$,<br>b) equality $0 \cdot x = 12$ is not a linear equation because $a = 0$,<br>c) equality $12 \cdot x = 12$ is a linear equation because $a = 12$,<br>d) equality $0 \cdot x = 0$ is not a linear equation because $a = 0$. |

**Figure 7.** Simple deduction example in the Slovak textbook (Šedivý et al., 2008, p. 5)

As mentioned above, simple deductive reasoning is defined as a single deduction from one or more premises. In the example given below, from the Slovak textbook, students are asked to decide whether the given equalities are linear and why (see **Figure 7**). In the solution, students decide whether the given equation is linear or not by checking the equation in the form of $ax = b$. Students check the single premise, whether $a$ is equal to zero or not, and decide on the linearity of the equation (in the text before the problem, the linear equation is described as an equation of the form $ax = b$, where $x$ is the unknown, $a$, $b$ are numbers, and $a \neq 0$).

In addition, the Norwegian and Turkish textbooks contain simple deduction tasks that use technology. The following items are part of a narrative where the function and variable of the concept are explained. The students learn that the function values are uniquely determined and that for a graph to represent a function, only one function value should be found for each value of the variable. The following examples illustrate how one can use a dynamic geometry program to examine this condition and conclude whether the graphs shown represent a function or not. They were, therefore, coded as simple deductions using technology (**Figure 8**).

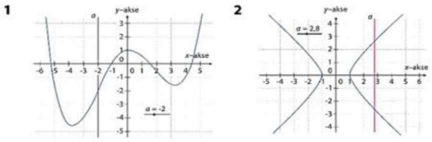
| Original Task: Norwegian Textbook | Translated Version |
|---|---|
| **1** Med et dynamisk graftegningsprogram kan du flytte den vertikale linja $x = a$ til venstre og høyre. Du ser at linja skjærer grafen bare i ett punkt. Grafen er en funksjon.<br>**2** Med et dynamisk graftegningsprogram kan du flytte den vertikale linja $x = a$ til venstre og høyre. Du ser at linja skjærer grafen i to punkter. Grafen er ikke en funksjon.<br> | **1** With a dynamic graph drawing program, you can move the vertical line $x = a$ to the left and right. You see than the line intersects the graph only at one point. The graph is a function.<br>**2** With a dynamic graph drawing program, you can move the vertical line $x=a$ to the left and right. You see that the line intersects the graph at two points. The graph is not a function.<br>[1 image where the graph is a function]<br>[2 image where the graph is not a function] |

**Figure 8.** Simple deduction using technology examples in the Norwegian textbook (Tofteberg et al., 2020, p. 163)

| Original Task: Turkish Textbook | Translated Text |
|---|---|
| **ÖRNEK 2**<br>Hesap makinesi yardımıyla $\sqrt{137}$ ve $-\sqrt{30}$ sayılarının irrasyonel sayı olup olmadığını inceleyelim.<br>**ÇÖZÜM**<br>Hesap makinesinde 137 yazalım ve karekök tuşuna basalım.<br>$\sqrt{137} = 11,7046999107...$ sayısının kesir kısmı 7046999107... olduğundan devirli olmadığı görülür. Bu durumda $\sqrt{137}$ ifadesi irrasyonel sayıdır.<br>Hesap makinesinde 30 yazalım ve karekök tuşuna basalım. Sayı negatif kareköklü olduğundan ± tuşuna basalım.<br>$-\sqrt{30} = -5,4772255750...$ sayısının kesir kısmı 4772255750... olduğundan devirli olmadığı görülür. Bu durumda $-\sqrt{30}$ ifadesi irrasyonel sayıdır.<br> | **PROBLEM**<br>By using the calculator, let's investigate whether $\sqrt{137}$ and $-\sqrt{30}$ are irrational or not.<br>[calculator images]<br>**SOLUTION**<br>Input 137 on your calculator and press the square root button.<br>Since the decimal part of the $\sqrt{137} = 11,7046999107...$ is 7046999107... , it is not repeating. Thus, $\sqrt{137}$ is irrational.<br>Input 30 on your calculator and press the square root button. Since the number is a negative square root, press the button $\pm$.<br>Since the decimal part of the $-\sqrt{30} = -5,4772255750...$ is 4772255750... , it is not repeating. Thus, $-\sqrt{30}$ is irrational. |

**Figure 9.** An example of one-step simple deduction involving technology in the Turkish textbook (MoNE, 2021, p. 83)

In the Turkish textbook example (see **Figure 9**), students decide that the given numbers are irrational since the numbers' decimal part is non-repeating, indicating a one-step simple deduction. This task is also the one involving technology (i.e., the use of a calculator).

Mathematizing refers to the transformation of the word problem defined in the real world into a mathematical expression. Analysis revealed that RP items categorized as "mathematizing" are mostly located in the Turkish textbook. More specifically, 23 items were classified as mathematizing, whereas this number is five for the Norwegian textbook and eight for the Slovak textbook. Below, one example from the Turkish textbook is presented (see **Figure 10**). In the question, the maximum length of the common side of the given rectangular piece of land, which is divided into two is being asked. The length of the common side is found by finding the greatest common factor of the two areas through algebraic expressions. Hence, the real-world problem situation is transformed into a symbolic form.

**Original Task: Turkish Textbook**

ÖRNEK 2

Recep ile Halil'in tarlalarının bir kenarı ortaktır. Recep'in tarlası 72 dam², Halil'in tarlası ise 60 dam² dir. Recep ile Halil'in tarlalarının kenar uzunlukları tam sayı olduğuna göre bu tarlaların ortak olan kenarlarının uzunluğunun en çok kaç dam olduğunu bulalım.

ÇÖZÜM

Tarlaların ortak kenar uzunluğu 60 ve 72 sayılarını tam bölen bir sayı olmalıdır. Bu kenar uzunluğunun en büyük olması için:

Ardışık bölmeyi kullanarak 60 ve 72'nin EBOB'unu bulalım.

| | | |
|---|---|---|
| 60 | 72 | 2 |
| 30 | 36 | 2 |
| 15 | 18 | 2 |
| 15 | 9 | 3 |
| 5 | 3 | 3 |
| 5 | 1 | 5 |
| 1 | | |

EBOB (60, 72) = 2 · 2 · 3 = 12 olur.

Tarlanın ortak kenarının uzunluğu en çok 12 dam'dir.

**Translated Text**

**PROBLEM**

Recep and Halil have (rectangular) lands that share one side in common. The area of Recep's land is 72 dam² and the area of Halil's land is 60 dam². Given that the side lengths of the lands are integers, what could be the maximum length of the common side of those two (rectangular) lands in dam?

**SOLUTION**

The length of the common side of those lands needs to be a factor of 60 and 72. To have the length to be maximum, let's find the GCF(60, 72).

| | | |
|---|---|---|
| 60 | 72 | 2 |
| 30 | 36 | 2 |
| 15 | 18 | 2 |
| 15 | 9 | 3 |
| 5 | 3 | 3 |
| 5 | 1 | 5 |
| 1 | | |

GCF (60, 72) = 2 x 2 x 3 = 12

Hence, the maximum length of the common side of the lands is 12 dam.

**Figure 10.** Mathematizing example from the Turkish textbook (MoNE, 2021, p. 27)

**Original Task: Slovak Textbook**

*PROBLÉM*

Pán Kováč nakupoval zemiaky na zimu. Mal dve možnosti nákupu:

1. Ak ich nakúpi v tržnici, zaplatí 18 Sk za jeden kilogram zemiakov.
2. Ak pôjde autom do poľnohospodárskeho družstva, tam zemiaky predávajú po 11 Sk za kilogram. Navyše však zaplatí 170 Sk za benzín.

Najmenej koľko kilogramov zemiakov mal pán Kováč nakúpiť, aby bolo preňho výhodnejšie zájsť do poľnohospodárskeho družstva?

*RIEŠENIE*

Vnučka pána Kováča, Betka, počíta:

počet kilogramov zemiakov ....... $x$
cena v tržnici za $x$ kg ............ $18x$ Sk
cena na družstve za $x$ kg .......... $11x$ Sk
výdaje na benzín ................ 170 Sk
spolu pri nákupe v družstve ....... $11x + 170$ Sk

Hľadám také $x$, pre ktoré platí: $18x > 11x + 170$

$18x > 11x + 170 \quad / -11x$

$7x > 170 \quad / :7$

$x > 24\frac{2}{7}$

**Translated Text**

**PROBLEM**

Mr. Kováč wants to buy potatoes for the winter. He has two purchase options.

1. He buys in the market, where one kilogram of potatoes costs 18 crowns (Sk).
2. He will go by car to the agricultural cooperative, where they sell potatoes for 11 crowns per kilogram. But he also has to pay 170 crowns for the gasoline.

At least how many kilograms of potatoes should Mr. Kováč buy so that it would be more profitable for him to go to the agricultural cooperative?

**SOLUTION**

Mr. Kováč's granddaughter, Betka, calculates:

| | |
|---|---|
| the number of kilograms of potatoes | $x$ |
| the price in the market for $x$ kg | $18x$ Sk |
| the price at the cooperative for $x$ kg | $11x$ Sk |
| gasoline expenses | 170 Sk |
| together when shopping at the cooperative | $11x +$ 170 Sk |

I'm looking for $x$ for which it holds: $18x > 11x + 170$

(here ends the part of the solution coded as mathematising).

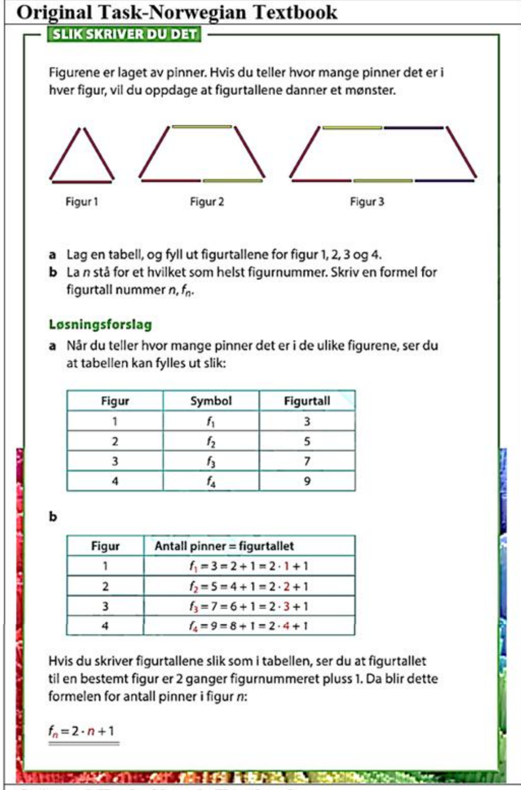
**Figure 11.** Mathematization example from the Slovak textbook (Šedivý et al., 2008, pp. 43-44)

Similarly, in the following example from the Slovak textbook, a real-life shopping problem is translated into symbols. The reasoning problem is solved by solving the given inequalities (see **Figure 11**).

The category of reasoning with empirical arguments/specific cases is the largest category of the Turkish textbook, accounting for 44 RP items of which 43 include more than two different types of representations. The number of items in the Norwegian and Slovak textbooks is close to each other; 13 items in the Norwegian textbook and 18 items in the Slovak textbook are identified under this category. Those items are related either to making claims and generalizing or justifying a claim by using empirical arguments.

An example from the Norwegian and Slovak textbooks coded as reasoning with empirical arguments involving justification of a claim is presented below (see **Figure 12**). As seen in the Norwegian textbook, the reasoning starts with the specific number of sticks, and these specific cases produce a generalized formula for finding the total number of sticks.

**Original Task-Norwegian Textbook**



SLIK SKRIVER DU DET

Figurene er laget av pinner. Hvis du teller hvor mange pinner det er i hver figur, vil du oppdage at figurtallene danner et mønster.

Figur 1 Figur 2 Figur 3

**a** Lag en tabell, og fyll ut figurtallene for figur 1, 2, 3 og 4.
**b** La $n$ stå for et hvilket som helst figurnummer. Skriv en formel for figurtall nummer $n$, $f_n$.

**Løsningsforslag**

**a** Når du teller hvor mange pinner det er i de ulike figurene, ser du at tabellen kan fylles ut slik:

| Figur | Symbol | Figurtall |
|---|---|---|
| 1 | $f_1$ | 3 |
| 2 | $f_2$ | 5 |
| 3 | $f_3$ | 7 |
| 4 | $f_4$ | 9 |

**b**

| Figur | Antall pinner = figurtallet |
|---|---|
| 1 | $f_1 = 3 = 2 + 1 = 2 \cdot 1 + 1$ |
| 2 | $f_2 = 5 = 4 + 1 = 2 \cdot 2 + 1$ |
| 3 | $f_3 = 7 = 6 + 1 = 2 \cdot 3 + 1$ |
| 4 | $f_4 = 9 = 8 + 1 = 2 \cdot 4 + 1$ |

Hvis du skriver figurtallene slik som i tabellen, ser du at figurtallet til en bestemt figur er 2 ganger figurnummeret pluss 1. Da blir dette formelen for antall pinner i figur $n$:

$f_n = 2 \cdot n + 1$

**Translated Text**

The figures are made of sticks. If you count how many sticks there are in each figure, you will discover that the figurative numbers make a pattern.

[image of sticks]

**a** Make a table and fill out the figurative numbers for figures 1,2,3 and 4.
**b** Let $n$ represent a random figurative number. Write a formula for the $n$-th figurative number, $f_n$.

Suggested solution

**a** When you count how many sticks there are in the different figures, you see that the table can be filled out as:

[Table of figurative numbers and their symbols for figures 1, 2, 3 and 4]

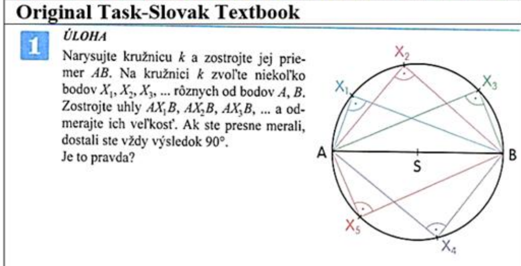
**b**

[Table showing how the figurative numbers are calculated for figures 1, 2, 3 and 4]

If you write the figurative numbers in the table, you see that the figurative number of a specific figure is 2 times the figure number plus 1. Then this is the formula for the total number of sticks in figure $n$:

$$f_n = 2 \cdot n + 1.$$

**Original Task-Slovak Textbook**



1 *ÚLOHA*

Narysujte kružnicu $k$ a zostrojte jej priemer $AB$. Na kružnici $k$ zvoľte niekoľko bodov $X_1, X_2, X_3$, ... rôznych od bodov $A$, $B$. Zostrojte uhly $AX_1B$, $AX_2B$, $AX_3B$, ... a odmerajte ich veľkosť. Ak ste presne merali, dostali ste vždy výsledok 90°. Je to pravda?



**Translated Text**

**TASK**

Draw a circle $k$ and construct the diameter $AB$. On the circle $k$, select several points $X_1, X_2, X_3$, ... different from the points $A$, $B$. Construct the angles $AX_1B$, $AX_2B$, $AX_3B$, ... and measure their size. If you measured accurately, you always got a result of 90°. Is it true?

[image of the circle $k$ with diameter $AB$ and selected points $X_1$ to $X_5$, angles $AX_1B$ to $AX_5B$ are labeled as right angles]

**Figure 12.** Examples of reasoning with empirical arguments in the Norwegian (Tofteberg et al., 2020, p. 97) and Slovak textbooks (Šedivý et al., 2007, p. 116)

In the next example from the Slovak textbook, the student has to verify the validity of the assertion formulated at the end of the task (the angle subtended from a diameter is a right angle) in several specific cases. In contrast to the task from the Norwegian textbook, in this case, the aim is not to discover a statement based on several specific cases but to verify the validity of an already formulated statement in some specific cases.

The last category in the framework is developing conclusions/justifying/refuting through deductive reasoning. Proof by generic example, counterexample, and systematic enumeration are included in this category as specific cases of deductive reasoning. Deductive reasoning that does not fall under one of the above cases is categorized as "other."

According to **Table 4**, the Slovak textbook is rich in items on deductive reasoning. More specifically, 74 items (54 consisting of at least two different representations) fall under this category. The Norwegian textbook contains 27 items, seven of which involve more than two representations. Similar to the Slovak textbook, most of the items in the Norwegian textbook fall under the category of "other," and the textbook has three generic and two counterexamples under this category. Compared to other countries, Turkish textbook have the least number of items in this category. More specifically, there are 14 items, of which 13 are categorized as "other,“ and one counterexample in the Turkish textbook. The findings also reveal that all the items in this category have at least two representations. Similarly, most of the items (57 out of 74) in the Slovak textbook have more than two representations. However, most of the items in the Norwegian textbook (20 out of 27) have reasoning using single representations.

The example below (see **Figure 13**), taken from the Turkish textbook, was coded as developing conclusions/justifying/refuting through deductive reasoning, since there is an inference of the conclusion from the known information, using logical rules. More specifically, the ratio between the side lengths of the similar figures is used to infer the conclusion that the two rectangles are similar. The task presented above requires calculating the probability of a certain event. The solution uses tree graphs to determine the number of possible and all favorable outcomes of the investigated event. Then, based on the (classical Laplace) definition of probability, the ratio of these two numbers is calculated. Thus, the result is obtained by deduction from the definition

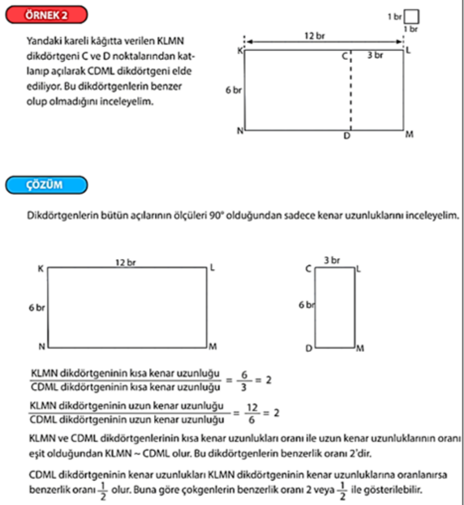
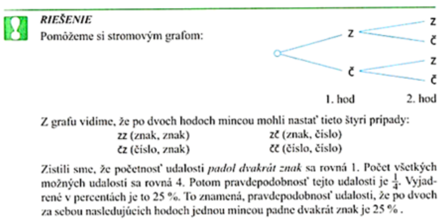
| Original Task-Turkish Textbook | Translated Text |
|---|---|
| <br>ÖRNEK 2<br>Yandaki kareli kâğıtta verilen KLMN dikdörtgeni C ve D noktalarından katlanıp açılarak CDML dikdörtgeni elde ediliyor. Bu dikdörtgenlerin benzer olup olmadığını inceleyelim.<br>ÇÖZÜM<br>Dikdörtgenlerin bütün açılarının ölçüleri 90° olduğundan sadece kenar uzunluklarını inceleyelim.<br>$\frac{\text{KLMN dikdörtgeninin kısa kenar uzunluğu}}{\text{CDML dikdörtgeninin kısa kenar uzunluğu}} = \frac{6}{3} = 2$<br>$\frac{\text{KLMN dikdörtgeninin uzun kenar uzunluğu}}{\text{CDML dikdörtgeninin uzun kenar uzunluğu}} = \frac{12}{6} = 2$<br>KLMN ve CDML dikdörtgenlerinin kısa kenar uzunlukları oranı ile uzun kenar uzunluklarının oranı eşit olduğundan KLMN ~ CDML olur. Bu dikdörtgenlerin benzerlik oranı 2'dir.<br>CDML dikdörtgeninin kenar uzunlukları KLMN dikdörtgeninin kenar uzunluklarına oranlanırsa benzerlik oranı $\frac{1}{2}$ olur. Buna göre çokgenlerin benzerlik oranı 2 veya $\frac{1}{2}$ ile gösterilebilir. | **PROBLEM**<br>The rectangle KLMN given on the grid paper is folded and unfolded along points C and D, and, the rectangle CDML is obtained. Let us investigate whether the given rectangles are similar or not.<br><br>**SOLUTION**<br>Since every angle of the rectangle is a right angle, we focus on the sides.<br><br>$\frac{\text{the short side of the rectangle KLMN}}{\text{the short side of the rectangle CDML}} = \frac{6}{3} = 2$<br><br>$\frac{\text{the long side of the rectangle KLMN}}{\text{the short side of the rectangle CDML}} = \frac{12}{6} = 2$<br><br>Since the ratio of the short sides and the ratio of the long sides of the rectangles KLMN and CDML are equal we have KLMN ~ CDML. The similarity ratio of these two rectangles is 2.<br>The ratio of the side lengths of the rectangle CDML to KLMN the similarity ratio will be $\frac{1}{2}$. Thus, the similarity ratio of the given polygons can be shown as 2 or $\frac{1}{2}$. |
| **Original Task-Slovak Textbook** | **Translated Text** |
| <br>*RIEŠENIE*<br>Pomôžeme si stromovým grafom:<br>Z grafu vidime, že po dvoch hodoch mincou mohli nastať tieto štyri pripady:<br>zz (znak, znak) zč (znak, čislo)<br>čz (čislo, znak) čč (čislo, čislo)<br>Zistili sme, že početnosť udalosti *padol dvakrát znak* sa rovná 1. Počet všetkých možných udalostí sa rovná 4. Potom pravdepodobnosť tejto udalosti je $\frac{1}{4}$. Vyjadrené v percentách je to 25 %. To znamená, pravdepodobnosť udalosti, že po dvoch za sebou nasledujúcich hodoch jednou mincou padne dvakrát znak je 25 %. | **EXAMPLE**<br>Calculate the probability of the event "a tail fell twice" if we toss one coin twice in a row.<br><br>**SOLUTION**<br>Let's use tree graphs.<br>[tree graph with the parent nodes *z* (head) and č (tail) for the first toss, in both cases the child nodes are *z* and č for the second toss]<br><br>From the graph, we can see that after two coin tosses, these four cases could occur:<br><br>zz (head, head), zč (head, tail), čz (tail, head), čč (tail, tail)<br><br>We found that the frequency of the event "a tail fell twice" is equal to 1. The number of all possible outcomes is equal to 4. Then the probability of this outcome is $\frac{1}{4}$. Expressed as a percentage, it is 25%. This means that the probability of the event that after two consecutive flips of one coin, the tail will land twice is 25%. |

**Figure 13.** Examples of deductive reasoning in the Turkish (MoNE, 2021, p. 234) and Slovak textbooks (Šedivý et al., 2008, p. 93)

and considerations of the number of possibilities. Since the solution requires not only the use of the definition of probability but also other considerations, we qualify this way of reasoning as developing conclusions through deductive reasoning rather than simple (one-step) deduction.

## DISCUSSION AND CONCLUSION

This study examined the ways of reasoning addressed in $8^{th}$ grade mathematics textbooks from three countries: Turkey, Norway, and Slovakia. As we investigated RP tasks, we particularly focused on examples with solutions to identify the type of reasoning that the task explicitly presents. However, the written solutions of the tasks that explicitly indicate particular ways of reasoning do not guarantee that those tasks will be performed in the same way. Regarding this issue, Bieda (2010) has argued that the number of RP opportunities is less in the classroom execution than in the written curriculum. Still, we believe that the possibility of solved tasks presenting the identified way of reasoning in the implemented curriculum is greater than the possibility of unsolved tasks; therefore, in the present study, we kept our focus on the solved examples.

The preliminary findings indicate that the percentage of RP tasks among solved tasks is about 50% in each country's textbook, which indicates that almost half of the solved tasks involve a mathematical claim and argumentation about that claim. This finding is promising regarding the potential contribution of textbooks in students' achievement (Oates, 2014) since it indicates that there are a considerable number of RP tasks in $8^{th}$ grade mathematics textbooks. These results support the idea of Stylianides (2009), who argued that the percentage of RP tasks could be higher in reform-oriented textbooks. The Slovak textbook has the highest percentage of RP items (58.50 %), and the Slovak national curriculum, valid from 2008, requires students to create simple hypotheses, investigate their truth, and develop their reasoning ability. In the description of the area "logic, reasoning, & proofs" (one of the five areas into which the content of the mathematics subject is divided), it is stated: "In the thematic area Logic, reasoning, proofs, which intertwines with the entire mathematical curriculum, students develop their ability to argue logically, reason, look for errors in reasoning and argumentation, express themselves accurately and formulate questions" (Bálint et al., 2010, p. 2). Such requirements are not new for mathematics teachers or textbook authors – similar demands were present in curricula before 2008. Thus, the higher number of RP items in the Slovak textbook could be interpreted as a reflection of these ideas.

Similar results were observed in the Turkish textbook, where 57.84 % of the solved tasks are identified as RP tasks. We found this type of reasoning aligns with the constructivist perspective that the current mathematics curriculum in Turkey and so the aligned textbook target (MoNE, 2018). As the curriculum indicated, building connections between mathematics ideas, developing sense-making, and improving problem-solving skills are highlighted in the middle school mathematics curriculum. Similar evidence was obtained from Norway: the results of the Norwegian textbook analysis indicate that 43% of the solved tasks are identified as RP items. In Norway, a new national curriculum for mathematics was introduced in 2020, replacing the one that had been in effect since 2006. The curriculum is further specified through a list of competencies connected to the core elements to be achieved by the end of each school year, except for grade 1. One of these five core elements is "Reasoning and Argumentation," where "reasoning" is defined as "being able to follow, evaluate and understand mathematical chains of thought," and "argumentation" as "give reasons for their approaches, reasonings and solutions and prove that these are valid" (Utdanningsdirektoratet, 2019, p. 3). The textbook analyzed in this study is one of the first to be updated according to the new curriculum.

Thus, we can conclude that RP are expected to play a central role in teaching and learning school mathematics in the three countries investigated, supporting the view that reform-oriented middle school textbooks involve RP tasks (Davis, 2012). Furthermore, this parallel observation across three countries' 8th grade mathematics textbook showed us that they all reflected the reform movements in mathematics education in their textbooks, but not to a great extent. That is, in all three countries, half of the solved tasks did not involve any reasoning; rather, procedural application of what has been introduced as a mathematical idea. Although step-by-step problem-solving procedure may help students to get practice in the target objective, students' understanding of the mathematical claims in those steps and the argumentation behind the procedure is important to develop analytical thinking (Stacey & Vincent, 2009; Stylianides, 2009). Thus, although the percentage of RP items in three countries textbooks seems to be higher compared to the other studies (e.g., Bieda et al., 2014; Davis, 2012), we only focused on the solved examples in this research. Thus, considering the whole book's content, much more attention could be given to RP items in all three countries textbooks.

Although the percentages of RP items in the three textbooks are close to each other, the types of reasoning that the tasks belong to differ according to the perspectives of the textbooks. Therefore, we discuss below our findings in light of the textbooks' own orientations. For the Turkish 8th grade math textbook, we observe two ways of reasoning that are more dominant than others. The first one is reasoning with empirical arguments/specific cases and simple one-step deduction. The tasks containing reasoning with empirical arguments/specific cases involve inductive thinking and aim to develop a mathematical claim through argumentation and/or examining the solution steps of a specific case. As mentioned earlier, the Turkish textbook has a structured design involving an introductory task, an activity, solved tasks, unsolved problems, and a set of evaluation problems for each unit. Considering the order of different types of tasks in the Turkish textbook, worked examples are placed after activities and before unsolved practice problems. Therefore, having a relatively higher percentage of reasoning with empirical arguments/specific cases was not surprising because they were used to set a mathematical claim, which was then used in another solved problem or unsolved problem deductively. The second most frequent way of reasoning in the Turkish math textbook is simple one-step deduction. We believe the lower percentage of developing conclusions/justifying/refuting through deductive reasoning (12%) is due to the emphasis on simple deduction. When we compare the percentages of those two ways of reasoning, we observe that the Turkish textbook aims to demonstrate how to use deductive reasoning in a simpler way to solve a mathematics problem. It is also important to note here that those solved tasks indicating simple one-step deduction are placed after the solved tasks presenting reasoning with empirical arguments/specific cases. Hence, the mathematical claim is first set by the empirical argumentation or examination of specific cases and then used in simple one-step deduction problems.

One-step deduction problems are also popular in the Norwegian textbook, where a few more than half of the Norwegian RP items (53%) are coded as "one-step deduction." The second biggest category is "developing conclusions/justifying/refuting through deductive reasoning" (27%). The fact that simple deduction is emphasized more than complex deductive reasoning might be explained by how RP are highlighted in the competency goals, as listed in the new curriculum. Although the definition of argumentation as a core element creates an expectation that pupils engage in mathematical proving, the word "proof" does not appear again in any of the competency goals for grade 2 to grade 10. According to Valenta and Enge (2020), who analyzed the new curriculum in terms of formulations related to RP, there is no formulation that points directly towards proof and proving in the competency goals, while there are several formulations about the more general term "argumentation." So, although the new curriculum clearly introduces the aspects of RP, there is no intention of exposing pupils to formal mathematical proofs.

Interestingly, the total proportion of Slovak items coded as deductive, with either one or more steps, was found to be the same as in the Norwegian textbook. The difference is that the Slovak textbook has far longer chains of deduction than one-step items. The reasons for such a significant number of tasks using deductive reasoning in the Slovak textbook do not follow directly either from the content of education specified in the national curriculum or from the objectives formulated by this document. As mentioned above, the national curriculum places emphasis on creating simple hypotheses and developing students' abilities to reason, but there is no specific focus on the type of reasoning. In addition, RP tasks practically do not occur in the national exam testing based on the national curriculum aiming to get a picture of pupils' performance at the end of the ninth-grade. Thus, it could be deduced that the national curriculum leaves quite a lot of space for textbook writers to decide in what way–as far as RP is concerned–they will use the mathematical topics intended for grade 8. At least two of these topics–geometric constructions and probability–are suitable for the use of deductive reasoning, and the authors of the analyzed textbook prefer to use them in this way. It can, therefore, be said that the representation of deductive argumentation largely reflects the philosophy of the authors of the textbook; it is in accordance with the belief of a part of the Slovak community of teachers that school mathematics at the end of the lower secondary education should resemble "scientific" mathematics in form.

Although these differences are observed among the textbooks, there are some similarities. Specifically, "appeal to authority" tasks are very limited in the selected textbooks, which, in fact, could be observed in implementing the tasks in the classroom. Thus, the analysis of the three textbooks does not contradict the observations regarding other textbooks, as Stacey and Vincent (2009) stated that empirical reasoning and deductive reasoning were the most frequently observed RP explanations and therefore could be considered the core ways of reasoning in math textbooks. On the other hand, those differences were parallel with other comparative textbook studies (e.g., Vicente et al., 2022) that some textbooks support students' reasoning more than others, which they claim ultimately reflected in students' mathematics achievement. Although all three countries' textbooks involve nearly 50% of solved tasks involving reasoning and proof, our findings showed that they did not provide a wide range of opportunities for supporting different types of reasoning, the implications of which we discuss below.

## Implications, Limitations, and Recommendations

The literature indicates that textbooks play an important role in how mathematics lessons are presented (e.g., Hiebert et al., 2003; Sievert et al., 2021a, 2021b) and how students perform in mathematics achievement tests (e.g., Siegler & Oppenzato, 2021; Törnroos, 2005; Xin, 2007). Therefore, it is important to examine the ways of reasoning a textbook may present and emphasize, as well as to compare the textbooks of different countries in terms of their ways of reasoning. This study targeted this purpose and demonstrated the ways of reasoning in solved RP tasks in the textbooks of three countries, namely Turkey, Slovakia, and Norway.

Our findings indicate several implications for textbook developers and for teachers who use textbooks as the main resource in planning their lessons (Davis, 2012; Mullis et al., 2012; Stacey & Vincent, 2009; Stylianides, 2009; Vicente et al., 2022). First, the distribution of the ways of reasoning needs to be considered when preparing textbooks, which suggests textbook developers are aware of different ways of reasoning and the potential of each way of reasoning in mathematics. This is important because underrepresenting some ways of reasoning while overemphasizing others may lead to students not having sufficient opportunity to develop certain ways of thinking, ultimately influencing their mathematical achievement. Although we observed differences in each country's textbook regarding the dominant way(s) of reasoning, this study is limited to only three countries, and only one textbook was selected for each country. To consolidate and/or extend our findings, we recommend further studies investigating the phenomenon with a more extensive data set. In addition, we suggest further studies investigating the ways of reasoning across different mathematics topics in textbooks, in addition to international comparisons involving several textbooks from several countries.

Second, our findings indicate that teachers should not use all the tasks and the underlying ways of reasoning uncritically. On the contrary, it is important that they develop the skill to identify the possible ways of reasoning involved in various tasks. Therefore, we suggest that teacher educators could design courses for pre-service teachers that target their knowledge and skills in examining and using mathematics textbooks concerning RP. Considering that those reasoning tasks could only serve their purpose if the teacher implements them by allowing students to make mathematical claims and reason through claims and argumentation. Therefore, teachers' critical role is inevitable, and our study is limited in its design as a document analysis. We highly recommend more extensive studies with research design incorporating different methodological ways of data collection, such as observing teachers' implementation of the textbook tasks and asking them to design a reasoning-based lesson using selected textbooks. Our study was also limited to solved examples because those tasks could allow us to see whether there was an argumentation along with the mathematical claims, although a considerably large part of the textbooks involving activities and unsolved tasks were ignored in the analysis of the current study. However, if teachers are involved in further studies, the unsolved tasks may also be included in the analysis, and the ways of reasoning those unsolved tasks present can be examined by observing teachers' use of them in the classroom. Hence, in further studies, analysis of complete textbooks might provide a more accurate picture of RP opportunities offered to pupils.

Third, we suggest that both textbook developers and teachers be aware of the number of representations involved in tasks. Textbooks involving tasks with multiple representations are not sufficient, and teachers have a critical role in relating multiple representations to find the way of reasoning addressed by the tasks. In this sense, both teacher education courses and professional development for in-service teachers focusing on these aspects of RP tasks are recommended.

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