Teachers' Beliefs about Mathematical Knowledge for Teaching Definitions

Reidar Mosvold
University of Stavanger

Janne Fauskanger
University of Stavanger

Previous research indicates the importance of teachers' knowledge of mathematical definitions—as well as their beliefs. Much remains unknown, however, about the specific knowledge required doing the mathematical task of teaching involving definitions and the related teacher beliefs. In this article, we analyze focus-group interviews that were conducted in a Norwegian context to examine the adaptability of the U.S. developed measures of mathematical knowledge for teaching. Qualitative content analysis was applied in order to learn more about the teachers' beliefs about mathematical knowledge for teaching definitions. The results indicate that teachers believe knowledge of mathematical definitions is an important aspect of mathematical knowledge for teaching, but they do not regard it as important to actually know the mathematical definitions themselves.

Keywords: mathematical knowledge for teaching, teacher beliefs, mathematical definitions

In his presidential address at the 1985 Annual Meeting of the American Educational Research Association, Lee Shulman presented his theories concerning the different aspects of teachers' professional knowledge (Shulman, 1986). A number of attempts have been made by researchers afterwards to build upon these ideas (e.g., Graeber & Tirosh, 2008). In mathematics education, the efforts of Deborah Ball and her colleagues at the University of Michigan (see e.g., Ball, Thames, & Phelps, 2008) are among the most promising (Morris, Hiebert, & Spitzer, 2009). They have formulated a practice-based theory of what is often referred to as 'mathematical knowledge for teaching' (MKT), and they have also created measures of teachers' MKT (e.g., Hill, Schilling, & Ball, 2004). The MKT measures—as well as the MKT framework—have been developed from studies of mathematics teaching in the U.S.

In the last couple of years, researchers have made attempts to translate, adapt and use MKT items in other countries (for a review, see Blömeke & Delaney, 2012). Among the first attempts was that of Delaney (2008), who adapted and used a set of MKT items for use in Ireland. Researchers who have translated and used MKT items in other countries after this normally build upon his results and suggestions (e.g., Mosvold, Fauskanger, Jakobsen, & Melhus, 2009). Several researchers have—in their attempts to analyze the challenges of adapting MKT items for other countries—pointed at possible cultural differences in the tasks of teaching. Since MKT is conceptualized in practice, Cole (2012) argued, the question of whether or not the tasks of teaching are independent of environment and cultural context is a logical one to ask. In their study of Norwegian teachers' perceived difficulties with the adapted MKT items, Fauskanger and Mosvold (2010) also indicated that there might be cultural issues involved.

One particular task of teaching that has received attention in previous research is that of "choosing and developing useable definitions" (Ball, Thames, & Phelps, 2008, p. 400). In his study of Indonesian teachers' mathematical knowledge for teaching geometry, Ng (2012) found that the MKT measures discriminated between teachers who adopted inclusive and those who adopted exclusive definitions rather than between knowledgeable and less knowledgeable teachers. He argued that there might be some cultural differences between the use of inclusive geometric definitions between Indonesian teachers and U.S. teachers; he also argued that using the measures were useful for providing a better understanding of what teachers need to know in order to do the work of teaching in Indonesia (ibid.).

Mathematics teachers all over the world face demands related to choosing and developing definitions that are appropriate for use among their students, and Zazkis and Leikin (2008) suggest that teachers' knowledge of mathematical definitions and their concept images affect their instructional decisions, the explanations they provide in the classroom, the way they guide their students, and how they conduct mathematical discussions. To plan future professional development it is asserted that teachers' beliefs about teaching knowledge may influence their interpretation of their experiences (e.g., Ravindran, Greene, & Debacker, 2005). Given these results from previous research, we found it relevant to make an effort to learn more about teachers' beliefs about the mathematical knowledge needed for teaching definitions in a different cultural context. With this as a background, we approach the following research question: What do teachers' reflections on MKT items reveal about their beliefs concerning mathematical knowledge for teaching definitions?

In order to answer this question, we analyze the reflections given by Norwegian teachers in focus-group interviews where MKT items were used to focus the discussions. Before we approach this, however, we first need to make some clarifications about beliefs related to teaching knowledge and how they relate to other types of beliefs. Then we need to discuss how these beliefs relate to knowledge in general and MKT in particular. We also need to elaborate on our focus on that particular task of teaching concerning definitions in relation to the more general research on teachers' knowledge of mathematical definitions. These issues are addressed in the next section.

Theoretical Influences

Philosophers have pondered about beliefs and knowledge—and the connection between the two—for centuries. The result of the philosophers' reflections on these issues is manifest in the branch of philosophy called epistemology—which has a particular focus on discussions concerning knowledge and beliefs. Within the field of educational research in general and mathematics education in particular, there has been a vast amount of research related to beliefs. In his overview of research in this area, Philipp (2007) presented some of the terms that have been used when these issues have been investigated in mathematics education research: affect (including emotions, attitudes and beliefs), beliefs systems, conceptions, identity, knowledge and values. All of these concepts—including that of beliefs—have been used with various meanings by different researchers.

Beliefs About...

Before approaching the concept of beliefs about teaching knowledge—which is our focus in this article—we need to make some clarifications concerning the more general concept of beliefs. Mathematics teachers' beliefs have often been grouped into beliefs about the nature of mathematics, about mathematics teaching and about mathematics learning—as presented in Table 1.

Table 1
Categories of teachers' beliefs (adapted from Beswick, 2012, p. 130)

Beliefs about the nature of mathematics	Beliefs about mathematics teaching	Beliefs about mathematics learning
Instrumentalist	Content focused with an emphasis on performance	Skill mastery, passive reception of knowledge
Platonist	Content focused with an emphasis on understanding	Active construction of understanding
Problem solving	Learner focused	Autonomous exploration of own interest

The three categories of Ernest (1989)—as presented in the left column of Table 1—have been widely used as a description of beliefs about the nature of mathematics. In the instrumentalist view, mathematics is seen as "an accumulation of facts, skills and rules to be used in the pursuance of some external end" (Ernest, 1989, p. 250). The Platonist view sees mathematics as a body of pre-existing knowledge. Finally, in the problem solving view, mathematics is regarded as a dynamic human invention.

Almost three decades ago, Thompson (1984) claimed that the connection between teachers' beliefs about mathematics and their teaching practice had been largely ignored. She called for research with a focus on this connection between beliefs and practice, and a number of studies with such a focus subsequently emerged (e.g., Cooney, 1985; Raymond, 1997; Skott, 2001); several of these studies had a focus on inconsistencies between beliefs and practice. Following Thompson's initiative, there has been an increased interest in beliefs about the nature of mathematics; there has also been a continually increasing focus on beliefs about mathematics teaching and learning. Van Zoest, Jones and Thornton (1994) distinguished between three important aspects in research on beliefs about mathematics teaching (see the middle column of Table 1), whereas others (e.g., Ernest, 1989) distinguished between beliefs concerning three aspects of mathematics learning (see the right column of Table 1).

Beliefs and Knowledge

The relationship between knowledge and beliefs makes up a long-standing discussion (Pehkonen, 2008), and a main difficulty has been to distinguish beliefs from knowledge (Thompson, 1992). There appear to be differences as well as similarities between students' knowledge and beliefs (Op' Eynde, De Corte, & Verschaffel, 2002); research on teachers' knowledge and beliefs indicates that this is also the case here (Forgasz & Leder, 2008). In her

attempt to sort out the connection between teacher knowledge and teacher beliefs, Thompson (1992) pointed out that the difficulties involved in changing teacher performance are intimately connected with what teachers believe and know. Her approach has had significant impact on the direction of research in this area. Furinghetti and Pehkonen (2002) emphasized the close connections between knowledge and beliefs, and they argued that beliefs should be considered as part of teachers' personal knowledge. In another attempt to clarify between the concepts, Kuntze (2011) used the term 'professional knowledge'—in which beliefs were included. Many researchers distinguish between these two concepts, but some argue that beliefs and knowledge are strongly related. Beswick (2011, 2012) argued for the equivalence of beliefs and knowledge; she also suggested that beliefs about mathematical content and pedagogy should be included in the MKT framework. Philipp (2007), on the other hand, maintained that beliefs are closely related to knowledge, but a distinction should be made between the terms. In this article, we follow Philipp's suggestion and distinguish between knowledge and beliefs. We focus on the beliefs teachers have about knowledge needed for teaching, and we consider this to be an aspect of teachers' personal epistemology.

Beliefs about Teaching Knowledge

Teachers' personal epistemology includes beliefs about knowledge—commonly referred to as epistemological beliefs (Hofer, 2002)—and these epistemological beliefs are considered important; Schommer-Aikins and colleagues (2010) proposed that teachers' epistemological beliefs have a potential impact on students' learning in all academic levels. Even though the origin of studies concerning students' epistemological beliefs can be traced four decades back—Perry's (1970) seminal work has often been referred to—the actual term 'epistemological beliefs' was first used by Schommer (1994). She used the term in reference to "beliefs about the nature of knowledge and learning" (Schommer, 1994, p. 293). In research regarding epistemological beliefs, there is, however, little agreement concerning the actual construct. Some argue that epistemological beliefs are domain specific, and some argue that they are not (e.g., Buehl, Alexander, & Murphy, 2002). There is also disagreement about how the construct is connected with other related constructs (ibid.). Although several competing models of the nature of epistemological beliefs have been proposed, general epistemological beliefs seem to refer to "individuals' belief about the nature of knowledge and the processes of knowing" (Hofer & Pintrich, 1997, p. 112); sometimes it is also used with reference to learning and teaching (Op't Eynde et al., 2006). In their attempt to clarify the research in this area, Hofer and Pintrich (1997) proposed that epistemological theories are composed of "certainty of knowledge, simplicity of knowledge, source of knowledge, and justification for knowing" (ibid., p. 133). Work on disciplinary beliefs indicates that epistemological beliefs might vary from one discipline to another (e.g., Hofer & Pintrich, 1997).

Despite the importance and amount of research related to teachers' beliefs, relatively few studies focus on teachers beliefs' about teaching knowledge in general (Buehl & Fives, 2009; Fives & Buehl, 2008); even fewer studies focus on teachers' beliefs about the content of their mathematical knowledge for teaching in particular. Consequently, the body of knowledge to be considered is—due to the complexity and multidimensionality of teachers' knowledge (see

next section)—of importance in studies of teachers' beliefs about teaching knowledge (Buehl & Fives, 2009). Prior research emphasized the importance of studying teachers' beliefs about teaching knowledge, because these beliefs may influence how and what they learn from participating in professional development (e.g., Ravindran, Greene, & DeBacker, 2005); beliefs about teaching knowledge may also influence teaching practices (e.g., Sinatra & Kardash, 2004). Fives and Buehl (2010) proposed that teachers' beliefs about what they need to know constitute a distinct domain. Bendixen and Feucht (2010) supported this, and they maintained that this "provides additional depth to our understanding of teachers' personal epistemology" (p. 567).

Distinct beliefs about different aspects of teaching knowledge exists, such as the source of teaching knowledge, the stability of teaching knowledge and the structure of teaching knowledge (Buehl & Fives, 2009). In the present article we focus on practicing teachers' beliefs about a fourth aspect: the content of teaching knowledge (as in Fives & Buehl, 2008)—in particular teachers' beliefs about the knowledge needed to teach mathematical definitions.

Mathematical Knowledge for Teaching

Several frameworks for teachers' knowledge have been developed (e.g., Ball, Thames, & Phelps, 2008; Blömeke, Hsieh, Kaiser, & Schmidt, 2014; Rowland, Huckstep, & Thwaites, 2009). For the purpose of this article, we focus on the MKT framework only. This has been regarded as one of the most promising frameworks of teacher knowledge (Morris, Hiebert, & Spitzer, 2009), and the items we used to focus the group discussions were developed within this framework.

It is evident that teachers need to have some knowledge of the content they are supposed to teach. It is also generally agreed upon that teachers' knowledge need to go somewhat beyond the content they teach; their knowledge must be deeper than a plain knowledge of the content of the curriculum. The burning question is, however, what characterizes the content knowledge needed for teaching a subject like mathematics. Building upon Shulman's (1986) ideas concerning the existence of a domain of content knowledge that is unique to the teaching profession, Ball, Thames and Phelps (2008) made an effort to contribute to the further development of our understanding of this particular kind of knowledge. Shulman and his colleagues developed typologies to describe the various aspects of teachers' professional knowledge, and they focused in particular on what they referred to as "pedagogical content knowledge". This domain of knowledge connects the knowledge of content with teaching practice, and this—Ball and colleagues (2008) argue—is why it is so popular.

At the University of Michigan, they started investigating the work of teaching mathematics in the Mathematics Teaching and Learning to Teach project (MTLT). In this project, they started with practice in order to learn more about the knowledge needed by teachers in order to teach mathematics. The results provided a foundation for what they refer to as "a practice-based theory of mathematical knowledge for teaching" (Ball et al., 2008, p. 395). In these classroom studies, the researchers focused on the work of teaching mathematics rather than on teachers. They also focused on the mathematical demands of teaching, and these "tasks of teaching" are regarded as specific to the work of teaching

mathematics; the mathematical tasks of teaching are also strongly connected with the MKT items. Hill and colleagues (2004) elaborated on this when they explained how item writing served different purposes for the researchers in Michigan. Item writing served the purpose of exploring the nature and composition of subject-matter knowledge of mathematics for teaching and MKT in particular. The item writing process was used to develop the tasks of teaching. On a more practical level, they hoped that the creation of these measures would lead to increased understanding of—and renewed interest in—the content knowledge of teachers (ibid.).

Building upon the results from the MTLT project, the researchers at the University of Michigan started developing survey measures of the content knowledge needed for teaching mathematics as part of the Learning Mathematics for Teaching project (LMT). In Figure 1 an example from the public released LMT items that focuses on definitions is presented. Among the items that were discussed by the teachers in our study, one of the items had a focus on whether or not 1 is defined as a prime number. The item in Figure 1 is not the exact same, but we let it serve as an illustration since it also has a focus on the definition of prime numbers.

2. Ms. Chambreaux's students are working on the following problem:

Is 371 a prime number?

As she walks around the room looking at their papers, she sees many different ways to solve this problem. Which solution method is correct? (Mark ONE answer.)

a) Check to see whether 371 is divisible by 2, 3, 4, 5, 6, 7, 8, or 9.

b) Break 371 into 3 and 71; they are both prime, so 371 must also be prime.

c) Check to see whether 371 is divisible by any prime number less than 20.

d) Break 371 into 37 and 1; they are both prime, so 371 must also be prime.

Figure 1. Item 2 from the released LMT items (Ball & Hill, 2008, p. 4).

Mathematical Knowledge for Teaching Definitions

Knowledge of mathematical definitions is part of MKT, and it is represented in a task of teaching that Ball and colleagues (2008) formulated as "choosing and developing usable definitions". When reviewing an item like the one above, it becomes apparent that knowledge of definitions might relate to all areas of the MKT framework. Mathematical definitions are relevant for research in mathematics education in general, and the study of mathematical definitions is strongly connected with that of mathematical proofs (Knapp, 2006; Leikin & Zazkis, 2010). In the TIMSS 1999 Video Study—an international comparison study of mathematics teaching in seven countries—the results indicated cultural differences in the way teachers focused on mathematical definitions (Hiebert et al., 2003). Hiebert and colleagues (ibid.) found, among other things, that teachers from Hong Kong SAR had a stronger focus on presenting definitions than teachers from other countries.

Definitions have developed throughout the history of mathematics, and it was on the basis of the genetic approach—where a main idea is that learners should follow the path in which

discoveries were originally made—that De Villiers (1998) suggested that students should be engaged in defining concepts rather than learning about definitions. Zazkis and Leikin (2008) followed up on this when they argued that definitions of mathematical concepts as well as the processes of defining are fundamental aspects of teachers' subject matter knowledge. They continued to argue that teachers' knowledge of mathematical definitions and their concept images affect their instructional decisions, the explanations they provide in the classroom, the way they guide their students, and how they conduct mathematical discussions (e.g., Zazkis & Leikin, 2008). Leikin and Zazkis (2010) found that prospective mathematics teachers' knowledge of definitions is situated in the content domain of mathematics. They claimed that it reflects the nature of school mathematics textbooks and of the school curriculum and they found a gap between the mathematics learned in university courses and school mathematics. It is therefore not surprising that Ball, Thames and Phelps (2008), in their presentation of mathematical tasks of teaching, listed "choosing and developing useable definitions" (p. 400) as one of the challenges that are distinctive to the work of teaching mathematics. This goes beyond the ability to recite the actual definitions and into the area of understanding variations of definitions—whether congruent or non-congruent (Usiskin & Griffin, 2008)—and understanding mathematically accurate yet useful definitions and its trajectory.

Methods

In our efforts to learn more about teachers' beliefs about the content of their teaching knowledge, we arranged focus-group interviews. Focus groups have the potential to initiate "concentrated conversations that might never occur in the «real world»" (Morgan, 1998, p. 31). Such focused discussions could give realistic accounts of what teachers think about the adapted MKT items "because they are forced to think about and possibly revise their views" (Bryman, 2004, p. 348). The initial aim with these interviews was to investigate whether or not our adaptation of the MKT measures was successful by bringing in the voices of the test-takers (Fauskanger, Jakobsen, Mosvold, & Bjuland, 2012). In our previous analyses of these interviews (e.g., Fauskanger, 2012; Fauskanger & Mosvold, 2010), we learned that the practicing teachers also discussed different aspects of the knowledge they found relevant and irrelevant for their work as teachers—including aspects related to mathematical definitions (Fauskanger, 2012). For the purpose of this article—and in order to learn more about the Norwegian teachers' beliefs concerning MKT definitions—we decided to make a new analysis of the transcripts focusing on what was actually discussed related to definitions.

Participants

Fifteen teachers participated in seven semi-structured focus-group interviews, and these teachers were selected from a convenience sample of schools and teachers. All the participants had a special interest in mathematics and mathematics teacher education. The first two interviews were held at the university, whereas the other five were held at the teachers' respective schools. The first group consisted of two experienced teachers, whereas the second group consisted of three inexperienced teachers. The participants in these two groups were selected on the basis of their level of experience and special interest in mathematics education, and were all from different schools. In the next five interviews, pairs

of teachers from five schools were selected for participation in collaboration between the school principals and the researchers; these five schools were selected out of the total sample of 17 schools that participated in our pilot study.

In the first focus-group interview (FGI1), Eric and Eve participated. Both were experienced mathematics teachers. In the second interview (FGI2), three inexperienced teachers participated: Ingrid, Ingeborg and Ingfrid. In the third focus-group interview, the two teachers from Beta School were both responsible for mathematics teaching in their school. Betty was teaching mathematics in Grade 6 at the moment, whereas Benjamin had an administrative position and was not teaching that year. Both teachers in the fourth interview—at Zeta High—had finished their teacher education not long ago. The teachers from Zeta High were given the following nicknames in our data: Zachariah and Zelda. In Kappa High—which was where the fifth interview was held—Karen and Ken participated in the interviews. Matthew was one of the participating teachers from Mu School in the sixth interview, and he had lots of experience as a teacher. His colleague, Mary, was less experienced. In the seventh and final focus-group interview—which was held at Nu High—Nigel and Nora participated in the interview. Nigel had 15 years of experience as a teacher, whereas Nora had been working as a teacher for four years. Both had taught mathematics every year of their teaching careers.

Instrument and Procedure

Before the interviews, we used a form (Elementary form A, MSP_A04) of items from the LMT project to measure the teachers' MKT. These items had been translated and adapted for use among Norwegian teachers (Fauskanger et al., 2012; Mosvold et al., 2009). The form consisted of 30 item stems and 61 items and contained the following three sets of MKT items: number concepts and operations (27 items), geometry (19 items), and patterns, functions and algebra (15 items).

When they had finished the test, the teachers were given a short break. After this break, the selected teachers were interviewed in focus groups of two or three teachers. The interviews were designed to study our adaptation of the MKT measures, and questions were asked about the following: a) teachers' background, b) general considerations of the MKT measures, c) particular considerations in relation to the MC format, d) comments on the mathematical topic, structure and difficulty item by item, and e) comments and reflections that supplement the other issues discussed in the interviews (Fauskanger et al., 2012).

Data Analysis

The focus-group interviews were recorded and transcribed verbatim; these transcriptions were analyzed using a combination of two different approaches to qualitative content analysis. As part of the data reduction—and in order to learn more about what the teachers said about definitions—a summative qualitative content analysis was first applied to the data (Hsieh & Shannon, 2005). We began by identifying all that was discussed related to MKT items focusing on definitions, and all that was said related to definitions when discussing other items as well. Both authors of this article carried out independent analysis of the data to ensure reliability. One carried out content analysis with the aid of the computer software

NVivo10 (QSR International), whereas the other carried out his analysis using open source tools for text analysis. Both authors searched the transcripts for occurrences of the word 'definition' and derived terms. In this part of the analysis, we defined the utterance as a coding unit; the context unit was defined as two utterances before and after the utterance in which the key word appeared (Krippendorf, 2004). When reading the transcripts, we discovered that words like 'concept' and 'formula' were used more or less as synonyms of 'definition'. We therefore searched the transcripts for these terms as well. In our separate analyses, we ended up with an almost perfect overlap of excerpts from the transcripts. These excerpts (the context units) have been subject to further qualitative content analysis and discussion below. In this second part of the data analysis, we used a more conventional content analysis (Hsieh & Shannon, 2005), and categories were developed inductively. In the results section some of the transcripts have been slightly adapted to avoid gap fillers and repetitions.

Results

When analyzing our interview data to investigate what teachers' reflections on MKT items reveal concerning their beliefs about MKT definitions, we ended up with two partially overlapping categories. Some teachers seemed to believe that knowledge of definitions is an important part of their MKT. Other teachers seemed more reluctant, and—although they might believe that knowledge of definitions is important—they argued that teachers do not actually need to remember the mathematical definitions or formulas in order to be good teachers. Below is a presentation and discussion of the results from our analysis.

Knowledge of Definitions is an Important Part of Teachers' MKT?

The teachers discussed definitions, concepts as well as formulas, and algorithms in all the interviews. There were negative statements concerning definitions in all the interviews, and there were positive statements about definitions in all but one interview. Further analysis of these statements revealed different aspects of teachers' beliefs concerning MKT definitions.

1) Definitions are important. In most of the interviews, teachers made statements indicating a belief that knowledge of definitions is an important aspect of teachers' knowledge. In the discussions between the interviewer and the two teachers from Beta School—they discussed a testlet item focusing on non-existing geometrical figures (testlet 17 in our form)—we can see how one of the teachers emphasizes knowledge of definitions:

> 153. Interviewer: You suggest, in a way, more of the kind of tasks that focus on definitions, and less of the kind of tasks that focus on calculations, then?
> 154. Betty: Yes, I think that is correct.
> 155. Benjamin. Definitions are incredibly important as a prerequisite, because if you don't have clear definitions and know a little about it, then you will easily be out of track.
> 156. Betty: And, what was said after the TIMSS study, what I have heard anyway, is that we score low on concepts. So, I believe it is more important to be clear about this than to be able to calculate correctly. [FGI3, Beta School, March 2, 2009]

Just prior to this, the teachers have been discussing the previous couple of items. In relation to an item focusing on special cases in geometry (testlet 15), the teachers have just argued that knowledge of definitions and concepts are important. When discussing item 17 in the dialogue above, Benjamin argues that a teacher will be out of track if he does not have clear definitions (155). These teachers' beliefs seem to include knowledge related to defining concepts as a prerequisite for teaching "on the track".

Benjamin here—arguing that knowledge of definitions is important—appears to be in line with research on mathematical definitions (e.g., Zazkis & Leikin, 2008). He contends that knowledge of definitions is an important prerequisite for teachers, and this is also in concurrence with the way Ball and colleagues (2008) present the task of teaching related to definitions. The actual task of teaching is formulated as "choosing and developing useable definitions". In order for a teacher to be able to do this, knowing the actual definitions is necessary.

2) Remembering definitions is not important. Although teachers in all the interviews appeared to believe that knowledge of definitions is important for mathematics teachers, not everyone seemed to agree with Benjamin's views. Several teachers maintained that remembering the actual definition is less important for them, and some of the teachers said explicitly that knowing the formula or definition is not an important aspect of teachers' knowledge.

When discussing a testlet focusing on student-made definitions and how they would meet the students' suggestions, the teachers from Zeta High said:

> 193. Zachariah: Yes, there you have definitions again (…). How do you define polygons and parallelograms versus rectangles [inclusive definitions] (…) What is the established [definition]?
> 194. Interviewer: Mmm.
> 195. Zachariah: The point is, I do not have [know the definition] (…).
> 196. Interviewer: So you are uncertain about the definition (…) Like, what is the formal definition?
> 197. Zachariah: Some [definitions] are OK (…), like equilateral right-angled triangle…
> 198. Zelda: When I, yes… If I study these students' proposals [presented in the MKT items discussed] to plan my teaching the next day, I would have looked it [the definitions] up in a book (…) I do not go round remembering this. Maybe when I have taught for 20 years I will have looked it up enough times to remember it, but right now I do not have room for this information. [FGI4, Zeta High, March 5, 2009]

The teachers at Zeta High seem to believe that remembering definitions is not an important part of their MKT (198), and Zelda's apparent base for this argument is that she can always look up the definitions in books when preparing her lessons (198). On the other hand, Zachariah seems to believe that it is beneficial to remember some definitions—like that of the equilateral right-angled triangle (197). A possible explanation might be that Zachariah's belief that it is not important to remember the definitions is related to his lack of knowledge on this—and the belief might then be interpreted as a kind of defense mechanism. Another possible explanation is that he says: "some are OK" because they are easy to remember or because they are relevant for his students.

This brings us into a discussion concerning the nature and properties of knowledge (e.g., Hofer & Pintrich, 1997), and it initiates a discussion of whether or not it is possible to know a definition without actually remembering it (Zazkis & Leikin, 2008). Some of the other teachers in our study had a clear opinion about this.

To be able to engage students in defining concepts rather than learning about definitions—as emphasized by De Villiers (1998)—teachers need to know definitions of mathematical concepts as well as the processes of defining (Zazkis & Leikin, 2008). If teachers hold the belief that knowing definitions is not an important part of their MKT, they might struggle to learn the definitions and engaging students in this particular way might be impossible.

Choosing and Developing Useable Definitions

Ball and colleagues (2008) formulate the task of teaching that relates to definitions by using the keywords: choosing, developing and useable. In our interviews, the teachers made some statements that are related to this. It appears from our analysis of the interview data, that some teachers believe mathematical definitions are more important in the higher grades—and that the mathematically correct definitions could be confusing to their younger students.

1) Adjusting to different groups of students. In their discussion of a testlet related to definitions of quadrangles—the same item that was discussed by Zachariah and Zelda above—the teachers from Kappa High said:

> 76. Karen: I think they [the MKT measures] should have been differentiated... As an example if one can have a rectangle that is not a parallelogram and that stuff [definitions of quadrangles]. (...). But we do not have [teach] it [definitions of different quadrangles] for the younger ones [students] we teach.
> 77. Ken: No, exactly. [FGI5, Kappa High, March 9, 2009]

This statement from Karen (76)—when seen in its context—can be interpreted as an argument against a focus on definitions in the lower grades. Leikin and Zazkis (2010) described as part of teachers' pedagogical content knowledge their "ability to match the teaching of definitions and defining with a particular classroom and to attend to students' ability levels, affective needs and motivation" (p. 454). In the excerpt above, however, it seems more like Karen argues that they do not have to teach definitions of different quadrangles with their students, and the teacher therefore does not need to know about this. With reference to the MKT framework, however, one might argue that teachers need to know the mathematical definitions if they are going to be able to choose and develop definitions that are appropriate for their students. The teachers' knowledge does, however, have to go beyond the content of the particular grade level they are teaching (Ball, Thames, & Phelps, 2008).

In this connection, it should also be brought into discussion that the demand for teachers' knowledge concerning mathematical definitions needs to be seen in relation to possible cultural differences in teachers' emphasis on learning definitions by heart. Mathematics curricula vary in their emphasis on knowing and remembering definitions across countries

(e.g., Ng, 2012), and such cultural differences in the content domain might also be reflected in cultural differences regarding teachers' beliefs about the content of teaching knowledge.

2) Inclusive definitions are confusing. When discussing whether or not the suggested definitions of quadrangles would be useable among their students, the teachers from Zeta High argued:

> 204. Zachariah: Hmm, in the case of our students, I would never have said that a parallelogram could—in any kind of definition—be mixed with a rectangle. When I immediately say that they'd be completely confused. Whether that is the right definition, I don't know that. I don't know the answer to that right now. But when I explain what a rectangle is, then I say that: this is a rectangle where you have two sides/edges that are equally long, two [more] sides/edges that are equally long, but the ratio between the two are not always the same. In a parallelogram you have the shift (…) If I start to bring in definitions claiming it might be like this, and it might be like that—but not always like that—but if we touch it from this angle....
> 205. Interviewer: Yes. Do you agree with what he said?
> 206. Zelda: Yes, I have skimmed the cream a little now, no need to go deeper into it than what is usually needed to solve the tasks. That might be something you explain individually to those who handle it... [FGI4, Zeta High, March 5, 2009]

Zachariah—when discussing the inclusive definition of quadrangles above—seemed to believe that one particular definition is correct (194). In this excerpt, however, the same teacher appears to open up to the possibility that there are cultural differences when it comes to mathematical definitions (204). This might be interpreted as a belief concerning the nature of mathematics, but it might also be interpreted as an indication of cultural differences concerning the use of definitions. In any case, this is only a minor observation and it was the only occurrence of such a discussion in our interviews. Since the knowledge required for teaching seems be more culturally based than pertaining simply to mathematical knowledge (Stylianides & Delaney, 2011), however, cultural aspects related to MKT definitions are important to study further.

Concluding Discussion

Research on mathematics teachers' knowledge has been thriving for decades, and a large amount of studies build upon the foundations laid by Shulman (1986). The attempt by Ball and her colleagues at the University of Michigan to develop a practice-based theory of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008) is widely acknowledged, and their theory represents an important extension of our understanding of mathematics teachers' knowledge. The theory has been criticized, however, and one issue that has received criticism is the lack of inclusion of beliefs (e.g., Beswick, 2011, 2012). Despite the large amount of research concerning beliefs and knowledge, researchers have still not reached a consensus regarding the relationship between the two. Some argue that the two are closely connected (e.g., Furinghetti & Pehkonen, 2002), whereas others propose that a distinction should be made between the two (e.g., Philipp, 2007). In this article, we have followed Philipp's (ibid.) advice and distinguished between beliefs and knowledge.

When regarding beliefs and knowledge as two distinct categories, it makes sense to investigate beliefs about knowledge. Beliefs about knowledge—often referred to as epistemological beliefs—have been studied by researchers for a long time (e.g., Perry, 1970; Schommer, 1994). We build upon the suggestion by Buehl, Alexander and Murphy (2002) that epistemological beliefs are domain specific, and we thus argue that it makes sense to study teachers' beliefs about MKT. Fives and Buehl (2010) proposed that teachers' beliefs about knowledge they needed as teachers represented a distinct domain of teacher beliefs. We support that, and we have tried to take this idea one step further in this article.

Previous research on mathematics teachers' beliefs have often focused on teachers' beliefs about: i) the nature of mathematics, ii) mathematics teaching, or iii) mathematics learning (Beswick, 2012). In this article, we propose an extension of these categories, and we suggest that beliefs about the knowledge needed for teaching mathematics should also be included (see table 2).

Table 2
Extension of Beswick's (2012) categories of teacher beliefs

Beliefs mathematics	Beliefs about mathematics teaching	Beliefs about mathematics learning	Beliefs about MKT
Instrumentalist	Content for performance	Mastery of skills	Remembering content
Platonist	Content with understanding	Construction of understanding	Understanding content
Problem solving	Learner focused	Autonomous exploration	Adjusting and differentiating

In our analysis, we have focused on teachers' beliefs about the mathematical knowledge needed to teach definitions. Most of the teachers in our study expressed beliefs about the importance of such knowledge. In their discussions, however, differences appeared regarding their understanding of what this meant. One teacher, Benjamin, argued that teachers need to "have clear definitions and know a little about it". "Have" and "know" means different things for different teachers, and this relates to Ernest's (1989) categories of beliefs about mathematics learning (second column from the right in table 2). Some teachers expressed beliefs supporting the idea that knowledge of definitions includes remembering them, whereas others, like Zachariah, did not seem to believe that knowing the actual definitions is important.

Teachers like Zachariah might hold beliefs that indicate an emphasis on understanding the content more than simply mastering the skills and remember facts. Zachariah and his colleague Zelda also seemed to be more concerned about adjusting the definitions to their particular groups of students. Zachariah argued that some definitions—like inclusive definitions—can be confusing for students, and we can interpret this as a belief concerning MKT that implies a focus on adjusting and differentiating the content. Ball, Thames and Phelps (2008, p. 400) presented "choosing and developing useable definitions" as a mathematical task of teaching, and this might include adjusting them in order to be more appropriate to students. This also fits well with the beliefs expressed by Karen and Ken. They

argued that inclusive definitions—although mathematically correct—are not necessarily appropriate to introduce to students' in lower grades. Both of these examples also indicate a connection between teachers' beliefs about teaching and their beliefs about MKT.

Based on the results from our analysis of these teachers' beliefs concerning this specific aspect of MKT, we suggest that a more general category of teacher beliefs should also be considered for inclusion in an extended version of Beswick's (2012) table. We have labeled the category "Beliefs about MKT", and we propose a set of sub-categories (see the right column of table 2). We suggest that the beliefs in the same row are still theoretical consistent across the table, and we suggest that the beliefs in the same column constitute a continuum. This does not imply, however, that individual teachers' beliefs are consistent across categories (Beswick, 2012).

In this study, we analyzed data from focus-group interviews with Norwegian teachers who had been measured with a set of adapted MKT items. This approach differs from a traditional use of MKT items, and it also differs from a more traditional approach to investigating teachers' epistemological beliefs (e.g., Fives & Buehl, 2008). We suggest, however, that such an approach might be useful to investigate further. When asking teachers to comment on items that have been developed to measure MKT, the context for discussing beliefs about MKT has been clearly defined. The discussions that naturally emerge in such a context—e.g. the discussions about definitions in particular—can, we argue, provide interesting information about the teachers' beliefs concerning these particular issues. There is, however, a need for further research in order to investigate whether or not the more generalized categories that we have suggested can also be found when analyzing beliefs about other aspects of MKT. Such studies can also delve deeper into the discussions concerning the role of beliefs in relation to MKT.

Finally, we want to make a comment regarding the cultural issue. This study was made in a Norwegian context, and other researchers, like Ng (2012) and Cole (2012), have suggested that there are cultural differences in the use of definitions and in how student developed algorithms are emphasized. Such differences might also influence teachers' beliefs about MKT, and further research is needed in order to learn more about the influence of such cultural differences in teaching practice on teachers' beliefs about MKT. This is also related to an even bigger question about possible cultural differences in MKT as such.

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Authors

Reidar Mosvold, **Associate Professor**, Department of Education and Sports Science, University of Stavanger, 4036 Stavanger, Norway; reidar.mosvold@uis.no

Janne Fauskanger, **Assistant Professor**, Department of Education and Sports Science, University of Stavanger, 4036 Stavanger, Norway; janne.fauskanger@uis.no