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The didactical phenomenology in learning the circle equation

Clement Ayarebilla Ali [1]*

[1] Department of Basic Education, Faculty of Educational Studies, Univerisity of Education, Winneba, Kumasi, GHANA
***Corresponding Author:** ayarebilla@yahoo.com

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ABSTRACT

Realistic mathematics education (RME) has proven to be an effective model for mathematics elsewhere. However, students and teachers still grapple to confront the circle equation to learn and teachers to teach. This study explored students' didactical phenomenological discourses in the circle equation as an alternative. The mixed methods research design was used to collect both quantitative and qualitative data from 50 senior high school students purposely selected from one senior high school. The instruments of data collection were the questionnaire and interview guide. The purpose of the questionnaire and interview guide was to validate and corroborate. The quantitative analyses contain categorical independent and continuous dependent variables and were explored by reliability statistics, simple and multiple analysis of variance tests of independence. On the other hand, the qualitative transcriptions of students' own perceptions. The results on the types of equations showed low didactical phenomenology as many variables fell below the .20 minimum internal consistency's criteria. However, the results of the tasks showed high acceptable didactical phenomenology. We therefore concluded that RME could be extended to other domains of mathematics. Thereafter, comprehensive recommendations were advanced for theory, method, research, practice, policy, and context.

Keywords: circle equation, didactical phenomenology, horizontal mathematization, realistic mathematics, vertical mathematization

INTRODUCTION

The problem of teaching and learning school mathematics with realistic classroom instruction cannot be overemphasized. Realistic mathematics education (RME) is a domain-specific instruction theory in which rich realistic situations are given prominent positions in the teaching and learning processes and the situations serve as sources the development of mathematical concepts, tasks, procedures, and contexts (Fauzan, 2002). Thus, students are offered a variety of problem situations to imagine and make something real from the real world, fantasy world of fairy tales or formal world of mathematics, as long as the problems are experientially real in the students' minds (Da, 2022; Drijvers, 2022; Hasbi et al., 2019; Matusov, 2011; Van den Heuvel-Panhuizen & Drijvers, 2011, 2020; Yilmaz, 2020). Also, RME is a socio-constructivist compatible and complementary collaboration, which points to the critical roles of the classroom culture in constructing and reconstructing mathematics tasks, model puzzles and didactical phenomenology. Thus, the pedagogical content tools of emergent modelling and design are applied in the classroom instructions. However, such interactions rarely exist in the teaching and learning of equations of the circle (Ali, 2019; Ali & Agyei, 2016).

In addition, teachers, students, and researchers in mathematics education barely conceive the concepts of horizontal and vertical mathematisations. Suffice us to explain that horizontal and vertical mathematizations explain the differences between transforming general social problems into mathematical problems and processing within the mathematical systems. Horizontal components relate to transferring and transforming real world problems to mathematically stated problems. This means the world around us abounds in mathematics concepts. The ability of the teacher and the student to conceptualise the world problems and transform them into mathematics is called horizontal mathematics. But vertical components relate to the mathematical processing and refurbishing of the real-world problems transformed into mathematics problems. In other words, horizontal mathematization leads from the world of life to the world of symbols, where one which lives also acts (and suffers), whereas in the vertical, symbols are shaped, reshaped and manipulated, mechanically, comprehendingly, and reflectively (Roth, 2016). Some activities involving strong horizontal components are identifying specific mathematics in contexts, discovering relations, transferring real world problems to mathematical problems, and transferring real world problems to known mathematical models. And activities involving strong vertical components are representing relations in formula, providing variety of models, and integrating the models to formulate new mathematical concepts (Da, 2022; Larsen, 2018; Rodriguez & Fernandez, 2017; Van den Heuvel-Panhuizen et al., 2016; Yilmaz, 2020).

Didactical Phenomenology

Despite the fact that studies (Da, 2022; Hausberger, 2020; Rodriguez & Fernandez, 2017; Van den Heuvel-Panhuizen & Drijvers, 2020; Viirman, 2014; Yilmaz, 2020) have discussed phenomenology of mathematical concepts, structures and ideas in relations to the phenomena for which they were created and extended in the teaching and learning processes situations, the teaching and learning of mathematics in general and that of equations of the circle in particular still suffer wanton setbacks. Teachers and students are unable to localize classrooms and interactions appear too parallel to realism of local scenarios. However, didactical phenomenology, if well conceived, shows teachers the places students can explore to successfully tackle the mathematics tasks. In the long run, the alternative aim of teaching and learning mathematics to benefit the society and the individual could be fruitfully attained.

Secondly, even though didactical phenomenology is grounded in a phenomenology (i.e., mathematical concepts, structures, and ideas) of mathematics, focuses on the relations between mathematical thought things (nooumenon) and didactical phenomenon (Hasbi et al., 2019; Van den Heuvel-Panhuizen & Drijvers, 2011; Van den Heuvel-Panhuizen et al., 2016), the processes of describing the phenomena (nooumenon), organizing, and creating phenomena still remain a mirage for many teachers and students. This breeds incoherent connections and poor interactions among the classroom milieus. Ultimately, mathematics tasks fail to connect teachers and students to real-life situations. This does not enable students to reflect on future practices as a result of indemonstrable usefulness to daily lives.

Thirdly, even though didactical phenomenology is the study of relations between the phenomena that mathematical concepts represent and the concepts themselves (Al-Jupri & Turmudi, 2009), teachers fail to interpret and suggest ways of identifying possible instructional activities to support individual activities and whole-class discussions. There is no gain saying that this does not allow students to engage in progressive mathematization and create congenial classroom environments for collectively renegotiations (Arnold, 2012). However, didactical phenomenology that focuses on the mathematical concepts, procedures and tools could help students to mathematize the phenomena, devise ways of solving practical problems, and analyze everyday applications (Drijvers, 2022; Gravemeijer, 2008).

Again, in the original conception of didactical phenomenology (Freudenthal, 1983), the descriptions of mathematical concepts, structures, and ideas were being created within the classroom, while considering students' learning processes outside the classroom. This ensured that teachers and students never landed themselves into unscientifically structured curricula, readymade anti-didactic mathematics, and dormant participation. This really made mathematics a human activity, reality activity and mathematising constructs. Such practices were categorized into procedural, conceptual, and dual textbooks' activities (Van den Heuvel-Panhuizen et al., 2016). However, these gains have been eroded over years with the procedural activities (mechanistic) being glued to memorization of mathematical facts, the conceptual activities being restricted to conceptual understanding of mental calculations, and the dual practically missing in the equations. Hence, students fail to design their own tasks, develop methods and work on their own paces (Davis & Chaiklin, 2015). The least talk about technology tools in the context of cultural interactions in the classroom the better (Ali & Davis, 2016; Drijvers, 2022; Loc & Tien, 2020; Rodriguez & Fernandez, 2017). Even though didactical phenomenology has huge potentials of employing sophisticated digital technology tools to develop new techniques for teaching and learning, it has virtually been narrowed down interactions with conceptual understanding structures only to the detrimental of enhancing new didactical situations in mathematizations. This has not provided enough alternatives to mechanistic teaching approaches, and teaching and learning still remains at where mathematical content is well structured and embossed with rigid procedures (Van den Heuvel-Panhuizen et al., 2016). Therefore, riding on the back of this powerful RME model, the following questions were promulgated for the study:

1. How do students interact in the types of circle equations and tasks in the center and radius in the didactical phenomenology?
2. What statistical significances can students attain in the interactions using didactical phenomenology in for solving types of equations, centers, and radii of the circle
3. What are the likely covariates confronting students' interactions in the didactical phenomenology?

METHODOLOGY

The embedded mixed methods triangulation research design was used to address the research questions in which qualitative transcriptions were used to support the quantitative statistical significances (Creswell, 2014). In this design, applications of vertical and horizontal mathematizations to the basic principles of didactical phenomelogy first adopted quantitative instruments (i.e., questionnaire) and explored students' gender, class, and grade of school as independent variables to assess their influences on the five basic equations of the circle as dependent variables. Subsequently, the unstructured qualitative interview guides transcribed students' own didactical phenomenology experiences with the five circle equations (Harwell, 2011). The main purpose of embedding the qualitative data into the quantitative was to support and corroborate the statistical significances. The quantitative instruments were questionnaire, and the qualitative instruments were interview guides. In the questionnaire, the items were structured based on the various topics in equations of the circle. These were types of equations, and centers and radii of equations. Students were to determine their knowledge and skills in these three topics. Concurrently, the interview guides were structured according to the types of equations and the tasks involving solving for the center and radius of the circle equation. Here, students were given tasks sheets to describe types of circle equations, and centers and radii of the equation of the circle (Castro-Rodríguez et al., 2022; Creswell, 2016; Hasbi et al., 2019; Loc & Tien, 2020; Yilmaz, 2020).

Table 1. Reliability statistics in exemplified types of equations of the circle

Type	Standard	Radius	General	Diameter	Tangent	Normal
Standard	***.576***					
Radius	.092	***.868***				
General	.072	.181	***.738***			
Diameter	.109	.104	.197	***.622***		
Tangent	.132	.109	.169	.167	***.933***	
Normal	.148	.128	.108	.188	***.334***	***.908***

The purposive sampling procedure sampled 50 elective mathematics students in a senior high school and explored their didactical phenomenology through the questionnaire and interview. The sampling technique targeted only students who have pursued elective mathematics in their programme of study (Etikan et al., 2016). Even though the purpose sampling technique was based on elective mathematics, the individual students were chosen based on simple random technique of generating random numbers for the selections (Cohen et al., 2011). Consequently, two sets of instruments of data collection were used for the processes. The first set was questionnaire containing quantitative psychometric test. In the method, the questions were structured in multiple-choice of four items each. The answer was structured in such a way that one could select 'excellent answer' 'best answer', 'very answer', 'good answer', and 'average answer'. The main essence of the structured questionnaire was to restrict verbose responses and obtain cleanest data. The second instrument was an open unstructured interview guide. The items in the guide were open to students to express themselves about their experiences with the instruments of the questionnaire. The purpose of the interview guide was to validate and corroborate the quantitative data (Cohen et al., 2011).

Obviously, two stages were explored in the analysis of the data. The first stage was the quantitative data. The quantitative analysis contained the categorical independent and continuous dependent variables. The quantitative analysis explored both the simple reliability tests to compare the groups of students' didactical phenomenology. These reliabilities were pooled from inter-item correlations between items of the questionnaire (Caswell, 2011). The basic statistics were reliability statistics that refers to the extent to which a scale produces consistent results, if the measurements are repeated a number of times. Thus, if the association in reliability analysis was high with any of the tasks and equations of the circle, we concluded that the scale yielded consistent results in the didactical phenomenology. Although there are different types of reliability, the internal consistency concept was applied in this analysis. This measured the reliability of a summated scale in the tasks and equations. That is, if the scales reflected the construct they intended to measure, we could conclude that the didactical phenomenology helped conceive the tasks and equations of the circle. And if the reliability coefficient of each particular construct was more than 2, we could conclude that it was statistically significant (Pallant, 2011). On the other hand, the qualitative transcriptions of students' own experiences were the verbal corroborations of the quantitative independence tests (Castro-Rodríguez et al., 2022; Creswell, 2014).

Although validity was used a statistic by itself, we also took into consideration the cardinal tenet of validity of the study. Validity, the extent to which concept were accurately measured, satisfied the content validity, where the instrument adequately covered all the content in equations of the circle at the high school level. The instrument addressed construct validity, where we could sufficiently draw inferences about tests related to concepts of equations of circle, and equally addressed criterion validity, where we believed that any other instrument used could arrive at the same variable (Chan & Idris, 2017; Heale & Twycross, 2015).

The research satisfied the seven ethical requirements advanced for a combined quantitative and qualitative research. These are social value (i.e., contribution of the expected results provided knowledge to the teaching and learning of equations of the circle), the scientific validity in which the research followed every single step of a scientific, equitable selection of participants ensured justice and equity, favourable risk-benefit ratio ensure minimal risk in relation to the benefits, independent assessment by head and mathematics teachers of the school, informed consent clearly, and respect of research participants to safeguard their confidentiality. The coding of the data helped maintained all the ethical issues (Pérez et al., 2017).

Research question 1: How do students interact in the types of equations and tasks in the didactical phenomenology?

RESULTS

Table 1 presents the results of the six dependent types of equations of the circle exemplified. These equations are standard, radius, general, diameter, tangent, and normal. It was discovered that the Cronbach's alpha based on standardized items was only 0.590 and statistically significantly different [$F(5, 495)=43.358$, $p=0.000$]. However, the mean inter-item correlations revealed that most variables fell below the 0.20 minimum internal consistency's criteria. This suggested that there were minimal interactions in the equations of the circle. This outcome is not surprising in the research area as most students only encounter learning of the equations of the circle by the general equation only. However, with introduction of the didactical phenomenology, it is expected that the status quo will change.

Table 2 presents the results of the ten dependent exemplified tasks in equations of the circle used for the study. Every equation contains a type of center. While radii are uniquely positive and greater than one, circle centers vary from one equation to the other. The center can be the origin (0, 0) or any point (h, k). We expect that either both axes are positive or negative or positive. In **Table 2**, it was discovered that Cronbach's alpha based on standardized items was 0.744. In addition, the mean inter-item correlations revealed that most variables had between 0.213 and 0.585 internal consistency and were regarded internally consistent. This means that the didactical phenomenology accurately measured the constructs well, from which we could infer that there were statistically significant.

Table 2. Reliability statistics in exemplified tasks in equations of the circle

	Center (0, 0)	Center (h, 0)	Center (0, k)	Center (-h, -k)	Center (h, k)	Center (-h, -k)	Center (-h, -k)	Center (h, k)
Center (0, 0)	1.000							
Center (h, 0)	**.213**	1.000						
Center (0, k)	.188	**.371**	1.000					
Center (-h, -k)	.127	**.300**	**.368**	1.000				
Center (h, k)	.153	**.250**	**.244**	**.349**	1.000			
Center (-h, -k)	.148	**.264**	**.252**	**.303**	**.302**	1.000		
Center (-h, -k)	.146	**.214**	**.231**	**.227**	**.273**	**.402**	1.000	
Center (h, k)	.173	.168	.191	**.311**	**.331**	**.373**	**.585**	1.000

Pre-Interview Transcriptions in Mathematization and Didactical Phenomenology

The transcripts of mathematization and didactical phenomenology were based on the RME theory of mathematics education that offered both pedagogical and didactical philosophies on teaching and learning from the Netherlands tradition. There were two important connections of the dialogue to the reality of mathematics. One connection was the human activity as it moved mathematics very closed to the students' daily life situations, and the other connection was the experientially real mathematics problems and applications of the concepts as the students' progress through self-developed model. On the conceptual meaning of the sign, the students produced varied but similar contextual understanding. For instance, student 'A' said a sign is an identity given to an object, student 'B' explained the sign as something used to represent or replace a long sentence, and student 'C' advanced that it is letter that represents a constant. When they were prompted and probed to outline common signs and symbols, they used in solving the equations of the circle, student 'A' insinuated on the equal to, addition, squares and constants, and student 'B' followed a similar pattern but added equal to, minus, square root, and multiplication. The most interesting aspects of the conversations were on the sins and symbols they mostly preferred to use. Student 'A': *I like the addition sign most because it makes calculations of the general equation easier*. Student 'B': *I prefer x and y because they are mostly known by teachers, textbooks, and computers*.

Similarly, on the meaning of a tool in mathematics, student 'A' associated it to a device that helps in doing something without being drawn part of the process so that one obtains an obtain accurate measurements or values, and student 'B' likened a tool to an instrument used in measuring quantities or aids in calculations. Unlike signs and symbols, the commonest tools in the senior high school were the metre rule, protractor, pair of dividers, pair of compasses, set squares, calculator, eraser, and pencil. Despite the long list of tools, the students enumerated, they mostly used metre rule, pair of compass, protractor and pencil in measuring and solving equations of the circle. The following figure displays the number of students who comfortably mathematized the signs, symbols, artefacts, and tools in solving tasks in equations of the circle.

The students' dialogues in the interactions with the signs, symbols, artefacts, and tools compensated for the differences in the categorizations from the stages of definitions, organizations, and representations to the external or material means of psychological and cultural regulations, operations and behaviours mediated their cultural acts and thought processes in mathematizing these signs and symbols. As students progressed from signs, artefacts, and tools to instruments, they accounted for higher psychological, cultural roles and classroom functions of the physical and psychological. Mathematization and didactical phenomenology then helped them to remodel the whole content and structure of psychological operations and altered the ways they experienced the contextual phenomena. It was very clear that the signs, symbols, artefacts, and tools helped catalyzed students' psychological thoughts about socio-cultural roles and Vygotsky's genetic law of cultural developments on the social learning (Da, 2022; Hasbi et al., 2019; Radford & Sabena 2015).

Secondly, the dialogues showed that the relations between two or more mathematization were not inherent or natural but established through didactical phenomenology. In fact, the students related the sign-symbolic notations to the tools and instruments to compute for the center ad radius in the mathematical graphs and the physical phenomena they were exploring. This helped to link signs and symbols (Bishop et al., 2002; Loc & Tien, 2020).

Pre-Interview Transcripts in Exemplified Equations of the Circle

The transcripts in exemplified equations of the circle explored the students' algebraic and geometric knowledge in the general, center-radius, diameter, standard, standard tangent, and standard normal equations of the. This helped their knowledge, skills, understanding and utilization to extended in learning equations of the circle. Notably, almost all the students referred to both standard and general equations of the circle as $x^2+y^2+2gx+2fy+c=0$ even though they alluded to some similarities and differences in the centers and radii. In solving equations whose centers are $C(0, 0)$, $C(h, 0)$, $C(0, k)$, $C(-h, 0)$, $C(0, -k)$, $C(h, k)$, $C(-h, k)$, $C(h, -k)$, and some students came out with the approximate solutions while others were simply overwhelmed and perplexed with the most captivating and interesting interrelationships and interconnections of the solutions! For instance, given the equation $x^2+y^2+8x=0$, the center is $C(-4, 0)$ and the radius is four units. This was related to an equation $x^2+y^2+6y=0$, whose center is definitely $C(0, -3)$ and the radius is also three units, the equation $x^2+y^2+8x+6y=0$, whose center is $C(-4, -3)$ and the radius is five units, and the equation $x^2+y^2-8x-6y=0$, whose center is $C(4, 3)$ and the radius is also five units. By extension, the equations $3x^2+3y^2+36x+12y+84=0$ and $3x^2+3y^2-36x-12y+84=0$ had the centers $C(-6, -2)$ and $C(6, 2)$, respectively with the same radius $2\sqrt{3}$ units.

The active mathematization followed by the didactical phenomenology, it was shown that irrespective of a particular type of equation of the circle, the center and the radius remain the same. That is, while the center of the circle is affected by the sign and size of the coefficients of x and of y, the radius is always positive. This helped students to interact uniformly in solving the tasks in equations of the circle. However, the standard deviations between the groups were quite small but statistically significant as witnessed in the between-groups.

Pre-Interview Transcripts in Exemplified Tasks of the Circle

The structured pre-interview showed that students *scarcely* interacted with one another in assimilating the teacher's demonstrations, procedures, and algorithms. Even in worst cases, students *never* provided for each other any routine procedures with the variety of signs, symbols, artefacts, tools, instruments and computers or calculators or other technologies for the teaching and learning of equations of the circle. The students openly expressed these views with the statements:

> "Sometimes if we do not know the topics, we consult our friends to find the information. Some teachers we contact sometimes intimidate and scare us. And if we do not question them well the teachers quickly tell us to do research on the issues concerned" (Student 'B').

Again, the students *scarcely* interacted with their teachers in creating open, informal, congenial, democratic, and free atmospheres, where sources of information, materials and resources could have been provided. Worst still, teachers *never* monitored each student's participation and progress in order to remediate and give immediate feedbacks through formal and informal means. A student categorically commented that *it is not always all the time that the teacher addresses our problems. Scarcely do we see the teachers to help us solve our problems.* Even many students *scarcely* interacted with mathematics content in establishing and maintaining logical and unified themes during teaching and learning. If they had this opportunity, they would have transferred the techniques, methods, and procedures in the mathematics content in orderly, sequentially, and logically arranged themes and topics with the signs, symbols, artefacts, tools, instruments and technologies within their environments. Even in the classroom, the students *never* suggested diverse solution paths to the mathematics tasks and worked out examples to the teachers, apart from what was already contained in the textbooks and the teachers' planned assessment and evaluation procedures. A frustrated student emphatically insinuated that the textbooks were not in the order in which the syllabi have been structured. The topics have not been orderly arranged and many contents have scarce explanations even with the difficult language, and where some textbooks have been orderly arranged, they did not understand the contents.

The story was not significantly different with the interactions with technologies. Most students strongly advanced that they *scarcely* interact with the technologies enshrined in signs, symbols, artefacts, tools, and instruments. Even the scarce technologies they interacted with *never* related the signs, symbols, artefacts, tools, and instruments to equations of the circle. And this did not ensure smooth, logical, and coherent transference of knowledge from one task to another. They never challenged and provided clues, innuendos and technical supports to the students as portrayed in the open statements. Even the scientific calculators and graph sheets ware not standard and did not usually correspond to the solutions they obtained in the tasks. The signs and symbols were not transferable due to lack of linkages.

Research question 2: What statistical significances have students attained in the interactions using didactical phenomenology in equations and tasks of the circle and what are the likely covariates should students during the interactions?

Results of Mathematization

The utilization of mathematization provided new mathematical concepts, structures, and ideas in relation to the phenomena and contexts in equations of the circle. The reality of didactical phenomenology placed students strategically to apply, explain and relate mathematics knowledge with signs and symbols. The students described the authentic settings of the mathematics tasks and established connections between reality and mathematics, and created possibilities to further construct mathematical new realities, situations, and contexts. This explained the statistically significance differences [$F(4, 466)=2.555$, $p=0.038$, partial eta squared=0.021] in the ways students presented and performed in the mathematization and didactical phenomenology. Even though the interaction effects were not statistically significant, gender and general equation as covariates statistically reduced the nonsignificant differences. The estimated marginal means supported the claims that gender and general equation did control and statistically removed the effects from the mathematization and didactical phenomenology.

Freudenthal's mathematization activities (Loc & Tien, 2020; Van den Heuvel-Panhuizen et al., 2016; Yilmaz, 2020) were utilized in the ten components in transferring the tasks to mathematically stated problems (horizontal), and processing and refurbishing the real-world tasks that have been transformed into mathematics (vertical). This really engendered the students to translate and transform the horizontally mathematized tasks from the real-world situations to the world of signs and symbols. In effect, it mechanically shaped, reshaped and manipulated the equations of the circle mathematics through comprehensive vertical mathematization. The convergence of the horizontal and vertical mathematizations broadened specific mathematics to general contexts of the signs, symbols, artefacts, and tools. For instance, student discovered that *a sign is an identity given to an object*, *it is something used to represent or replace a long sentence, and it is a letter that represents a constant*. The mathematizations visualized the general contexts by discovering and recognizing relationships and regularities in the exemplified equations of the circle. For instance, in outlining signs and symbols commonly explored to solve tasks in equations of the circle, the students discovered that *equal to*, *addition*, *squares* and *constants*, *square root* and *multiplication* were mostly preferred. Again, the *x* and *y* variables commonly represented the equations of the circle. In fact, the students' mathematizing activities in strong vertical components represented and connected these signs and symbols in providing the solutions to the tasks.

Also, students encountered the four different approaches, namely mechanistic, structuralist, empiricist and realistic encountered in mathematization during the processes of horizontal and vertical mathematization. For instance, the mechanistic provided systems of rules for students to verify and apply. The structuralist phenomenology helped students to provide organized, closed deductive systems and stressed vertical mathematization in the metre rule, protractor, divider, pair of compasses, set square, calculator, eraser, and pencil. The empiricist phenomenology helped students to assemble the actual three classroom tools, namely metre rule, calculator, and pair of compasses. Students preferred these three instruments for the reasons that there were easy to use, well known and mostly acquired by students, parents/guardians, schools, and municipalities. The empiricist component helped students to refer to texts and authorities for problem formulation and solution direction. The realistic component fully incorporated both vertical and horizontal mathematizations in solving the tasks (Hausberger, 2020; Van den Heuvel-Panhuizen & Drijvers, 2011).

Results of Didactical phenomenology

The one-way between-groups ANCOVA conducted in mathematization and didactical phenomenology (signs and symbols) showed that after adjusting for gender differences, there were significant differences in mathematics content and technology interactions [F(3, 488)=3.494, p=0.016, partial eta squared=0.021; F(3, 487)=4.853, p=0.040, partial eta squared=0.017], and the interactions between mathematic content and gender, and technologies and gender [F(3, 488)=2.955, p=0.032, partial eta squared=0.018; F(3, 487)=2.714, p=0.044, partial eta squared=0.017] even though gender itself was not statistically significant F(1, 488)=0.793, p=0.374, partial eta squared=0.002]. This means that gender significantly reduced the differences in the dependent mathematization, and didactical phenomenology components as supported and explained by the partial eta squared values and the estimated marginal means.

Again, the two-way between-groups ANCOVA conducted showed after adjusting for gender and general equation of the circle, significant differences [F(4, 466)=2.555, p=0.038, partial eta squared=0.021] even though there were no statistical differences the main effects [F(18, 466)=0.735, p=0.775, partial eta squared=0.028], and the covariates, gender and general equation [F(1, 466)=2.733, p=0.099, partial eta squared=0.006; F(1, 466)=0.027, p=0.038, partial eta squared=0.000]. These two results shows that gender and general equation legitimately statistically reduced the differences as supported by the partial eta squared values and the estimated marginal means. The utilization of ANCOVA and MANCOVA tests helped to explain and relate mathematics knowledge in a variety of ways. The students' ability to interact with signs and symbols during teaching and learning opportunities and possibilities to further construct mathematical new concepts, situations and contexts cannot be overemphasized. This means students could employ new methods and strategies to solve equations of the circle.

Results of Samples Tasks in Equations of the Circle

In solving the tasks in equations of the circle, the students utilized much the concept of a circle as sets of points (called loci) that satisfying relationship $(x-h)^2+(y-k)^2=r^2$ to achieve the statistically significant differences (Horsman, 2018). There were significantly statistical differences in students in the exemplified equations of the circle and the interaction effects after statistically reducing, controlling, and removing the effects of covariates. This was supported by the transcriptions in **Figure 1**.

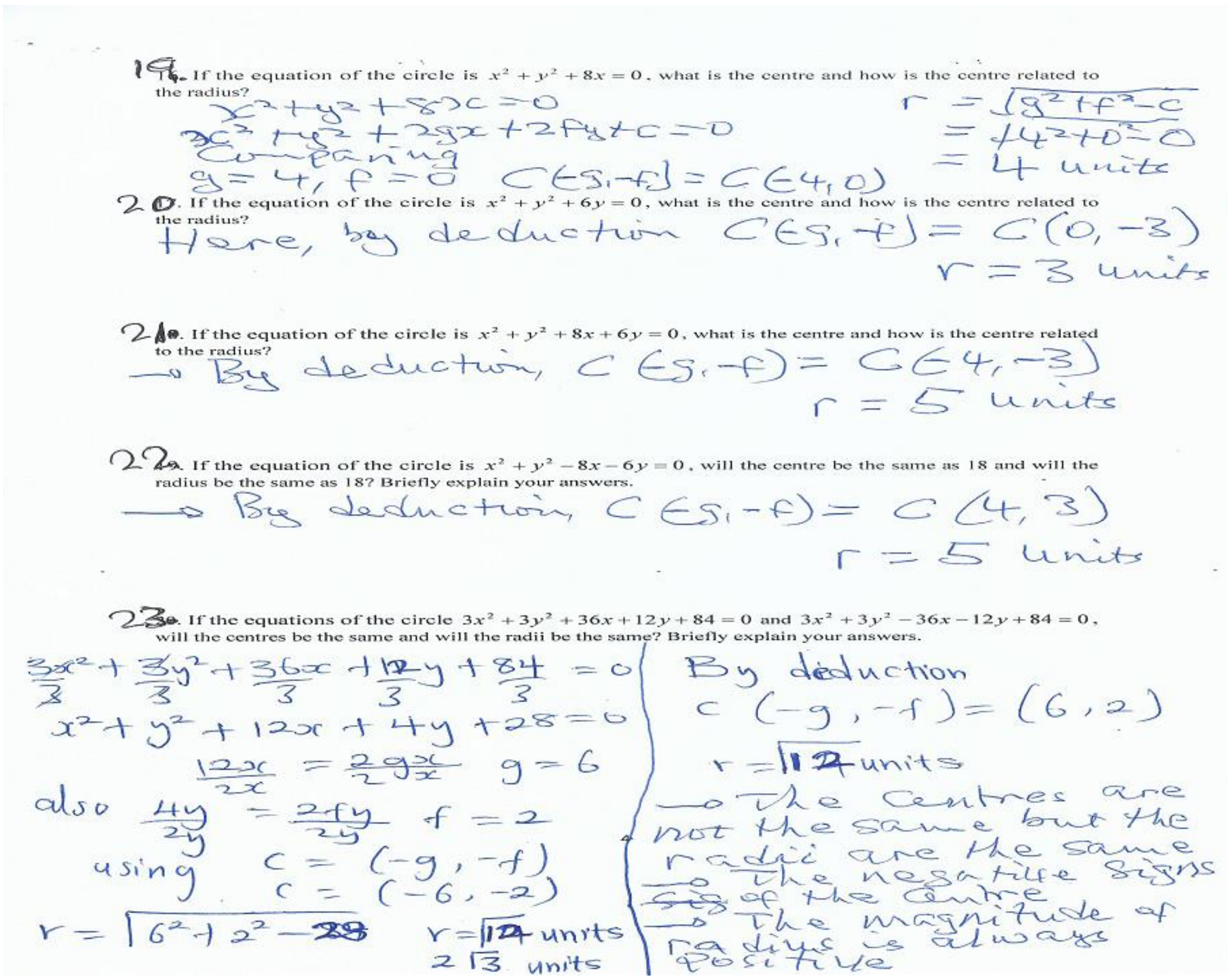
In **Figure 1**, the equations of the circle were grouped into conceptual structures, real-world conceptual structures, multiple-choice and word problems. The declarations for solving the equation, $x^2+y^2+6x+4y=0$ was the general equation of the circle, where $x^2+y^2+6x+4y=0$ was compared with $x^2+y^2+2gx+2fy+c=0$ for the center $C(-g, -f)$ and radius $r=\sqrt{g^2+f^2-c}$. However, another didactic transposition utilized the standard equation of the circle $(x-h)^2+(y-k)^2=r^2$ for the center $C(h, k)$ and radius r after double-completion of squares involving the variables x and y (Stitz & Zeager, 2013; Whitney & Reno, 2015). The standard equation stemmed from the fact that an equation of the circle is a special case of equations of the ellipse. Since, every equation of the circle is basically determined by its radius r and its center $C(h, k)$, then the distance from $P(x, y)$ on the circumference to $C(h, k)$ at the center is the radius r.

Also, the findings shows that if the standard equation of the circle is expanded with the center $C(h, k)$, then the polynomial became $x^2+y^2-2hx-2ky+h^2+k^2=r^2$, where the second-degree terms x^2+y^2, the linear terms $-2hx$ and $-2ky$ and the collection of h^2+k^2 can be into one general equation $x^2+y^2+2gx+2fy+c=0$. Here, the second terms, the linear terms and the radius were compared with x^2+y^2, $2gx+2fy$, and r^2, respectively to arrive at the new center $C(-g, -f)$ and radius $r=\sqrt{g^2+f^2-c}$. An alternative didactical phenomenology was based on the midpoint strategy or method, where the circle has two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ as the endpoints of the diameter in a line segment containing the center, and half of the diameter is the radius, for which the center is $(h,k)=(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2})$ and the radius is $r=\frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. If $P_1(x_1, y_1)$ is tangent to the standard equation, then the tangent equation of the circle was derived as $xx_1+yy_1=r^2$ and the normal as $\frac{y}{y_1}=\frac{x}{x_1}$ (Stitz & Zeager, 2013; Whitney & Reno, 2015). The geometric, algebraic, and analytical contexts reduced the phenomena from the general Cartesian equation of the conic section, $Ax^2+Bxy+Cy^2Dx+Ey+F=0$, where at least one of A, B, and C is not 0 and the discriminant is B^2-4AC for which if and only if $B^2-4AC<0$.

The MANCOVA results in sampled equations of the circle, adjusting for the covariates, revealed no significantly statistical differences in the main variables but statistically significant in the covariates [Wilks' lambda=.641, F(146, 806)=1.376, p<.05, multivariate partial eta squared=.200]. This clearly shows that one or two tasks could not effectively and efficiently yield significant teaching and learning except when four or more tasks were combined. What was even more revealing is the fact that without technologies, no tasks attained statistically significant differences. On the other hand, students interactions was significantly statistical different [Wilks' lambda=.959, F(8, 808)=2.118, p<.05, multivariate partial eta squared=.021] as well as the interaction effects [Wilks' Lambda=.250, F(146, 806)=1.385, p<.05, multivariate partial eta squared=.200]. This was made possible by the reduction of the covariates.

Post-Interview Transcripts

The transcripts in both mathematization and didactical phenomenology closely examined students' daily life situations, experientially real mathematics problems and applications of the models as starting points of the didactical models. On the conceptual meanings of the signs and symbols, students co-constructed and jointly reproduced varied contextual understanding


19. If the equation of the circle is $x^2 + y^2 + 8x = 0$, what is the centre and how is the centre related to the radius?

$x^2+y^2+8x=0$

$x^2+y^2+2gx+2fy+c=0$

Comparing

$g=4, f=0$ $C(-g,-f)=C(-4,0)$

$r=\sqrt{g^2+f^2-c}$

$=\sqrt{4^2+0^2-0}$

$=4$ units

20. If the equation of the circle is $x^2 + y^2 + 6y = 0$, what is the centre and how is the centre related to the radius?

Here, by deduction $C(-g,-f)=C(0,-3)$

$r=3$ units

21. If the equation of the circle is $x^2 + y^2 + 8x + 6y = 0$, what is the centre and how is the centre related to the radius?

By deduction, $C(-g,-f)=C(-4,-3)$

$r=5$ units

22. If the equation of the circle is $x^2 + y^2 - 8x - 6y = 0$, will the centre be the same as 18 and will the radius be the same as 18? Briefly explain your answers.

By deduction, $C(-g,-f)=C(4,3)$

$r=5$ units

23. If the equations of the circle $3x^2 + 3y^2 + 36x + 12y + 84 = 0$ and $3x^2 + 3y^2 - 36x - 12y + 84 = 0$, will the centres be the same and will the radii be the same? Briefly explain your answers.

$\frac{3x^2}{3}+\frac{3y^2}{3}+\frac{36x}{3}+\frac{12y}{3}+\frac{84}{3}=0$

$x^2+y^2+12x+4y+28=0$

$\frac{12x}{2x}=\frac{2gx}{2x}$ $g=6$

also $\frac{4y}{2y}=\frac{2fy}{2y}$ $f=2$

using $c=(-g,-f)$

$c=(-6,-2)$

$r=\sqrt{6^2+2^2-28}$ $r=\sqrt{12}$ units

$2\sqrt{3}$ units

By deduction

$c\ (-g,-f)=(6,2)$

$r=\sqrt{12}$ units

The Centres are not the same but the radii are the same

The negative signs of the Centre

The magnitude of radius is always positive

Figure 1. Exemplified types of equations of the circle (Ali, 2019)

with the notations, objects, and alphabets to represent or replace algebraic expressions and geometric meanings. They further enumerated mathematics tools that transformed and translated the signs and symbols, and utilized the tools in the processes of measuring, computing, and manipulating the tasks in equations of the circle. The findings show that mathematization and didactical phenomenology completely changed and modified students' understanding in utilizing the signs and symbols in the post test dialogues.

Also, the instruments and technologies that have been transformed and translated from mathematization and phenomenological dialogues provided significant influences and impeccable mathematics knowledge to the students. So, the students' mathematical sets, calculators and computers provided both mathematical knowledge and applications of mathematical knowledge. In the transcripts, it is evident that the social interactions among the students helped to analyze the effectiveness and efficiency of the didactical phenomenology to the problem. It provided simplicity, precision, detailed explanation, and accuracy in in the concepts therein. As a confirmation to the statistical significances, the didactical phenomenology integrated new tasks in equations of the circle. The exemplified equations of the circle helped students to apply the didactical phenomenology model to improve upon their knowledge, skills, understanding and utilization. Students comfortably and without hesitation differentiated between the standard and general equations of the circle, solved for the centers and radii, and related and connected one task to another. The results on the errors showed that students interacted uniformly and optimally in their interactions.

Discussion on Mathematization and Didactical Phenomenology

The findings showed that both the utilization of the mathematization and didactical phenomenology provided new mathematical concepts, structures, and ideas in relation to the phenomena. This placed the reality of didactical phenomenology in a variety of tangible or imaginary signs, symbols, artefacts, and tools. The authentic settings of the mathematics tasks established connections between reality and mathematics from the constructions and measurements of the mathematical phenomenological realities. The ultimate results were the statistically significance differences we arrived at to suggest that the students interacted and related well after adjusting for the covariates. Again, the utilization of the mathematization and didactical phenomenology successfully evaluated students' interactions with teachers, mathematics content and technologies in solving tasks in equations of the circle. The findings showed that the students intertwined the vertical and horizontal mathematizations in order to evaluate and make decisions about the quality of teaching and learning of equations of the circle. The signs, symbols and tools in the calculators, computers, graph sheets and mathematical sets offered opportunities to the students to evaluate the didactical phenomenological choices to gain deeper conceptual understanding (Castro-Rodríguez et al., 2022).

Also of great importance was the theory of RME that helped students to conceptualize equations of the circle in both vertical and horizontal mathematizations. The students reinvented and reformulated the types and equations and tasks in equations of the circle using the mathematical signs, symbols, and tools. In this way, they conceived the types of equations and tasks to real-life contexts in the horizontal components, and made connections between the mathematical skills, concepts, strategies, methods and theories within the signs, symbols, and tools in the vertical components. These two related components of mathematizations ultimately allowed the students to link real-life situations to standard equations, general equations, and exemplified tasks. Therefore, the simple conclusion is that the students organized and solved the tasks using the signs, symbols and tools implicitly enshrined in the mathematics sets, graph sheets and rulers.

In particular, the utilizations of the mechanistic, structuralist, empiricist and realistic of the didactical phenomenology models helped students to construct and shape their mathematics knowledge. As the definitions and rules were enshrined in the signs, symbols and tools, the structuralist dialogue made deductions and generalizations thereof, the empiricist dialogue assumed actual classroom applications and utilizations, and the realistic dialogue consumed and subsumed the whole entire phenomenological process. This conclusion was evident in the transcripts about the reality of signs, symbols, and tools as human activities. As a confirmation of students' successes, the students refined and reformulated new conception of signs, symbols, and tools. For instance, a student said *a sign is an identity given to an object*, *it is something used to represent or replace a long sentence, and it is a letter that represents a constant*. We believed frank and well-crafted meaning of a sign was an ample demonstration that the students really understood and utilized them. In particular, they schematized, reformulated, and discovered the didactical phenomenology required to solve tasks in equations of the circle, by properly conceiving the contextual meaning of *equal to*, *addition*, *squares*, *constants*, *multiplications*, and *x* and *y*.

Discussion on Sample Tasks and Types of Equations of the Circle

In the students' worksheets and transcripts, it was revealed that they conceived a circle as a atypical or degenerate coordinate-free conic section of an ellipse that generate the standard equations as center of the origin ($x^2+y^2=r^2$) or center of the point (h, k) [$(x-h)^2+(y-k)^2=r^2$]. In the ANCOVA analysis, the two standard equations yielded statistically significant differences in the interaction and main effects as supported by the partial eta squared values and estimated marginal means. This means the students candidly and progressively conceptualized the types of equations well and subsequently solved the tasks in the equations of the circle. Particularly, the students transformed the general equation with center $C(-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$, and extended didactic phenomenology to the standard equations $(x-h)^2+(y-k)^2=r^2$ with the center $C(h, k)$ and radius r. Ultimately, the standard equation $(x-h)^2+(y-k)^2=r^2$ was compared with the second-degree terms (x^2+y^2), the linear terms ($-2hx$ and $-2ky$) and the constants (h^2+k^2) to generate $x^2+y^2+2gx+2fy+c=0$. Another novelty and significant giant strides the students made was finding the midpoint of any two endpoints $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ at the diameter to arrive at the center $(h, k) = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ and the radius $r = \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Upon further normalization, they subsequently obtained the tangent equation $xx_1+yy_1=r^2$ and the normal $\frac{y}{y_1} = \frac{x}{x_1}$. These enabled the students to create several didactical phonological inter-relationships among the diameter, tangent and normal equations on one hand, and diameter, standard and general equations on the hand.

Discussion of Significance of Didactical Phenomenology

The quantitative results demonstrated fairly statistically significant differences across the vertical and horizontal mathematizations. Therefore, the qualitative and quantitative results have implications for broadening the conceptualization didactical phenomenology beyond equations of the circle. The smooth flow and flexible strides within and between the theory and the subsequent transcripts give much credence to enlarged didactical phenomenology. Even within the theory, four carefully chosen components were selected to form conceptual framework. With wider and broader networks of didactical phenomenology, the theory can make enormous contributions to reconceptualize and restructure mathematical knowledge. The quantitative results and the interview transcripts have boosted the methodological prowess to championing embedded mixed triangulation research design. In this study, there were some elements of sequential and concurrent methods of data collection strategies. The stages in collecting and analyzing the quantitative data helped to reshape the mixed method research design. In particular, the interview transcripts made the design so spectacular (Da, 2022; Hausberger, 2020).

Although there are a few studies that have sequentially experimentally examined the didactical triad (Ali et al., 2017, 2018; Da, 2022; Hausberger, 2020; Rodriguez & Fernandez, 2017; Roth, 2016), there is barely no research on the didactical phenomenology solving problems in equations of the circle in second-cycle level of education. The findings of this study helped to synchronize several didactical connections in setting socio-culturally related contexts to solving tasks in equations of the circle. It has emerged from this study that didactical phenomenology can serve as a catalysis for spearheading all mathematical domains. Again, the design provided unique contributions to mixed methods research design by integrating quantitative and qualitative interview transcripts to confirm, support and explain the quantitative results. In this way, the study has contributed to the research utility and fruitfulness of data integration through mathematical didactics, which is very rare in mathematics education research.

Didactical phenomenology has practical implications for stakeholders wishing to implement the didactical phenomenology in teaching and learning of mathematical. Didactical phenomenology could be used to make locally manufactured technologies and resources. Local artisans, book authors and community leaders could build synergy in providing quality education (Ali et al., 2018; Hasbi et al., 2019; Hausberger, 2020). Based on the students' transcripts, locally manufactured and designed technologies seemed to have been prominently evident and required support from all stakeholders. This would enhance school-community collaborations and participation to foster well balanced academic successes.

Also, beyond orientations of metre rules, mathematical sets, calculators and smart phones, stakeholders may refashion and restrategize locally produced and improvised technologies to ensure that students use their locally made tools and technologies. Locally manufactured, produced, and designed technologies and resources can ease understanding of abstract mathematics concepts like equations of the circle. In that particular school of research, students' mathematics achievements in equations of the circle and broader academic performance could have improved. As a matter of fact, this would in turn help garner more local resources to increase interactivity, performance, utilization, and participation (Ali, 2019; Hasbi et al., 2019).

The contributions of this study could enhance relationships between students, teachers, mathematical content, technologies, and resources of the mathematics classroom. The findings imply that didactical phenomenology can enrich and impact policy directions in mathematics education. In policy, senior high schools in general could further develop, improve, and refine the didactical relationships to all kinds of mathematical thinking, judgment, reasoning, and concepts. The results of this study have implications for the mathematics content and textbook preparations. Content should support stronger didactical and pedagogical relationships. The lack of algebraic/geometric relationships can be improved by creating an integrated algebra/geometry technologies and resources in solving mathematics problems. Algebra and geometry domains are typically regarded by senior high school students as unique, separate, and distinct fields of study that usually perpetuate the more perverse challenges students encounter in mathematics. Therefore, students' algebraic/geometric interactions could be strengthened by creating these adaptive methods, strategies, and technologies (Davis & Chaiklin, 2015; Loc & Tien, 2020).

Also, senior high mathematics content can focus on topic sequencing and technology-related while making room for didactical phenomenology. The mathematics contents could focus and reflect on the textbooks and makes connections to lesson planning, instructional strategies, and assessment procedures. The equations of the circle could integrate to be used as focus of extending didactical phenomenology to much broader concepts for effective mathematical interactions (Da, 2022; Larsen, 2018).

CONCLUSION

In this study, it was concluded that students effectively interacted and achieved giant strides in transforming real life problems into classroom mathematics problems. The students equally interacted well in solving for the types of circle equations and tasks in the center and radius in the didactical phenomenology. The **Table 1** and **Table 2** are clear evidence to support to this conclusion.

Also, students made attained high significances in using the didactical phenomenology to solve the types of equations, centers, and radii of the equations. The students did not show adequate conceptual understanding of the tasks but also eloquently elaborated the processes and processes involved. The transcript in **Figure 1** is clear indication of the ability of the students.

Lastly, the findings revealed quite a number of plausible covariates that needed much attention. Largely, gender and general equation of the circle were the major covariates statistically reduced the nonsignificant differences. So, most often in the analyses, these two covariates were either removed or reduced to ensure improved statistical significances.

Recommendations

Consideration the numerous significances of didactical phenomenology that emerged from these findings, it was recommended that:

1. The framework should be tested on a wide range of mathematics topics. The aim of the framework is not just to achieve knowledge but to demonstrate the processes and procedures in generating the knowledge.
2. Policy maker makers and implementers should prioritise the didactical phenomenology as the core model for teaching and learning mathematics at all levels of education. The model breeds both quantitative and qualitative outcomes and must be accepted for contemporary classroom mathematics instruction.
3. In research and scholarship, this model is the fastest and the strongest tool to achieve results. Accepting both quantitative and qualitative methods means that the researcher is always at a good position to determine the successes of the findings.

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