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The effect of the error-based activities incorporating argumentation on errors in the external visual representations of pre-service teachers: An action research

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ABSTRACT

The aim of this study is to reveal external visual representations of pre-service mathematics teachers towards number sets, to identify reasons for errors in representations and to create a lesson plan that can overcome the errors. In this context, the study is an example of action research. The first stage of the study was devoted to revealing external visual representations of pre-service mathematics teachers. The second stage of the study was devoted to lesson plan that can overcome the existing errors obtained in the first stage. Error-based activities with argumentation-based learning was determined as teaching method in the lesson plan. As a result of the study, 30 different errors were determined in their external visual representations. It was determined that the lesson plan was effective in overcoming the errors in external representations and in creating appropriate internal representations.

Keywords: external representation, internal representation, visual representation, error-based activities, argumentation-based learning

INTRODUCTION

Number Sets and Difficulties Experienced in Number Sets

Number sets constitute one of the fundamental conceptual structures of mathematics and form the basis for understanding many advanced mathematical ideas. Historically, number systems have expanded in response to both practical needs and theoretical developments, progressing from counting and natural numbers to integers, rational numbers, real numbers, and finally complex numbers (Ifrah, 2000; Tall, 1992). In mathematics education, these number sets are typically introduced as a hierarchical structure in which each new set extends to the previous ones, such as natural numbers being a subset of integers, integers of rational numbers, rational numbers of real numbers, and real numbers of complex numbers.

Counting numbers and natural numbers emerge from early counting activities and provide the foundation for numerical thinking. Although these number sets appear intuitive, research shows that students often experience difficulties distinguishing between counting numbers and natural numbers, particularly regarding the role of zero and the hierarchical relationship between these sets (Gelman & Gallistel, 1978; Nunes et al., 2007). These early difficulties may later influence students' understanding of more advanced number systems.

The introduction of integers represents a significant conceptual shift, as it requires extending number concepts to include negative values. Studies indicate that students frequently struggle with the meaning of negative numbers and their position within the number system, often treating integers as a separate category rather than as an extension of natural numbers (Bofferding, 2014; Vlassis, 2004). Such difficulties reflect challenges in constructing inclusion relationships among number sets.

Rational numbers, which historically emerged from ratios, fractions, and proportional reasoning, have been identified as one of the most challenging number sets for learners (Behr et al., 1983; Ni & Zhou, 2005). Students commonly experience difficulties in recognizing rational numbers as numbers on the number line, understanding their dense nature, and relating fractions, decimals, and integers within a unified structure (Vamvakoussi & Vosniadou, 2010). These challenges often result in fragmented conceptions of rational numbers and their relationship to other number sets.

The concept of real numbers, formalized relatively late in the history of mathematics, encompasses both rational and irrational numbers and requires an understanding of continuity and completeness (Tall, 1992). Research consistently shows that irrational numbers are particularly problematic for students, who may perceive them as exceptional, isolated, or even unrelated to rational numbers (Fischbein et al., 1995; Zazkis & Sirotic, 2010). Difficulties related to real numbers frequently involve misunderstanding the inclusion relationship between rational and irrational numbers and failing to conceptualize the real number system as a unified whole (Tall & Vinner, 1981).

Complex numbers represent the most abstract extension of the number system and were historically met with resistance even among mathematicians (Tall, 1992). In educational contexts, students often perceive complex numbers as disconnected from previously learned number sets or as purely symbolic entities lacking meaningful interpretation (Arcavi & Hadas, 2000). Difficulties in understanding the relationship between real and complex numbers and in representing complex numbers geometrically have been widely reported (Panaoura et al., 2006).

Beyond difficulties related to individual number sets, a substantial body of research indicates that students experience persistent challenges in understanding the relationships among number sets, particularly with respect to inclusion, hierarchy, and density. Learners often perceive number sets as disjoint rather than hierarchically nested, construct reversed subset relationships, or fail to represent inter-set relationships altogether (Duval, 2006; Tall & Vinner, 1981). Several studies have also emphasized that these difficulties are closely related to the types of representations used in instruction and assessment (Peled, 1999; Voskoglou & Kosyvas, 2011; Zazkis & Sirotic, 2004). To address such challenges, researchers recommend emphasizing multiple representational forms during concept instruction, allowing learners to coordinate symbolic, visual, and conceptual perspectives (Kieren, 1993). These difficulties become particularly evident in students' external representations, such as number lines and Venn diagrams, which require the explicit articulation of relationships among number sets (Zazkis, 2005).

Although literature provides valuable insights into students' difficulties with specific number sets—most notably rational, irrational, and complex numbers, existing studies generally focus on isolated number sets or limited relationships between them. As a result, students' understanding of the number system as an integrated and hierarchical whole remains insufficiently explored. This situation points to a need for holistic research that simultaneously considers all number sets and systematically examines the relationships among them, enabling a more comprehensive understanding of students' conceptual structures and representational difficulties.

External Visual Representations

Mathematical connection is regarded as one of the basic constructs of mathematical thinking (Umay, 2007). Eli (2009) defined mathematical connection as a component of a schema or a group of related schemas in a mental network. Piaget and Petit (1971) stated that individuals' mathematical knowledge will find its final form with connection in their mental schemas. For this reason, one should pay attention to connection for more meaningful learning. Oktaviyanthi and Agus (2019) stated that mathematical connection is effective in students' learning mathematics.

There are differences in the grouping of the dimensions of the ability to make connections by researchers in literature. Narlı (2016) addressed mathematical connections under three headings. These headings are as follows: connecting mathematical concepts with each other, connecting mathematics with different disciplines, and connecting mathematics with daily life. Similarly, Van De Walle et al. (2012) addressed mathematical connections as connecting mathematical ideas with each other, connecting mathematics with different disciplines and the real world. Bingölbali and Coşkun (2016) added the dimension of making connections between different representations of the concept to the dimensions of mathematical connections. Multiple representations, which are considered as a sub-dimension of mathematical connections, have an important place in the in-depth understanding of mathematical concepts.

In order to deeply understand the connection between multiple representations, which is one of the indicators of the mathematical connection skill, it is useful to examine the concept of representation. The commonly used meaning of representation is the way of structuring something that can be expressed with something else, the configuration of pictures, signs, characters, symbols and objects that represent something, an equivalent presentation (Goldin, 2002; Seeger, 1988). There is no single representation used in mathematics. In mathematics, the use of representation is an important part of mathematical activities since it is mandatory to express a mathematical object, concept, theorem or idea (Dreyfus & Eisenberg, 1996; Stylianou, 2010).

Representations are classified as internal and external representations according to their formation and presentation in the mind (Janvier et al., 1993). Internal representation is the structuring of a concept or relationship in the mind of a person that cannot be directly observed. External representations, which are the reflection of internal representations, are understood as the expression of the concept in the mind with representations such as tables, graphs, charts, and symbols that serve to concretize the concepts in the minds of people (Janvier et al., 1993; Pape & Tchoshanov, 2001). There is a two-way interaction between internal and external representations (Goldin & Kaput, 1996). When internal and external representations establish a connection between two representations, conceptual understanding develops (Hiebert & Carpenter, 1992; Janvier et al., 1993). Based on the similar idea of Goldin and Shteingold (2001), it is possible to say that even if external representations are used correctly without knowing what they mean, they do not reflect conceptual understanding. The aim is to establish a connection between the external representations and the internal representations. At the same time, for a concept to be fully understood, a connection must be established with the internal representations related to that concept. Because for a mathematical concept to be fully understood, other concepts related to that concept must also be known well (Baergen, 2006). For this reason, it is important for students to have knowledge about different representations of the concept and to be able to make transitions between representations for conceptual understanding (Even, 1998; Ryken, 2009). This perspective has brought together the behaviorist approach, which accepts learning as observed behaviors, and the constructivist approach, which sees knowledge as a structure in the human mind, on a common ground (Goldin & Shteingold, 2001). When considered from this perspective, it can be said that the internal representation of a concept is related to how that concept is structured in the mind, while the external representation is related to how a representation is made for that concept.

External representations used for a concept can be represented in different ways (Duncan, 2010) and there is no superiority over each other (George, 1997). They can be shown with symbols, written expressions, graphs, numbers, diagrams, and drawings (Stylianou, 2010). Students are expected to choose the most appropriate representation of the concept (Lesh et al., 1987) and understand that representations are different ways of explaining the same situation (Ainsworth et al., 1997). By using multiple representations in mathematics education, the weakness of one representation is compensated by the strength of the other representation (Friedlander & Tabach, 2001). Students have more options in terms of understanding and problem solving, and thus they feel more comfortable (Schultz & Waters, 2000). In addition, as a result of the application of mathematical multiple representations in teaching, an approach suitable for different learning styles of students can be adopted (Hughes-Hallett et al., 2010). Multiple representations support mathematical understanding (National Council of Teachers of Mathematics [NCTM], 2000). Inadequate use of multiple representations by teachers in the classroom means that significant benefits cannot be provided in terms of understanding concepts and communicating mathematical ideas (Greeno & Hall, 1997).

Apart from the classification of internal and external representations, different classifications of representations have also been made. It is possible to say that these classifications are the classification of external representations produced by students. Bruner (1966) made a classification of actional, imaginary and symbolic representations. In actional representation, contact with concrete objects is at the forefront, in imaginary representation, visuality such as shapes, pictures and graphics are at the forefront, and in algebraic representation, algebraic expressions are at the forefront. In Lesh’s (1979) multiple representation model, real life situations, manipulative models, pictures, written and verbal symbols are included. Schoenfeld (2006) mentioned symbolic, graphic, numerical and verbal representations. Gilbert (2010) divided representations into two groups as symbolic and visual. Sign, discourse and symbol are stated as symbolic representations, and structures such as picture, drawing and graphics are stated as visual representations. Visual representations such as graphs and diagrams, which are included in these classifications and have unlimited examples, concretize abstract concepts (Rau & Matthews, 2017) and help them to be understood better (Singer, 2009). They are visuals that encode the properties and relationships of mathematical concepts or structures (Sedig & Sumner, 2006).

It is known that algebraic representations are given more importance than visual representations due to the nature of mathematics (Noss et al., 1997). In fact, mathematicians have also given importance to visualization and visual representations, and they have played a key role in their studies (Borwein & Jorgenson, 2001; Noss et al., 1997). It may be thought that it is secondary for mathematicians, but it is important for mathematics education (Borba & Villarreal, 2005). With mathematical visual representations, students develop a better image of the concept, the process of mathematical discovery and understanding develops (Zimmerman & Cunningham, 1991), and it becomes easier to eliminate mathematical difficulties (Borba & Villarreal 2005). It is recommended by mathematics educators to use visual representations more as a natural element of mathematics at every level of education (Gutiérrez, 1996). For this reason, students’ skills in visual representations, which are important for mathematics and mathematics education, should be tested, and it is beneficial to carry out instructional activities to eliminate the identified difficulties.

How Can Errors Seen in Students’ Visual Representations Be Overcome?

First of all, it is useful to question the connection between the internal representation and the errors observed in the students’ external visual representations related to the concept. If it is an error that is not related to the internal representation and is only due to carelessness or lack of knowledge, teaching can be easily done to eliminate the lack of knowledge. If the visual representation is drawn as an external reflection of an inappropriate understanding in the internal representation, the detected error has the potential to be a misconception. Misconceptions are systematic errors and are distinguished from other types of errors (Smith et al., 1994). Zembat (2010) stated that the error is a result of misconception. Although there is no comparative study on this subject in the literature, it is thought that the cognitive conflict approach will be more effective in overcoming misconceptions (Bingölbali & Özmantar, 2014). Therefore, it provides significant benefits if the students’ mistakes are considered in the teaching carried out (Borasi, 1996).

Even though misconceptions are undesirable, they can be turned into advantages with well-planned teaching (Zembat, 2010). Error-based activities come to the fore as a teaching method based on students’ mistakes. Parviainen and Eriksson (2006) defines error-based activities as the ability to learn from mistakes. Durkin et al. (2017) state that presenting students with erroneous examples containing misconceptions will be useful in correcting students’ mistakes and eliminating misconceptions. With this method, concept teaching is deepened by discussing the mistakes made by students. Error-based discussions encourage students to think critically in cognitive conflict environments and to conduct research within the problem-solving process (Borasi, 1994). In the literature, it is seen that students’ success increases in error-based activities (Durkin & Rittle-Johnson, 2012), contributes to meaningful learning (Barbieri & Booth, 2020), supports students’ gaining different perspectives (Bray, 2013; Gedik et al., 2017), supports fun and permanent learning (Kalaç et al., 2024).

According to Wood (1988), the only way to prevent the formation of a permanent misconception is to discuss the misconception and establish mutual communication. The free discussion of student misconceptions in the classroom is closely related to the classroom atmosphere that the teacher will provide (Yackel & Cobb, 1996). One of the most important methods that can provide this classroom atmosphere is argumentation-based learning. Argumentation-based learning aims to enable individuals to produce arguments, defend, support, question, and criticize the arguments they produce, and to convince both themselves and others in written or oral form (Bieda et al., 2020; Yackel & Cobb, 1996). This learning method is a dynamic social discourse process in which students defend their arguments in a scientific discussion process with reasons that support their conceptual understanding, the arguments they create in groups or individually (Newton et al., 1999; Rumsey & Langrall, 2016). In this process, peer learning comes to the fore. In argumentation-based learning, students reveal their mental schemas and try to question the arguments formed in the discussion environment. There are research results in the literature that argumentation-based teaching increases students' success (Duran et al., 2017) and facilitates learning (Whitenack & Knipping, 2002).

Why Should Pre-Service Teachers (PTs) Be Able to Use Visual Representations?

The use of representations is recommended for understanding and transferring mathematical ideas, solving problems, and modeling and interpreting daily life (NCTM, 2000). Mathematics teachers have great responsibilities to achieve the objectives of the curriculum. Because it is stated that teachers' understanding of representation affects the development of their students' skills in using representations (Stylianou, 2010). Visualization and visual representation have an important place in mathematics. Visualization can be used not only in explaining mathematics but also in learning and doing mathematics (Alcock & Simpson, 2004). Teachers must first be successful for students' success in visualization (Presmeg, 1986). Teachers and students should think about visual representations and engage in visualization to understand mathematical concepts (Kotsopoulos & Cordy, 2009). It is expected that PTs who will be mathematics teachers of the future have a correct understanding of representations in general and visual representations in particular. It should be tested whether PTs have these skills, and if there are any deficiencies, educational activities should be carried out in teacher training institutions. In this study, considering number sets, which are one of the basic subjects of mathematics, the external visual representations of PTs regarding number sets using Venn diagrams were revealed, classified, evaluated and examined according to their mathematical errors. A lesson plan that can overcome the errors that emerged was prepared, implemented and evaluated. In terms of the purpose of the study, it was thought that it contributed to the field from different perspectives. First of all, one of the most important visual representations used in teaching sets, which is the subject of the study, is the Venn diagram. By considering all number sets, the errors in the visual representation of PTs regarding the Venn diagram will be revealed in detail. It was thought that the detailed error list to be created will be useful in planning the teaching processes and will increase the quality of the relevant teaching. In the literature review, no research was found that focused on teaching to overcome student errors regarding number sets. In this study, teaching to strengthen the visual representations of PTs regarding number sets will be implemented and evaluated. It is thought that the positive results obtained will also be useful in overcoming the misconceptions experienced in different subjects of mathematics. In this study, the answers to the following research questions were investigated for educational purposes.

RQ1 What are the external visual representations of PTs regarding number sets?

RQ2 What difficulties do PTs have regarding the visual representation of number sets?

RQ3 What kind of training can be implemented to overcome the difficulties of the PTs?

RQ4 Is the training effective in overcoming the difficulties of the PTs?

METHOD

Model of the Research

The present study is designed as an action research study. The starting point of the research emerged from recurring difficulties observed among pre-service mathematics teachers in representing number sets. During the Teaching of Numbers course, PTs were asked to represent number sets using Venn diagrams. The analysis of these drawings revealed that only one PT was able to construct a correct representation, while the majority exhibited serious difficulties, particularly in representations involving irrational and complex numbers. In many cases, irrational and rational numbers, as well as real and complex numbers, were incorrectly represented as disjoint sets. Further examination of the drawings and follow-up discussions with the PTs indicated that their verbal explanations were largely shaped by their visual representations. This finding suggests that Venn diagrams serve as a powerful external visual representation for revealing PTs' conceptual understanding of number sets. Consistent with prior research, these results highlight the strong relationship between conceptual understanding and external representations (Goldin & Kaput, 1996; Hiebert & Carpenter, 1992; Janvier et al., 1993; Stylianou, 2010). Based on these observations, it was deemed pedagogically important to systematically identify PTs' errors related to number sets through their external visual representations and to design instructional activities that explicitly address these errors. Accordingly, a detailed action research study was conducted to both reveal these difficulties and to examine the effectiveness of an instructional intervention aimed at overcoming them.

When the literature on action research is examined, it is seen that the steps followed in action research may vary across studies (Fraenkel & Wallen, 2003; Johnson, 2008). In the present study, the research process consisted of understanding the problem, designing a lesson plan based on the identified difficulties, implementing the instructional intervention, and evaluating its effectiveness. Although qualitative data collection tools were predominantly used, quantitative approaches were also incorporated. The problem-identification phase of the study reflects characteristics of a qualitative case study, whereas the evaluation of the instructional intervention corresponds to a pre-experimental quantitative research design.

Research Group

The research group in the problem-understanding phase of the study consisted of a total of 104 pre-service mathematics teachers, including 44 first-year, 21 second-year, 19 third-year, and 20 fourth-year students, enrolled in the primary mathematics teacher education program of a state university in Türkiye during the fall semester of the 2023–2024 academic year. The inclusion of PTs from different grade levels was intentional. This choice aimed to examine whether the difficulties observed in the use of external visual representations related to number sets were specific to a particular year of teacher education or persisted across different grade levels. In addition, the study sought to explore how these difficulties might vary from the first year to the final year of the program.

In the instructional phase of the study, the research group consisted of 10 fourth-year students from the mathematics department. Criterion sampling, one of the purposeful sampling methods, was employed in selecting the participants. The criterion for participation was that the students were volunteers enrolled in courses taught by the first author. The selection of senior mathematics department students for the instructional phase aimed to examine whether the difficulties identified among pre-service primary mathematics teachers regarding number sets, despite the basic nature of the topic, were also present among students enrolled in a program with more intensive mathematics coursework.

Data Collection Process

The data of the study were collected using the Number Set Form (NSF), Number Set Opinion Form (NSOF), individual interviews, and focus group interviews. These data collection tools were purposefully selected to reveal PTs' conceptual understanding of number sets and the relationships among them through both external visual representations and verbal explanations. In particular, external visual representations such as Venn diagrams are considered powerful tools for making individuals' internal representations and conceptual structures visible, especially in abstract mathematical topics such as number sets (Duval, 2006; Goldin, 2002; Presmeg, 1986).

The data collection process was carried out in stages concurrently with data analysis. In the first stage, the NSF was administered to the PTs in their own classes within one lesson hour. In the NSF, the names of the number sets were provided, and the PTs were asked to draw the number sets using a Venn diagram by considering the relationships among them. This form was preferred because Venn diagrams enable the identification of conceptual errors related to set inclusion, intersection, and hierarchical structure, which may not be easily detected through traditional written or multiple-choice questions. No time limitation was imposed, and the PTs completed their drawings within approximately 20 minutes. The drawings were analyzed in terms of their conceptual accuracy.

To ensure that the interpretations of the drawings were grounded in the PTs' own reasoning, individual interviews were conducted with 10 PTs. These interviews aimed to clarify the participants' thinking processes while drawing the rational and irrational number sets. This approach made it possible to better understand the meanings PTs attributed to their visual representations and to avoid interpretations based solely on researchers' assumptions. Analysis of the NSF data revealed that PTs made numerous errors in their drawings, most of which were related to the relationships between the rational–irrational and real–complex number sets.

In the second stage, the NSOF was administered to first- and second-year PTs, who exhibited the highest frequency of errors, one week after the initial application. The NSOF included open-ended questions such as "What kind of relationship do you think there is between the set of rational numbers and the set of irrational numbers (if any)?" and "What kind of relationship do you think there is between the set of real numbers and the set of complex numbers?". This form was used to elicit PTs' verbal explanations and internal representations, which are considered the underlying sources of their external visual representations. By comparing PTs' written opinions with their drawings, the consistency between internal conceptual understanding and external representations could be examined.

The combined analysis of the NSF and NSOF data provided important insights for the preparation of the instructional plan. Based on the identified errors and misconceptions, it was decided that error-based activities and argumentation-based learning should be emphasized during the instructional process.

The instructional intervention was carried out with fourth-year mathematics students. The NSF and NSOF were applied as pre-tests to a class of 16 PTs in the same session. Following the analysis of the pre-test data, six drawings containing common and critical errors were selected to be used as instructional materials during the training. One week after the pre-test application, the training was conducted based on the prepared lesson plan. Six PTs did not participate in the training for various reasons; therefore, the evaluation of the training was based on data from 10 participants. The training lasted two hours and was implemented in the form of a focus group to encourage discussion, justification, and reflection. The focus group sessions were audio-recorded and supported with photographs. One week after the training, the NSF and NSOF were administered again as post-tests to evaluate the effectiveness of the instructional intervention.

Analysis of Data

Content analysis was used to analyze qualitative data. The drawings made by the PTs were categorized according to the types of errors in the drawings. The first and second author analyzed the drawings of the fourth-grade students together. A list of codes and categories to be used in content analysis was prepared. The PTs and the types of errors present were numbered. An electronic worksheet was created with student numbers in the column and error types in the row. Both researchers processed the data analyses separately on the worksheet. Afterwards, the coding made by the two authors were compared. It was determined that 11 out of 193 error types were coded differently. In this context, an agreement percentage of approximately 94% was determined in error type detection. An agreement was reached for the 6% that did not agree. There was no disagreement in the analysis of PTs' opinions. In addition, the coding and drawings were approved by a mathematics teacher as an external auditor. While some PTs made only one type of error in their drawings, it was determined that most of the PTs made errors that could be evaluated in more than one category. To ensure confidentiality, the real names of the PTs were replaced with pseudonyms. The sample drawings of the PTs for the coding obtained were often presented directly. In this way, it was aimed to increase the validity and reliability of the study.

FINDINGS

Identifying and Understanding the Problem

In the first phase of the study, the Number Set Form (NSF) was administered to identify PTs' external visual representations of number sets. The analysis revealed that only three PTs were able to produce mathematically correct Venn diagrams representing number sets. These PTs, all of whom were first-year students, constructed drawings consistent with formal definitions and hierarchical relationships among number sets. Their representations were therefore evaluated as appropriate visual representations. An example of the correct drawing is shown in **Figure 1**.

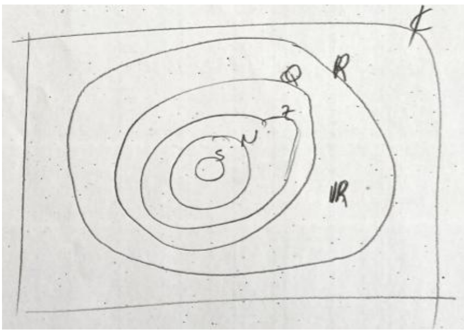


Figure 1. Correct drawing (Source: Authors' own elaboration)

The drawings of the remaining 101 PTs were found to be mathematically inappropriate. A total of 27 distinct error types were identified, which were grouped under six main categories. Across all drawings, 193 errors were detected. An overview of the distribution of these errors is presented in **Table 1**.

Table 1. Errors in PTs' visual representations of number sets

Characteristics of errors	Errors	1st grade	2nd grade	3rd grade	4th grade	Total
Errors related to the set of irrational numbers and its relationship with the set of rational numbers	Irrational and Rational numbers are two disjoint sets with no complement relation between them (E1)	31	16	6	5	57
	Irrational numbers include rational numbers (E2)	8	4	3	3	18
	Irrational numbers are in the complement set of real numbers (E3)	4	0	5	0	9
	Irrational number includes complex number (E4)	3	0	2	1	5
	Not specifying irrational numbers in visual representation (E5)	0	0	2	3	5
	Rational numbers include irrational numbers (E6)	0	2	2	0	4
	Irrational numbers include real numbers (E7)	1	0	1	1	3
Errors related to the relationship between the complex number set and the real number set	Complex numbers and real numbers are two disjoint sets (E8)	16	8	5	10	39
	Complex number is a separate set within the real number (E9)	1	2	1	4	8
	Not specifying complex numbers in visual representation (E10)	5	2	2	0	9
	Real numbers include complex numbers (E11)	2	3	0	0	5
	Complex numbers set is subset of real numbers set and irrational numbers set is proper subset of complex numbers set (E12)	1	0	0	0	1
	Natural numbers include complex numbers (E13)	0	0	1		1
Errors related to the relationship with rational numbers and other number sets	Rational numbers include real numbers (E14)	1	1	1	0	3
	Integers include rational numbers (E15)	2	0	0	1	3
	Rational numbers do not include natural numbers, integers and counting numbers (E16)	0	0	1	0	1
	Not including rational numbers (E17)	0	0	1	0	1
	Real and Rational numbers are the same set (E18)	0	0	1	0	1
Errors related to natural numbers and counting numbers and the relationship between them	Counting number includes natural numbers (E19)	0	1	1	1	3
	Natural numbers and counting numbers are in the same set (E20)	1	1	0	0	2
	Not including natural numbers (E21)1	1	0	1	0	2
	Not including counting numbers (E22)	1	0	1	0	2
Errors related to the relationship between integers and other number sets	Natural numbers include integers (E23)	1	1	1	0	3
	Integers and counting numbers are equal sets (E24)	1	0	0	0	1
	Intersections of integer and natural number are counting numbers (E25)	0	0	0	1	1
	Integer and Natural numbers are two disjoint sets (E26)	0	0	1	0	1
Not establishing a relationship between sets	Specifying sets separately and not expressing a relationship (E27)	5	0	0	0	5
	Total	83	41	39	30	193

As shown in **Table 1**, errors were predominantly concentrated in two categories: The relationship between irrational and rational numbers, and the relationship between complex and real numbers. This distribution indicates that PTs experience the greatest difficulties at key conceptual boundaries of the number system, particularly where abstract inclusion and extension relations are involved. In contrast, relatively fewer errors were observed in representations concerning rational numbers and other number sets, natural and counting numbers, and integers and other number sets. A small number of drawings did not express any relationship among number sets, suggesting limited relational reasoning. Based on this general distribution, the error categories are examined in detail below using representative visual examples.

As presented in **Table 1**, the most frequent errors identified in pre-service teachers' (PTs') visual representations were related to the relationship between irrational and rational number sets, with a total of 103 errors. This finding indicates that PTs experience substantial conceptual difficulties particularly at the boundary between these two number sets, which form complementary subsets within the real numbers. The concentration of errors in this category suggests that the distinction between rationality and irrationality, as well as their hierarchical position within the real number system, is not conceptually well established.

More than half of the errors made by the PTs (E1, f = 57) were related to the irrational number set and the relationship between irrational number and rational number. In most of the drawings, the PTs considered irrational numbers as a disjoint set from rational numbers without a complement relation. PTs made this mistake the most in their drawings. While most of the PTs made more than one mistake in their drawings, 10 PTs made only this mistake. In the drawings of the PTs, the set of irrational numbers was stated as any set in the complement of the set of rational numbers (In the drawings, mostly real numbers are considered as the universal set.). As is known, the set of irrational numbers is the complement of rational numbers with respect to real numbers. In these drawings, the set of irrational numbers is drawn not as the complement of the set of rational numbers but as a subset of the complement set. In these mathematically incorrect representations, the set of irrational numbers is represented by a closed figure like a Venn scheme. It was not understood whether this closed figure was used to indicate the entire complement set or not. For this reason, these PTs were interviewed and their opinions about the relationship between irrational numbers and rational numbers were obtained. As a result of the interview, it was determined that three PTs had a correct understanding of the concept of irrational numbers, but only used set representation to symbolically represent the region. Therefore, it is possible to evaluate these three representations as partially correct representations. The remaining seven PTs' perception of the relationship between irrational numbers and rational numbers was found to be only in the context of disjoint sets. The drawings of Aynur and Alaaddin and the conversation between them and the researcher are given below. Aynur stated that the representation in her drawing was symbolic and that the entire region outside the rational numbers and real numbers was the set of irrational numbers. Alaaddin, on the other hand, realized that his drawing was wrong due to his understanding of disjoint sets. By directing the researcher to the cognitive conflict, he reached the correct drawing by thinking about the concept of complement. The drawings and opinions of Aynur and Alaaddin are presented in **Figure 2**.

Researcher: What do you think is the relationship between irrational numbers and rational numbers?

Aynur: A rational number is not irrational. Irrational numbers include non-rational numbers such as root three.

Researcher: How did you draw the set of irrational numbers here?

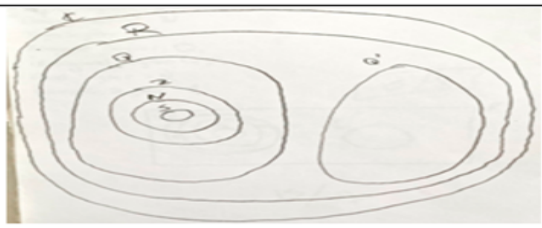
Aynur: Irrational numbers are a subset of real numbers, but they are a disjoint set different from rational numbers. Therefore, I drew them separately.

Researcher: You drew rational numbers as a big set and irrational numbers as a small set. Why did you draw them this way?

Aynur: I drew it symbolically, I didn't draw it thinking of its size, it just happened that way.

Researcher. Then where is its real place? Where does it include?

Aynur: It covers the places outside the rational numbers, being a subset of the real numbers. Those that are not rational but real.



Researcher: What do you think is the relationship between irrational numbers and rational numbers?

Aladdin: They are a subset of the set of complex numbers. They have no intersection. I think they are disjoint sets.

Researcher: You drew the set of irrational numbers like this. What did you draw with what in mind?

Aladdin: It was a new discovery for me. Non-rational numbers should be disjoint if they are irrational.

Researcher: You drew the set of irrational numbers in the same size as the set of rational numbers? Why did you draw it this way?

Alaaddin: I think the number of irrational numbers cannot be underestimated. It is an infinite set.

Researcher: Are the irrational numbers the remaining part of the set?

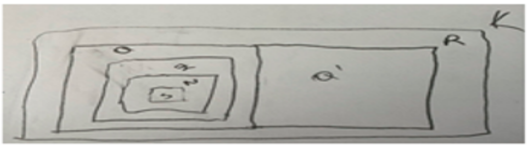
Aladdin: Yes, I thought so... I am questioning myself, I think there is a mistake in my drawing.

Researcher: Where do you think there is a mistake?

Aladdin: I cannot explain the parts of the real numbers that are not rational and irrational... The set of real numbers consists of two sets. Rational and irrational, there is no other option. When two disjoint sets are combined, they become real numbers.

Researcher: What should be the correct representation?

Aladdin:

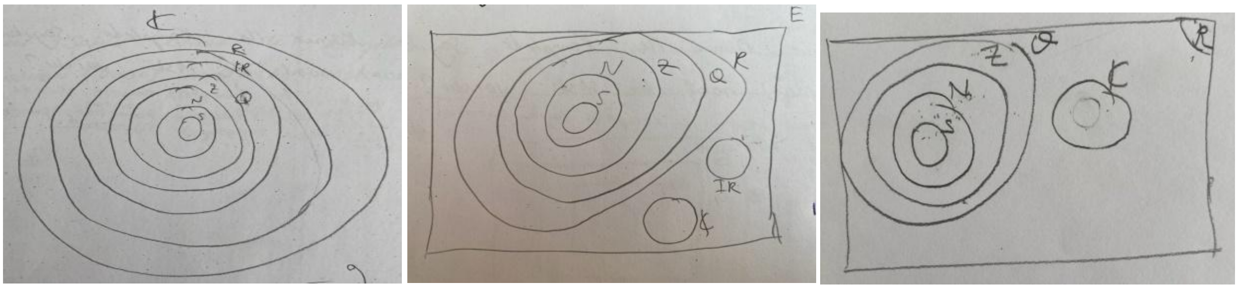


Researcher: Now, how do you think there is a relationship between the set of rational numbers and the set of irrational numbers?

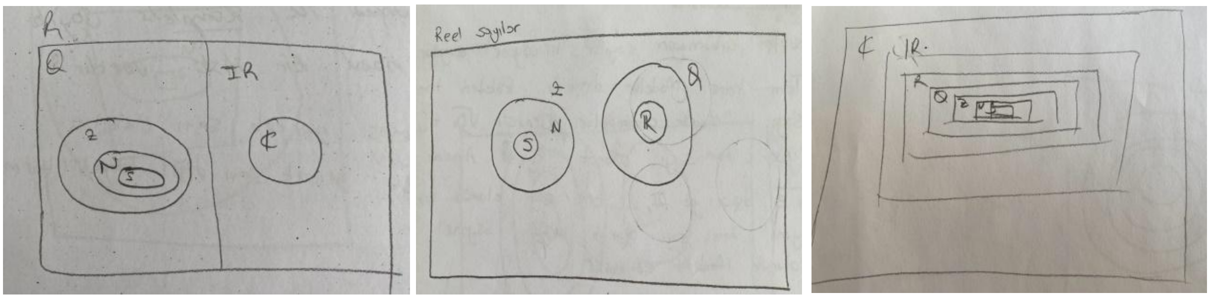
Aladdin: Their intersection is the empty set, their union is the real numbers.

Figure 2. Aynur and Alaaddin's drawings and the conversation between them and the researcher (Source: Authors' own elaboration)

Another frequently observed error involved representing irrational numbers as a superset of rational numbers (E2, f = 18, **Figure 3a**). In these drawings, rational numbers were placed entirely inside the irrational number set, indicating a reversal of the correct hierarchical structure. This visual structure suggests that PTs perceive irrational numbers as a broader or more inclusive category rather than as a distinct subset defined by non-rationality. Some PTs represented irrational numbers as belonging to the complement of the real numbers (E3, f = 9, **Figure 3b**), visually positioning the irrational set outside the real number set. This representation indicates a fundamental misconception about the nature of irrational numbers and suggests confusion between irrational numbers and non-real (or imaginary) quantities. Less frequent but conceptually significant errors included representing irrational numbers as containing complex numbers (E4, f = 5, **Figure 3c**) or omitting irrational numbers entirely from the diagram (E5, f = 5, **Figure 3d**). In the former case, the visual inclusion of complex numbers within the irrational set reflects an overgeneralization of "non-rationality" to encompass all non-real numbers. In the latter case, the absence of irrational numbers suggests either uncertainty about their status or an implicit assumption that they do not form a distinct set worthy of representation. Additional errors involved representing rational numbers as a superset of irrational numbers (E6, f = 4, **Figure 3e**) and constructing a linear progression in which irrational numbers were drawn as encompassing real numbers (E7, f = 3, **Figure 3f**). These representations reveal a tendency to conceptualize number sets as sequential or linearly ordered categories rather than as hierarchically nested structures. Examples of drawings in which such errors were detected are presented in **Figure 3a-3f**.



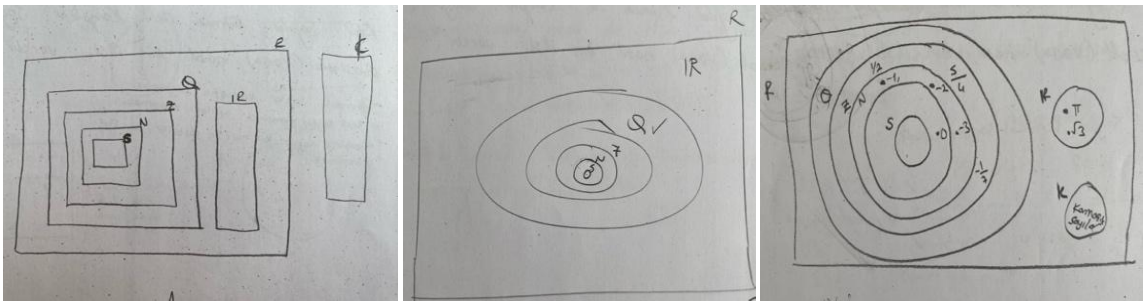
(a) (b) (c)



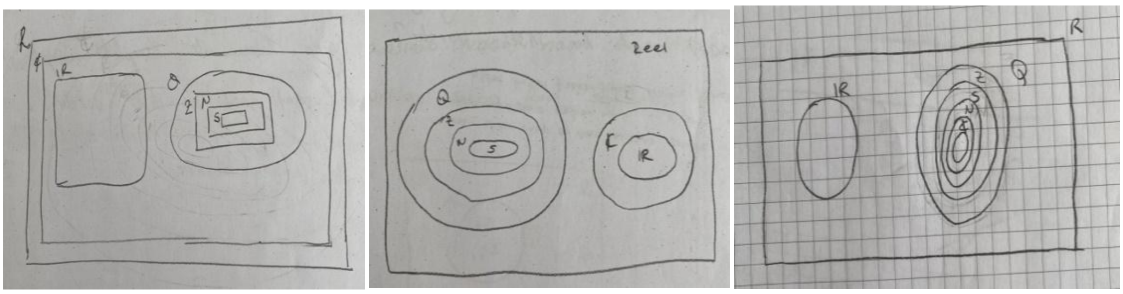
(d) (e) (f)

Figure 3. An example of a) E2, b) E3, c) E4, d) E5, e) E6, f) E7 (Source: Field study)

The second most frequent category of errors concerned the relationship between complex and real number sets, with a total of 63 errors. This distribution shows that PTs also experience notable difficulties in understanding the extension of the real number system to the complex numbers. The most common error was representing real and complex numbers as two disjoint sets (E8, f = 39, **Figure 4a**). In these drawings, PTs visually separated the two sets, suggesting that real numbers are not part of the complex number system. This indicates a lack of understanding that real numbers constitute a subset of complex numbers with zero imaginary part. Other PTs represented complex numbers as a subset within the real numbers (E9, f = 8, **Figure 4b**) or omitted the complex number set entirely (E10, f = 9, **Figure 4c**). The former error reflects a reversal of the correct inclusion relationship, while the latter suggests uncertainty regarding the necessity or role of complex numbers in the number system. Some representations depicted real numbers as progressively encompassing complex numbers (E11, f = 5, **Figure 4d**), indicating a linear or developmental perception of number sets. Rare but striking errors included drawings in which complex numbers were shown as containing only irrational numbers (E12, f = 1, **Figure 4e**) or even as being included within the natural numbers (E13, f = 1, **Figure 4f**). These extreme cases highlight deep conceptual confusion regarding both the nature and hierarchy of number sets. Example drawings regarding the errors are presented in **Figure 4a-4f**.



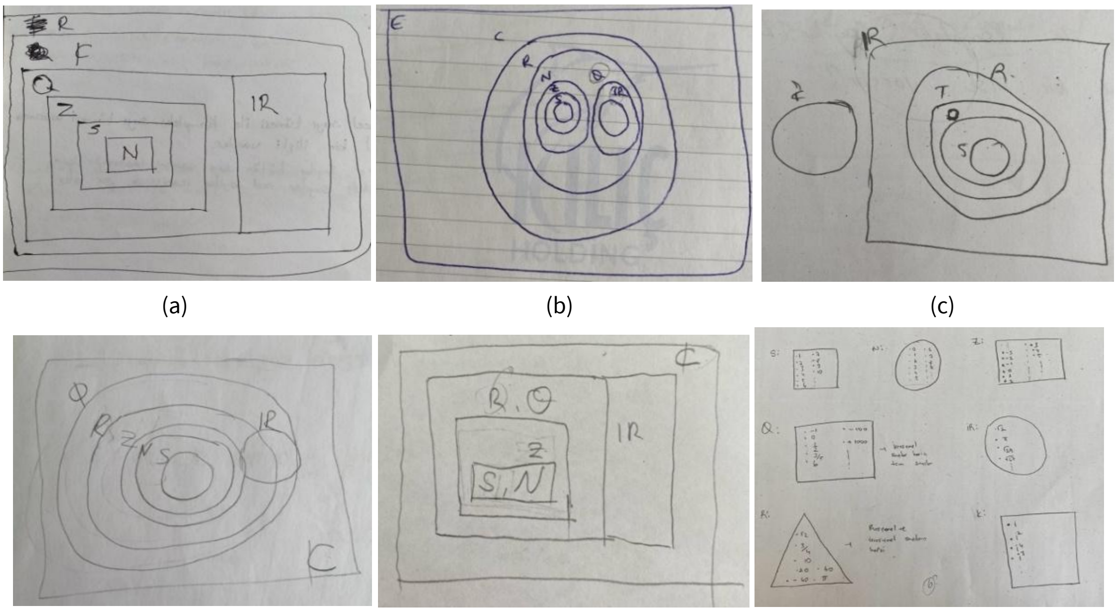
(a) (b) (c)



(d) (e) (f)

Figure 4. An example of a) E8, b) E9, c) E10, d) E11, e) E12, f) E13 (Source: Field study)

A smaller number of errors (f = 9) involved the relationship between rational numbers and other number sets. These included representations in which rational numbers were shown as containing real numbers (E14, f = 3, **Figure 5a**) or as being contained within integers (E15, f = 3, **Figure 5b**). Such drawings suggest that PTs rely on surface-level numerical familiarity rather than formal set definitions. Other errors involved excluding rational numbers from diagrams altogether (E17, f = 1) or representing real and rational numbers as identical sets (E18, f = 1), indicating blurred conceptual boundaries between these sets. Errors concerning the relationship between natural numbers and counting numbers totaled 9. Some PTs represented counting numbers as a superset of natural numbers (E19, f = 3, **Figure 5c**) or treated the two sets as identical (H20, f = 2, **Figure 5d**). Other PTs omitted one of these sets entirely (H21, f = 2; H22, f = 2), suggesting uncertainty about their formal definitions and distinctions. Errors involving integers were relatively rare (6 in total) but revealed important misconceptions. These included representing natural numbers as containing integers (E23, f = 3, **Figure 5e**) and treating integers and counting numbers as identical sets (E24, f = 1). Some PTs depicted integers and natural numbers as intersecting but not nested sets (E25, f = 1) or as completely disjoint sets (E26, f = 1), indicating inconsistent or fragmented understanding of set relationships. Finally, 5 errors were categorized as a complete failure to express relationships between number sets (E27, **Figure 5f**). In these cases, PTs listed or drew the sets separately without indicating any inclusion, intersection, or hierarchical relationship. This pattern suggests an absence of relational thinking regarding the structure of the number system. Examples of the most repeated incorrect drawings of these categories are presented in **Figures 15-20**.



(a) (b) (c) (d) (e) (f)

Figure 5. An example of a) E14, b) E15, c) E19, d) E20, e) E23, f) E27 (Source: Field study)

When the drawings of the PTs were evaluated in general, it was determined that most of the incorrect drawings (85%) were related to the relationship between irrational numbers and rational numbers and the relationship between complex numbers and real numbers, respectively. In particular, rational numbers and irrational numbers were considered as two disjoint sets without a complementary relationship, and similarly, complex numbers and real numbers were considered as two disjoint sets. When an evaluation was made on the basis of grades, it was determined that the most errors were in the first and second grades. Similarly, it was determined that the most common errors regarding the rational-irrational number and complex-real relationship, which are the most common error types, were determined to be in the first and second grades. These predominantly erroneous drawings of the PTs suggested that the PTs did not have a correct understanding of the relationship between irrational numbers-rational numbers and complex numbers-real numbers. To determine the PTs' understanding of these relationships, the opinions of the first and second grade PTs, in which the most errors were determined in this area, were taken. The PTs were asked about the relationship (if any) between rational numbers and irrational numbers and between complex numbers and real numbers. With this questioning, it was aimed to obtain an idea about the internal representations of the PTs, which are the source of their external representations.

When the opinions of PTs about the relationship between rational numbers and irrational numbers were analysed, it was found that the opinions were divided into nine categories in total. These categories and the sample opinions that led to the formation of these categories are presented in **Table 2**.

Table 2. PTs' opinions on the relationship between the set of rational numbers and the set of irrational numbers

Categories	Codes (f)	Example expressions
Intersection	The intersections of rational numbers and irrational numbers are empty set/disjoint (13).	İkisi de reel sayıların birer alt kümesidir. Ancak kesişim kümeleri boş kümedir (Ayrık kümelerdir) Both are subsets of real numbers. However, their intersection sets are empty sets. (They are disjoint sets.)

Table 2 (Continued). PTs' opinions on the relationship between the set of rational numbers and the set of irrational numbers

Categories	Codes (f)	Example expressions
Ratio	If it can be written as a/b, it is a rational number, if not, it is an irrational number (5).	Numbers that are not written as the ratio of two integers to each other provided that the denominator is not 0 are called irrational. Rational numbers are numbers expressed as the ratio of two integers to each other, provided that the denominator is not 0.
Exact value	The exact value of irrational numbers is not clear (6).	Rational numbers have exact value, irrationals do not.
Root	If the number goes beyond the root, it is rational, if not, it is irrational (5).	Rational numbers that go out of the root, irrational numbers that cannot go out of the root
Inclusion	Irrational numbers include rational numbers (6).	Irrational numbers include rational numbers
Complementary	Non-rational numbers are irrational numbers/Two sets are not the same (3).	Non-rational numbers are irrational. There is an opposite relationship.
Union	The union of irrational numbers and irrational numbers are real numbers (2)/complex numbers (1).	The two combine to form real numbers.
Multiple opinions	Intersection- union (1), ratio-intersection (3), root-ratio (2), ratio-intersection (2), complement-ratio (1), exact value-intersection (1), complement-intersection (2), ratio-exact value (2), root-exact value (1), ratio-root-exact value (1), complement-ratio-exact value (1)	Rational numbers are numbers that can be written as a/b. But irrational numbers are numbers that do not fit a representation. They do not include each other. They have no common elements.
No relationship	There is no relationship between irrational numbers and rational numbers (5)	There is no relationship between them.
No response	No response (2)	

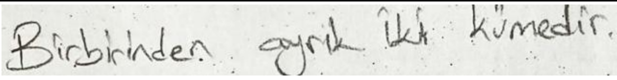
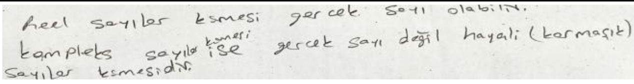
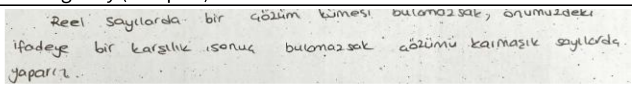
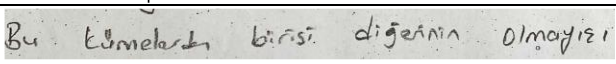
When **Table 2** is analyzed, it is revealed that the PTs evaluated their opinions in terms of the intersection, union, inclusion, compliment of set and in the dimensions of whether the numbers are written as a ratio, whether the exact value is known or not, and whether they are out of the root or not. 5 PTs stated that there was no relationship between two sets of numbers. Two PTs did not give an answer. When the opinions of the PTs who looked at the relationship in question from a single perspective were analyzed, it was observed that 13 PTs saw the two sets as two disjoint sets without a complementary relationship. The PTs with this opinion also drew the two sets in this way in their drawings. Five PTs stated whether the number can be written as a/b or not, whether it goes out of the root or not as the relationship between the two sets. As it is known, not all irrational numbers are in the form of a rooted expression. It can be said that this view is an extreme specialization. Six PTs have a misconception that irrational numbers include rational numbers. They drew their drawings in this way. Two of the three PTs stated that irrational numbers are non-rational numbers, and one PT stated that two sets are not equivalent to each other but did not mention which universal set was taken into consideration. Two of the three PTs stated that the union of two sets was the set of real numbers and one PT stated that the union of two sets was the set of complex numbers. They reflected the same view in their drawings.

After the opinions of the PTs about the relationship between the irrational number set and the rational number set, their opinions about the relationship between the real number set and the complex number set were taken. When the opinions were analyzed, it was determined that the opinions were grouped under six different categories. **Table 3** presents information about the opinions of PTs.

Table 3. PTs' opinions on the relationship between real numbers and complex numbers

Categories	Codes (f)	Example expressions
Correct inclusion	Complex numbers include real numbers (20)	Real numbers include all sets of numbers except complex, while complex numbers include all sets of numbers.
Incorrect inclusion	Real numbers include complex numbers (13)	The real numbers include everything, including complex numbers.

Table 3 (Continued). PTs' opinions on the relationship between real numbers and complex numbers

Categories	Codes (f)	Example expressions
Disjoint set	The intersection of two sets is the empty set (12)	 They are two disjoint sets.
No relationship	There is no relationship between the two sets (5)	 No relationship.
Real value	Real numbers are real, complex numbers are imaginary (4)	 The set of real numbers can be a real number. The set of complex numbers is not a real number but a set of imaginary (complex) numbers.
Root finding	If we cannot find a real root, we look for a complex root (2)	 If we cannot find a solution set in real numbers, if we cannot find a result for the expression in front of us, we solve in complex numbers.
Complement	One is what the other is not (1)	 One of these sets is the absence of the other.
No response	No response (2)	

When **Table 3** is analyzed, it is revealed that most of the PTs do not have a correct understanding of the set relationship between the set of real numbers and the set of complex numbers. 20 PTs were able to express that complex numbers include real numbers. They were able to reflect this understanding in their drawings. On the contrary, 13 PTs stated that real numbers include the set of complex numbers. 12 PTs stated that the set of real numbers and the set of complex numbers are two disjoint sets. They reflected this understanding in their drawings and drew the relationship between real numbers and complex numbers incorrectly. Four PTs stated that real numbers have real value and the value of a complex number is imaginary. Two PTs stated that if the real root is not found, the complex root is looked for. A PT also claimed that two sets are complements of each other. Almost all of these PTs could not draw the relationship between the set of real numbers and the set of complex numbers correctly. Only one PT in the real value category drew the relationship between real numbers and complex numbers correctly. Five PTs claimed that there was no relationship between the two sets, while two PTs did not express an opinion. These PTs could not make a drawing showing the relationship between real numbers and complex numbers in their drawings. The fact that the opinions of the PTs and their drawings were parallel to each other revealed that the deficiencies observed in the drawings were misconceptions beyond errors. Therefore, it can be said that most of the internal representations that PTs have about irrational numbers, rational numbers, real numbers, complex number sets and the relations between them are candidates for misconceptions. It can be said that most of the existing internal representations cannot provide a source for creating appropriate external visual representations.

Preparing a Lesson Plan

It has been revealed that the source of the errors in the external visual representations made by the PTs for number sets with the Venn diagram may be due to the errors seen in the internal representations. Therefore, it has been concluded that the training to be given should be aimed at overcoming the misconceptions in the internal representations. As a result of the literature review, it has been thought that it would be useful to use error-based activities and argumentation-based learning (EBAA) to overcome the errors seen in the visual representations. It has been thought that it would be useful to hold class discussions focusing on the relationships between sets before the activities to help the PTs develop appropriate internal representations. It has been decided that the following steps will be included in the lesson plan for training. The training is planned to last two hours.

- Discussions on strengthening internal representation: Discussion of the discovery process of number sets in the context of needs (one hour)
- Training with EBAA for interactions between internal and external representations (one hour).

Training with the Lesson Plan

Considering that the problem had similar characteristics with the group it was posed to, it was decided to apply the lesson plan to the students of the mathematics department where the first author was teaching. First, SKF and SKGF were applied to 16 pre-service mathematics teachers. As a result of the application, it was revealed that none of them could draw correctly, made mistakes like **Figure 1**, and had similar views in **Tables 2** and **3**. Since six of them stated that they would not be able to attend the next lesson, the application was carried out with 10 pre-service mathematics teachers. Training started a week later. In one hour of the training, number sets were discussed with class in the context of meeting needs. "Why do you think people need counting numbers?", "Why are natural numbers needed?", "Why are integers a need?", "Are rational numbers needed when there are integers?", "How did irrational numbers come into being?", "How were real numbers formed?", "Why were complex numbers needed?". Number sets were discussed in relation to each other with questions like these. After class discussing, six of the different incorrect drawings made by the PTs were used in the teaching. The six incorrect drawings were drawn on the board and discussed in order. The discussions started with the drawing examination by a PT who was chosen as a volunteer for each incorrect drawing and continued. The researcher did not express an opinion on the correct understanding until the end of the application. The incorrect drawings used in the error-based activity are presented in **Figure 6**.

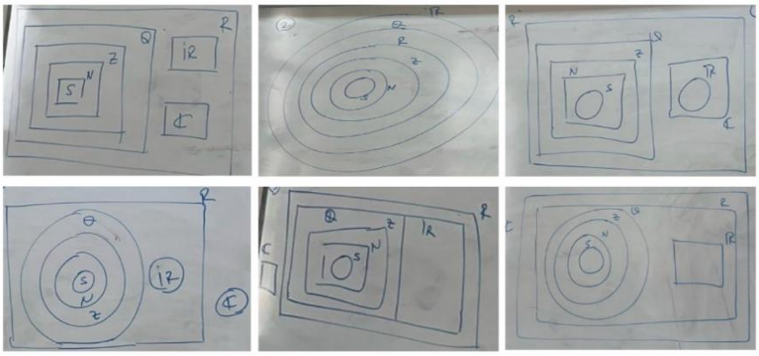


Figure 6. PTs' drawings used in the error-based activity (Source: Field study)

In the classroom, the drawings were analyzed by different PT. A discussion environment was created in the classroom about whether the drawings were erroneous or not, what the errors were, if any, and how the correct drawing should be made. In the first five examples, the focus was on finding their own errors. No information was given by the researcher, only the PTs were allowed to discuss among themselves. In the last example, a consensus was reached on what the correct drawing should be, considering the information obtained from the previous discussions. As an example, the discussion that took place in the classroom for the first and sixth drawing is presented below.

Researcher: Can you evaluate whether the drawing made in the first figure is correct or not?

PT1: He drew correctly until the rational number. What did he do here? ... He drew it as a different set outside the rational number...

Researcher: Do you think it is drawn correctly?

PT1: ... (there are murmurs in the class that it is drawn correctly, but no one wants to take the floor).

Researcher: What kind of numbers do you think are in this region (pointing to the region between rational numbers and irrational numbers)?

PT1: These gaps are both outside the rational number and inside the real number outside the irrational number. That is, real numbers that are not rational and irrational.

Researcher: What kind of numbers are these numbers?

PT1: ...There are no such numbers. Then this place is drawn incorrectly (The class voices out loud that it is wrong). Rational numbers and irrational numbers should form a whole and there should be no gaps.

Researcher: Can you evaluate whether the drawing in the sixth figure is correct or not?

PT2: Since the sets include each other, it is correct up to rational numbers. Complex numbers include real numbers. This is also correct. There is only a problem with irrational numbers.

Researcher: How should the set of irrational numbers be?

PT2: He should not specify it as a different set. Because we call irrational numbers the complement of rational numbers, this is all of the hatched region (he is scanning the relevant region on the board). So everywhere outside the set of rational numbers.

Researcher: Then how can a correction be made here?

PT3: Here, it would be enough to remove the set shape and just specify the name of the hatched region (He draws the shape that should be on the board) (see **Figure 7**)

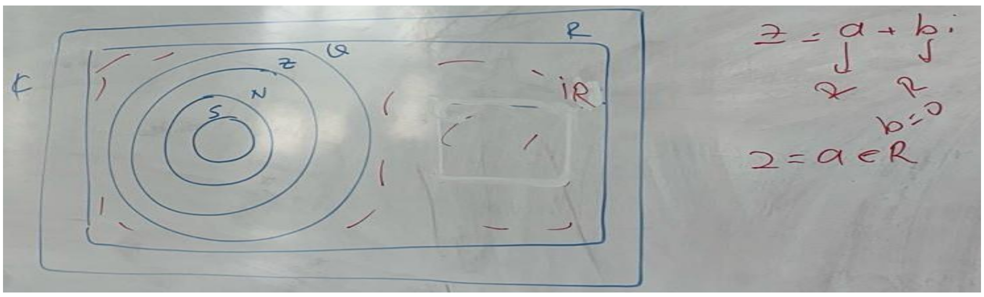


Figure 7. PT3's drawing (Source: Field study)

PT4: Teacher, why do complex numbers include real numbers? Can you tell me that?

Researcher: What are complex numbers?

PT4: Numbers written as $z = a + bi$.

Researcher: [The researcher writes the statements made by Ö4 on the board to clearly relate the algebraic representation to the visual representation]. What are the things we call *a* and *b* here?

PT4: *a* and *b* are real numbers.

Researcher: Can *a* and *b* take the value zero?

PT4: Since zero is a real number, they can. If *b* is equal to zero, then the number will only be $z = a$. So, it becomes a real number.

Researcher: Then what can be said about the values that complex numbers can take?

PT4: Then complex numbers can take both real number values and their own values. In that case, real numbers are also included in complex numbers, they are included.

When the argument structure of the first discussion was examined, it was seen that the class tended to accept that the drawing was correct by using their existing internal representations regarding the rational-irrational number relationship. The researcher added the rebuttal component to the argument with the question "What kind of numbers do you think are in this region?". A new argument was produced that resulted in the drawing being incorrect by evaluating the refuting component. As a warrant of this argument, the numbers in the gap in the drawing were given as "They are real numbers that are not rational and irrational". The internal representations of union and complement were expressed as baking for this warrant. It was concluded that no number could be in the gap in the drawing and it was convinced that the drawing was incorrect. In the second discussion, while determining the location of the set of irrational numbers, it was used as a warrant that irrational numbers are the complements of rational numbers with respect to real numbers. It was concluded that the drawing was incorrect and the correct drawing was made. Similarly, another preservice teachers made a valid reasoning as a result of algebraic arguments regarding why complex numbers include real numbers. The correct inclusion internal representation was reached.

Evaluation of the Training

Although the feedbacks were received that the PTs reached the correct understanding during the training, it was wondered whether the training reduced the student errors in the drawings concretely and what kind of changes occurred in the internal representations. For this reason, two weeks after the training, the PTs were asked to draw the number sets again and to give written opinions about the number sets. Firstly, the results of the NSF applied as pre-test and post-test were analysed. The errors in the drawings of PT before and after the training presented in **Table 4**, respectively.

Table 4. Changes in the drawings of PTs

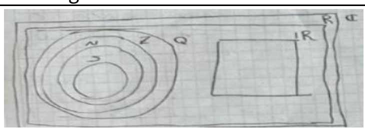
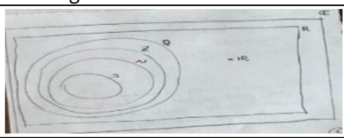
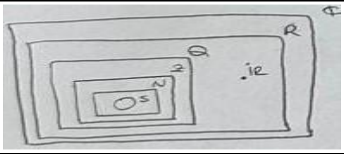
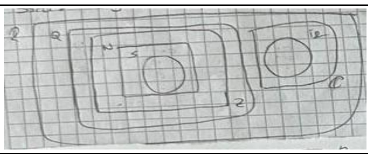
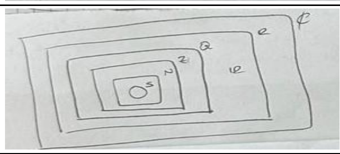
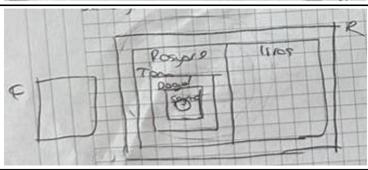
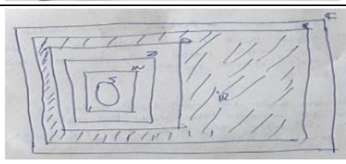
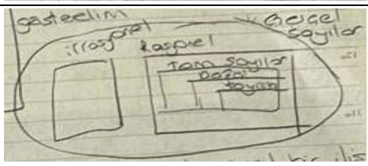
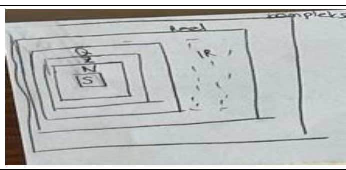
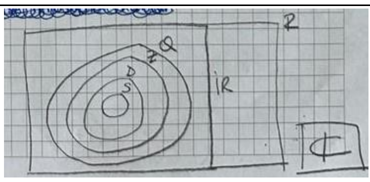
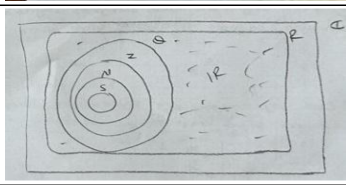
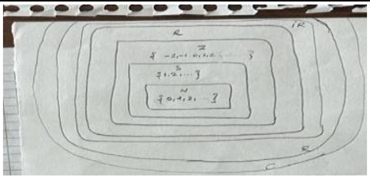
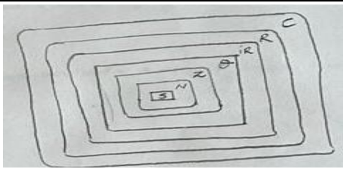
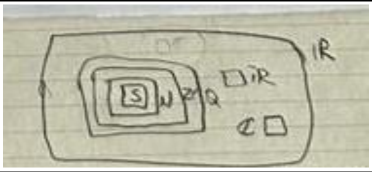
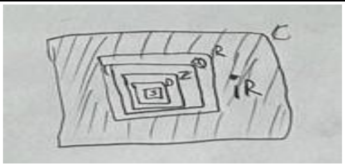
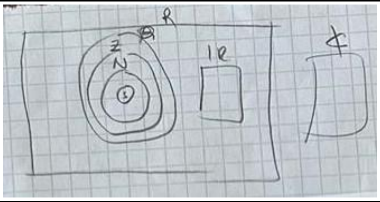
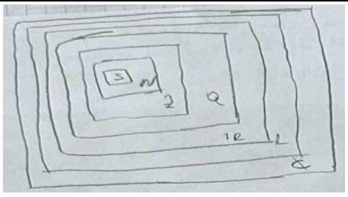
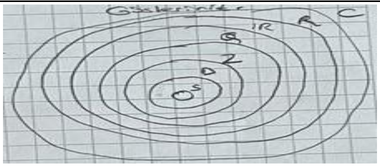
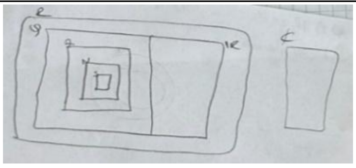
Before instruction		**After instruction**	
Errors	Drawings	Drawings	Errors
E1			-
E2, E7, E10, E14			-
E1, E9, E12			-
E8, E28			-
E1, E10			-
E2, E8			-

Table 4. Changes in the drawings of PTs

Before instruction		After instruction	
Errors	Drawings	Drawings	Errors
E2			E2
E1, E9			E29, E30
E1, E8			E2
E2			E8, E28

When **Table 4** was analysed, it was revealed that six out of 10 PTs were able to correct their misconceptions completely. From this point of view, it can be said that most of the PTs got rid of their misconceptions at the end of the teaching and had the correct concept image. Only one of the remaining PTs (MS7) continued to have misconceptions. The errors of three PTs were not observed but different errors were determined. Unlike the errors determined in **Table 1**, three new error types were determined. These are E28: There are real numbers other than irrational and rational numbers, E29: The complement of real numbers is irrational numbers, E30: Complex numbers are the union of real numbers and irrational numbers. In order to determine the change in the PTs' views on the relationship between the irrational-rational number set and the real-complex number set, the NSOF data implemented as pre-test and post-test were analyzed. The data of the analysis are presented in **Table 5**.

Table 5. The changes in the opinions of PTs

Before irrational&rational	After irrational&rational	Before real&complex	After real&complex
Ratio	Ratio	Correct inclusion	Correct inclusion
Ratio	Ratio	Equal	Correct inclusion
Ratio	Ratio-union	Root	Correct inclusion
Ratio-exact value	Complement-union	Correct inclusion	Correct inclusion
Ratio	İntersection-union-complement	Incorrect inclusion	Correct inclusion
Inclusion-ratio	Complement-union	Incorrect inclusion	Correct inclusion
Ratio-exact value	Inclusion	Correct inclusion	Correct inclusion
Intersection	Intersection-union (complex)	Incorrect inclusion	Correct inclusion
Exact value	Inclusion	No response	Correct inclusion
Ratio-exact value	Union-complement	Correct inclusion	Disjoint set

According to **Table 5**, two PTs matched the relationship between irrational numbers and rational numbers with the concept of ratio, and this idea did not change after the training. Four PTs (PT4, PT5, PT6, PT10) adopted the views of union and complement, which could be the correct understanding for drawing. The concept of complement, which had never been detected before, started to take place in the opinions of five PTs. Accordingly, it can be said that the training was useful in terms of creating a correct conceptual image and internal representations for number sets. For example, while PT6 thought that irrational numbers included rational numbers before, he reached the ideal understanding of union-complement after the training. Despite this, three PTs (PT7, PT8, PT9) could not develop a correct understanding as a result of the education applied parallel to their drawings in **Table 4**. It was determined that nine PTs demonstrated a correct understanding of the relationship between the set of real numbers and the set of complex numbers after the training. Only PT10 replaced his correct understanding with an incorrect understanding.

DISCUSSION

In the first phase of the action research, systematic efforts were undertaken to identify and clarify the nature of the problem. The findings revealed that PTs experienced considerable difficulty in producing appropriate Venn-diagram representations of number sets, committing a total of 30 distinct error types. In addition, a wide range of internal representations related to rational–irrational and real–complex number relationships were identified, accounting for a substantial proportion of these errors. The presence of such diverse representations within a mathematically well-defined domain supports the view that visual representations are shaped by individual characteristics and learning experiences (Alshwaikh, 2008).

A review of the literature indicates that relatively few studies provide detailed error analyses focusing specifically on number sets or their visual representations. From this perspective, the error typology developed in the present study constitutes a meaningful contribution to the field. At the same time, the comprehensive error list incorporates several error types previously reported in the literature, thereby situating the findings within existing research while extending it (Keçeli & Turanlı, 2013; Kidron, 2018; Nordlander & Nordlander, 2012). Most identified errors were concentrated in rational–irrational and real–complex number relationships, suggesting that PTs either lacked a sound conceptual understanding of these relationships or relied primarily on surface-level knowledge. Internal representations such as ratio, definite value, root, and real value appeared to be grounded largely in procedural experiences rather than conceptual reasoning. The strong alignment between PTs' verbal explanations and their drawings indicates that these errors were systematic rather than incidental, pointing to the presence of persistent misconceptions. In particular, misunderstandings related to irrational and complex numbers were especially prominent, a finding consistent with prior research documenting similar difficulties (Chin & Jiew, 2020; Fischbein et al., 1995; Keçeli & Turanlı, 2013).

When the identified difficulties are examined through the lens of set concepts, it becomes evident that some errors closely align with well-established findings in mathematics education research, others have been addressed only indirectly, and several are articulated explicitly for the first time in the present study. The analysis shows that a large proportion of errors stem from misinterpretations of hierarchical inclusion, complementarity, and disjointness among number sets. Errors such as treating rational and irrational numbers as disjoint sets (E1), constructing incorrect containment relations (E2, E6, E7), defining irrational numbers as the complement of real numbers (E3, E29), omitting certain number sets entirely (E5, E10, E17, E21, E22), and misrepresenting the real–complex number relationship (E8, E9, E11) closely correspond to earlier research emphasizing fragmented concept images and reversed inclusion reasoning (Fischbein et al., 1985; Tall & Vinner, 1981; Zazkis, 2005). Likewise, errors involving natural, counting, and integer sets (E19, E20, E23, E24, E26) reflect well-documented challenges in constructing coherent hierarchical structures within basic number systems (Nunes et al., 2007; Vlassis, 2004).

Beyond these established patterns, several errors partially overlap with previous research but are articulated more explicitly and systematically in this study through detailed analysis of visual representations. These include representing irrational numbers as containing complex numbers (E4), assuming that complex numbers consist solely of irrational real numbers (E12), excluding rational numbers from the natural–integer hierarchy (E16), interpreting natural and integer sets as intersecting rather than inclusive (E25), and assuming the existence of real numbers beyond rational and irrational numbers (E28). Although related conceptual ambiguities have been noted in earlier studies (Arcavi & Hadas, 2000; Ni & Zhou, 2005), such specific representational configurations have rarely been documented with comparable clarity. Finally, a small number of errors, such as viewing natural numbers as encompassing complex numbers (E13) or defining complex numbers as the union of real and irrational numbers (E30), appear to be largely novel. These errors reflect not only reversed hierarchical reasoning but also fundamental misapplications of set operations, particularly union and complement, indicating deep structural distortions in PTs' understanding of number systems.

By conceptualizing verbal explanations as internal representations and drawings as external representations, the findings demonstrate that internal representations exert a strong influence on external ones. This result supports the view that external representations serve as manifestations of internal conceptual structures (Janvier et al., 1993; Pape & Tchoshanov, 2001). Consequently, the development of accurate external visual representations requires prior attention to internal conceptual understanding.

The observed difficulties may be attributed to PTs' prior mathematics learning experiences, which often privilege algorithmic proficiency over conceptual understanding (Mamona-Downs, 2001). In particular, the concept of complement, central to understanding the rational–irrational relationship, is rarely emphasized in instruction, a finding consistent with earlier research (Doruk & Çiltaş, 2020). Another contributing factor may be the limited instructional emphasis on visual representations, as algebraic forms typically dominate mathematics teaching (Noss et al., 1997). Although visual representations may play a less prominent role for mathematicians, they serve a crucial pedagogical function in mathematics education (Borba & Villarreal, 2005). Visual representations support conceptual understanding, foster mathematical exploration, and help overcome learning difficulties (Borba & Villarreal, 2005; Zimmerman & Cunningham, 1991). Accordingly, the systematic integration of visual representations into mathematics instruction across all educational levels is strongly recommended (Gutiérrez, 1996).

Following the instructional intervention, most PTs did not repeat their earlier errors, indicating that the intervention was effective in supporting appropriate external visual representations. In addition, clear improvements were observed in internal representations. For instance, the previously neglected concept of complement was incorporated into representations of the rational–irrational relationship, and definition-based or ratio-based representations were transformed into mathematically accurate visualizations. Similar progress was observed in representations of the real–complex number relationship. These findings align with prior research highlighting the effectiveness of error-based activities, argumentation-based learning, and the explicit construction of conceptual connections (Duran et al., 2017; Gedik et al., 2017; Vale et al., 2011). Nevertheless, a small number of PTs continued to exhibit errors or developed new misconceptions, underscoring the resistance of deeply entrenched conceptions to conceptual change. This outcome is consistent with research showing that learners with stable misconceptions may resist change even after targeted instruction (Barnett & Ceci, 2002; Chen et al., 2020; Chi, 2005; Oliver, 2011).

Overall, the instructional process positively shaped PTs' representational practices. Classroom discussions encouraged connections among internal representations, while error-based activities prompted PTs to evaluate and restructure their external representations through cognitive conflict. These findings reinforce the view that internal and external representations interact dynamically (Goldin & Kaput, 1996) and that conceptual understanding develops through their meaningful coordination (Hiebert & Carpenter, 1992; Janvier et al., 1993). The model presented in **Figure 8** illustrates how argumentation-based learning supports transitions from internal to external representations, whereas error-based activities facilitate the reverse process.

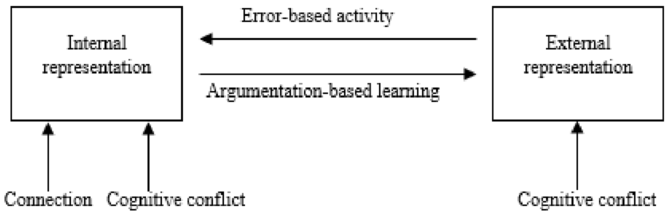


Figure 8. The effects of EBAA on internal and external representations (Source: Authors' own elaboration)

CONCLUSION AND RECOMMENDATIONS

This study investigated PTs' external visual representations of number sets through the use of Venn diagrams. The findings identified 30 distinct error types and a variety of conceptual images, some of which reflected misconceptions, particularly with respect to rational–irrational and real–complex number relationships. These findings point to substantial conceptual difficulties in a core and foundational area of mathematics.

Overall, the results indicate that PTs' errors in visual representations of number sets are systematic rather than random, reflecting relatively stable conceptual structures that range from well-documented misconceptions to partially developed and novel forms of reasoning. While the majority of errors align with existing literature on disjoint-set reasoning, reversed inclusion, and fragmented concept images, the identification of hybrid and operation-based misconceptions, especially those involving complement and union relationships, extends current knowledge of how learners internally organize number systems. By categorizing error types as overlapping with prior research, partially overlapping, or original, this study offers a more refined diagnostic framework that can inform both future research and instructional design in teacher education.

The findings further demonstrate that the instructional intervention played a significant role in supporting PTs' conceptual development related to number sets and their interrelationships. The reduction in recurring errors, together with improvements in both external visual representations and internal conceptual structures, suggests that targeted instructional practices can effectively address persistent misconceptions. In particular, the explicit integration of the complement concept and the shift from definition-based or ratio-based representations to mathematically accurate visualizations indicate deep conceptual change rather than superficial correction. Consistent with prior research, these results underscore the effectiveness of instructional approaches that view errors as learning opportunities, promote argumentation, and emphasize meaningful conceptual connections.

Finally, the findings suggest that the instructional intervention not only improved representational accuracy but also contributed to a deeper understanding of the dynamic relationship between internal and external visual representations. The model presented in **Figure 23** provides a novel explanatory framework by explicitly depicting bidirectional transitions between internal and external representations. Unlike earlier models that describe this interaction in general terms, the proposed model clarifies that argumentation-based learning primarily supports the externalization of internal representations, whereas error-based activities facilitate the internal reconstruction of meaning from external representations. By integrating these processes within a single instructional cycle, the model extends existing theoretical perspectives.

The findings of this study are limited by the research design, participant group, and data collection instruments employed. Future research could replicate and extend this work using diverse methodologies, participant populations, and data sources. In addition, the instructional intervention may be applied to other mathematical topics to examine its broader applicability and effectiveness. The identified error types could also be transformed into a diagnostic assessment tool and administered to larger samples to improve generalizability. Finally, sharing these findings with educators may contribute to more effective mathematics instruction by highlighting the systematic use of visual representations, error-based learning activities, and argumentation-based instructional practices.

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AI statement: The authors stated that no generative artificial intelligence or AI-based tools were used in the writing, editing, or data analysis of this article. All analyses, interpretations, and text writing were performed entirely by the authors.

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