

# Study of asymptotes in upper secondary mathematics education: A praxeological analysis of textbooks in Croatia

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## ABSTRACT

The notion of the asymptote is part of the upper secondary and university mathematics curriculum. Acknowledging the existing literature on student issues with asymptotes and the influential role of textbooks in mathematics education, we examined the knowledge of asymptotes provided in upper secondary mathematics textbooks in Croatia. Within the framework of the Anthropological Theory of the Didactic, we constructed a reference model for the study of asymptotes and conducted a systematic and detailed textbook analysis, focusing on praxeological organisation, ostensive manifestations, and the process of study. The analysis revealed that textbooks fail to connect knowledge and representations of asymptotes across different contexts, to organise the process of study in a mathematically and pedagogically coherent manner, and to address recognised student issues related to asymptotes. This study contributes to the research on asymptotes and textbook analysis, and proposes a reference model based on exploration using different representations, analogous work, and the construction of a generalised discourse.

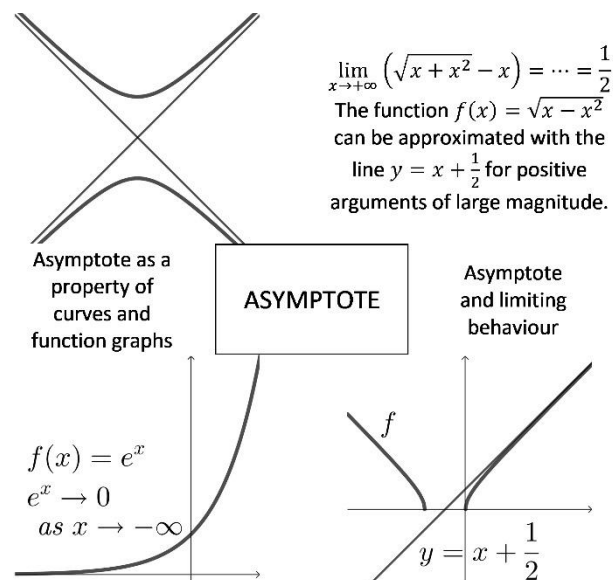
**Keywords:** Anthropological Theory of the Didactic, asymptote, praxeology, process of study, textbook analysis, upper secondary mathematics education

## INTRODUCTION

The notion of the asymptote originated in geometry, particularly in relation to the hyperbola, but has since found applications across a wide range of scientific fields. In modern mathematics, asymptotics contributes to number theory, algebra, analysis, statistics, and differential equations. Asymptotes are relevant both for identifying properties of function graphs and curves, and for applying asymptotic behaviour in problem-solving. As a result, the notion of the asymptote is an integral part of upper secondary and university-level mathematics education. It is commonly introduced in the context of specific classes of functions and curves, such as the exponential function and the hyperbola, and through the study of limits of functions (**Figure 1**).

The connection between asymptotes and fundamental mathematical notions, such as function and limit, along with the wide range of applications in advanced mathematics, makes the study of asymptotes both relevant and valuable for research in mathematics education. Ensuring vertical coherence requires that the knowledge acquired in upper secondary education lays a solid foundation for generalisations attainable at the university level. However, a potential inconsistency arises if students develop ideas unsuitable for, or even conflicted with, the formal knowledge of asymptotic behaviour. Recognising that textbooks serve as an important resource in mathematics education, this study examines the knowledge of asymptotes available in upper secondary mathematics textbooks in Croatia.

Herein we revisit issues related to student knowledge of asymptotes and the role of textbooks in mathematics education identified in the literature. Building on these findings, we present our reference epistemological model (REM) for the study of asymptotes in upper secondary education. Set in the theoretical framework of the Anthropological Theory of the Didactic (ATD), the REM served as a framework for a detailed textbook analysis. Although the study is situated within the Croatian educational context, the proposed methodology and findings have potential implications and generalisation for research on other mathematical notions and in other educational settings.



**Figure 1.** The roles of the asymptote in upper secondary mathematics education (Source: Authors' own elaboration)

## BACKGROUND OF THE STUDY

### Literature Review

Regardless of country, students have similar experiences and issues with the knowledge of asymptotes. Studies of asymptotes are typically linked to the treatment of graphs of elementary functions (exponential, logarithmic, rational, tangent, and cotangent functions), curves (such as hyperbola), and limits of functions, and working with asymptotes requires interpreting the end-behaviour of a curve or a function using verbal, graphical and symbolic representations (Čižmešija et al., 2017; Kajander & Lovric, 2009; Katalenić et al., 2024; Kidron, 2011; Mok & Johnson, 2000; Mpofu & Mudaly, 2020; Mpofu & Pournara, 2018; Mudaly & Mpofu, 2019; Nair, 2010).

Different aspects of the knowledge of asymptotes have been explored in the literature. Mpofu and Pournara (2018) found that secondary school students tended to misinterpret the asymptotes of curves and function graphs with the coordinate axes. For example, the horizontal asymptote of the transformed hyperbola  $y = 2 + \frac{1}{x}$  was unnoticed because it intersects the abscissa axis, which is the horizontal asymptote of the prototype hyperbola  $y = \frac{1}{x}$ . Multiple studies found that students connected the existence of vertical asymptotes of function graphs with points of discontinuity of the function (Kajander & Lovric, 2009; Mok & Johnson, 2000; Mpofu & Pournara, 2018; Sebsibe & Feza, 2019). For example, the rational function  $y = \frac{x^2-1}{x-1}$  was mistaken to have a vertical asymptote  $x = 1$ , because this argument is the zero of the denominator in the function formula, but the given function has a removable discontinuity for that argument. Research also showed that students described asymptotes of a curve or function graph as the lines that the curve or function graph monotonically approaches but never intersects (Kidron, 2011; McDowell, 2021; Mpofu & Mudaly, 2020; Mpofu & Pournara, 2018). This property holds for vertical asymptotes of function graphs and for horizontal asymptotes of certain classes of functions, but does not hold in general, e.g., the function  $f(x) = \frac{(x+1)^2}{x^2+1}$  intersects its horizontal asymptote  $y = 1$ .

The issues with the knowledge of asymptotes presented above were not limited to secondary students, but also university mathematics and non-mathematics students, and prospective and practising teachers showed similar difficulties (L. Berger, 2018; M. Berger & Bowie, 2012; Dahl, 2017; Flesher, 2003; Huillet, 2014; Katalenić et al., 2023, 2024; Nair, 2010). These issues transpose to the issues with the knowledge of limits, resulting with the perception that the limiting value is the boundary and unattainable value of the function or curve (Huillet, 2014; Nair, 2010; Roh, 2008; Szydlik, 2000). Thus, questioning the knowledge of asymptotes at the upper secondary level contributes to developing coherent knowledge at the university level regarding asymptotic and limiting behaviour.

Multiple studies have shown that textbooks influence student knowledge in different ways. Mathematics teachers rely on textbooks when planning and implementing lessons (Fan et al., 2013; Glasnović Gracin & Jukić Matić, 2016; Huillet, 2014; Jukić Matić & Glasnović Gracin, 2016; Pepin & Haggarty, 2001), and students also rely on the typical examples, procedures, and reasoning presented in the textbooks (Biza & Zachariades, 2010; Hardy, 2009; Nair, 2010; Raman, 2002). Further, research showed that textbooks offer limited examples and representations (Biza & Zachariades, 2010) and teachers uncritically follow the textbook instruction in their lessons (Huillet, 2014; Kajander & Lovric, 2009), hence students demonstrate issues relating to the content presented in textbooks (Kajander & Lovric, 2009; Nair, 2010).

Kajander and Lovric (2009) noted several textbook features contributing to potential student issues, such as the predominance of informal language; incorrect, imprecise or oversimplified definitions, statements and representations; and working with and discussing notions without a proper definition. Raman (2002) proposed that textbooks should provide students with structured

experiences connecting informal and formal activities and multiple representations in mathematics education. For example, in a study by Dahl (2017), students recognised that a curve can in fact intersect its asymptote, and they referenced textbook examples to support that claim.

### Theoretical Framework

ATD considers how knowledge originates, develops, and changes under the condition and constraints of different institutions (Bosch & Gascón, 2014; Chevallard & Sensevy, 2014). In the context of education, it means that a particular body of knowledge has different forms within the context of academic institutions, curriculum resources, and the classroom environment. Studying the differences between institutionally constituted knowledge brings insight into potential educational changes and interventions to overcome the issues they caused. ATD provides a means to question and examine the relevant knowledge in any educational setting, using a *praxeology* as a model for a body of knowledge, and a *process of study* as a model describing the construction of that knowledge (Barbé et al., 2005; Bosch et al., 2004).

A praxeology is a quadrupled entity  $[T/\tau/\theta/\Theta]$  consisting of the practical—*praxis*  $[T/\tau]$ , and the discursive—*logos* block  $[\theta/\Theta]$  (Barbé et al., 2005; Bosch et al., 2004; Chevallard, 2007; Winsløw, 2011). The *praxis* block consists of a *type of task*  $T$  that is a question or a problem to be solved, and *technique*  $\tau$  that is an action performed to reach the solution to  $T$ . The *logos* block consists of *technology*  $\theta$  that includes discourses—descriptions, explanations, justifications—of the practical activity or a particular mathematical notion, and *theory*  $\theta$  that supports the praxeology on the formal, abstract level. The practical and discursive activities manifest with observable or manipulable signs, visuals, gestures or instruments; thus, abstract notions are non-ostensive objects and their verbal, graphical, symbolic and other representations are ostensive objects (ostensives) (Arzarello et al., 2008; Bosch & Chevallard, 1999). A body of knowledge related to some notion consists of a single praxeology or a collection of praxeologies that include ostensive manifestation of the notion in their practical or discursive block, and this is typical to a given institution. Though *praxis* and *logos* are integral parts of a whole, multiple studies in secondary and university education have found that these blocks appeared to be unconnected within institutionalised knowledge (Barbé et al., 2005; Bosch et al., 2004; Čižmešija et al., 2017; González-Martín et al., 2013; Hardy, 2009).

The process of study of a body of knowledge is described with six moments: *the first encounter* as an introduction or re-introduction to a body of knowledge; *exploration of the type of tasks*; *work on technique*; *construction of the discourse*—the technological-theoretical block; *institutionalisation* of relevant pieces of knowledge; and *evaluation* of relevance of the body of knowledge (Barbé et al., 2005; Chevallard, 1999; García et al., 2006). A study process organised around all six moments contributes to the construction of praxeologies that are complete, connected, and supported with discourses. However, research has shown that the study process is typically organised around only two moments; e.g., first a teacher presents technological-theoretical constructs, and then students work on techniques by applying the given constructs to solve typical examples (Barbé et al., 2005; Chevallard, 1999; Corica & Otero, 2012; Serrano Martínez et al., 2020). Developing consistent and supported discourses—the technological-theoretical knowledge—is crucial for sustaining and connecting praxeologies, and this is achievable by engaging all moments in the study process.

Several studies within ATD implemented a praxeological analysis of the textbooks, such as studies by González-Martín et al. (2013) on the introduction of real numbers in Brazilian textbooks, Agustito et al. (2025) on two-variable linear equation systems in Indonesian textbooks, Utami et al. (2025) who compared Japanese and Indonesian textbooks on the learning of functions, or Chandra et al. (2025) on learning obstacles related to derivative in Indonesian textbooks. Few studies have focused on the process of study, such as studies by Corica and Otero (2012) on a university calculus course in Brazil, Barbé et al. (2005) on teaching limits in Spanish high schools, or Serrano Martínez et al. (2020) on bridging courses. Applying both these ATD tools enables a detailed and structured analysis exposing what is taught and modelled with praxeologies, and how it is taught and modelled with the process of study (Katalenić et al., 2024).

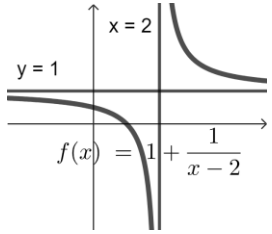
### REM for the Study of Asymptotes

Within ATD, REM is a hypothetical model created by a researcher with the purpose of studying a particular educational question (Chevallard & Bosch, 2014). The construction of a REM considers mathematical knowledge, curriculum requirements, and didactical knowledge based on the results of educational research. On the one hand, it needs to be aligned with formal, logical, and deductive mathematics structures, while on the other, it should uphold conditions and constraints of a particular educational context and provide possible solutions to questions or problems acknowledged in educational research (Barbé et al., 2005; García et al., 2006; Winsløw et al., 2014).

In upper secondary education, the notion of the asymptote emerges when graphing and evaluating functions or curves from certain classes of elementary functions or curves, such as exponential, logarithmic, rational and tangent functions, and the hyperbola. The asymptote is known as a line that the graph or curve approaches. The formal definition of an asymptote involves the Euclidean distance: given a curve, its asymptote is a line such that the distance from a point on the curve to the line tends to zero as the point tends to infinity along an unbounded part of the curve. As a result, formulas with a limit are derived for different types of asymptotes of a real function.

The REM for the process of study of asymptotes in upper secondary education includes praxeologies within two theories corresponding to the *Algebra of functions and curves* and *Calculus*. The praxeologies build around the following types of tasks: to *graph* a function or a curve ( $T_G$ ), *evaluate* a formula of a function or a curve ( $T_E$ ) or *evaluate* the limit of a formula of a function or a curve ( $T_L$ ), *determine* an asymptote ( $T_A$ ), and *develop* a formula for an asymptote of a function ( $T_D$ ). The tasks engage different techniques supported by different technologies constructed within and connected between the two theories, Algebra and Calculus.

**Table 1.** Reference epistemological model for the praxeological organisation in Algebra, example of a rational function

Praxeology	Task	Technique	Technology
Graph_A	T <sub>G</sub> Graph a function $f(x) = 1 + \frac{1}{x-2}$ .	τ <sub>G</sub> Transforming the prototype graph $y = \frac{1}{x}$ to obtain the graph of function $f$	θ <sub>AL</sub> Approaching line Verbal ostensive: The graph of function $f$ approaches the lines $x = 2$ and $y = 1$ .  Graphical ostensive: 
Evaluate_A	T <sub>E</sub> Evaluate a function $f(x) = 1 + \frac{1}{x-2}$ for selected arguments.	τ <sub>E</sub> Calculating the value of the formula for $x$ close to 2 and $x$ of large magnitude	θ <sub>AP</sub> Approximation Verbal ostensive: The values of function $f$ have large magnitude for arguments approximately equal to 2, and they are approximately equal to 1 for arguments of large magnitude. Symbolic ostensive: $f(x)$ is large for $x \approx 2$ and $f(x) \approx 1$ for large $x$
Determine_A	T <sub>A</sub> Determine the asymptotes of a linear rational function $f(x) = 2 + \frac{1}{x+1}$ .	τ <sub>G</sub> Transforming prototype graph $y = \frac{1}{x}$ τ <sub>E</sub> Calculating the value of the formula for $x \approx -1$ and $x$ of large magnitude	θ <sub>AL</sub> The graph of function $f$ approaches the lines $x = -1$ and $y = 2$ . θ <sub>AP</sub> $f(x)$ has large magnitude for arguments $x \approx -1$ , and $f(x) \approx 2$ for arguments of large magnitude.
Develop_A	T <sub>D</sub> Develop a formula for an asymptote of a linear rational function $f(x) = a + \frac{1}{x-b}$ .	τ <sub>G</sub> Transforming prototype graph $y = \frac{1}{x}$ τ <sub>E</sub> Calculating the value of the formula for $x \approx b$ and large $x$	θ <sub>AL</sub> Approaching line θ <sub>AP</sub> Approximation θ <sub>AF</sub> Property of class of functions The linear rational function $f(x) = a + \frac{1}{x-b}$ has vertical asymptote $x = b$ and horizontal asymptote $y = a$ .

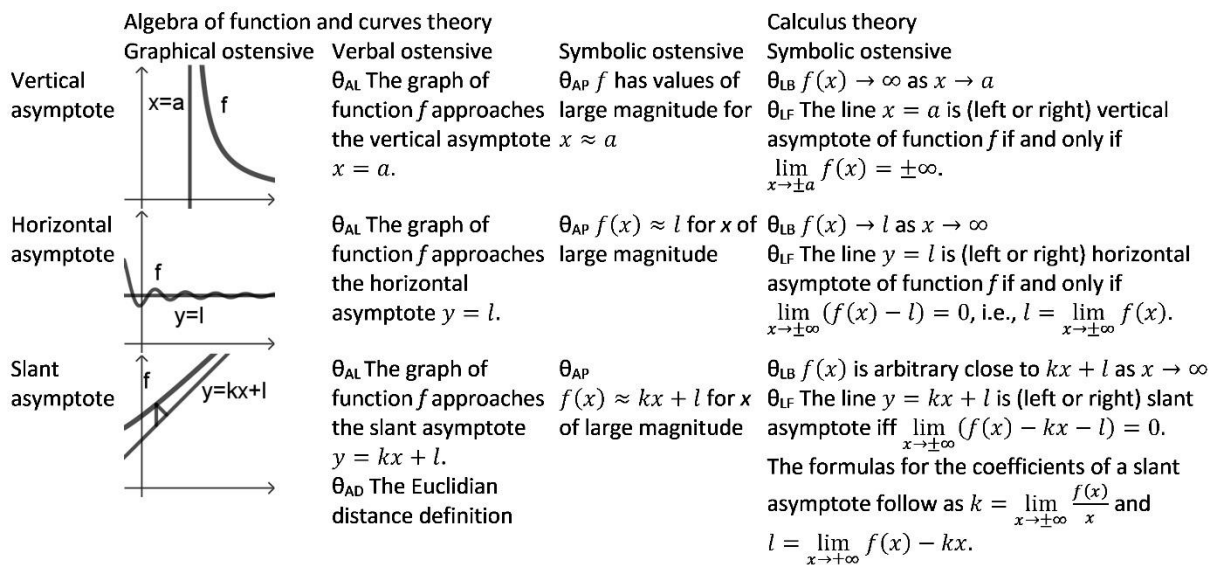
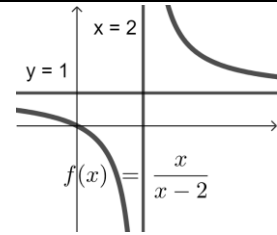
Praxeologies in Algebra of functions and curves (labelled with \_A) are given in **Table 1**. Given a class of functions/curves, a prototype function/curve has the simplest formula typical for that class of functions/curves, e.g., the function  $f(x) = 2^x$  is a prototype exponential function, and the function  $g(x) = 2^{x-2} + 2$  is also an exponential function, but not a prototype. The graph of a prototype function is the prototype graph for this class. Graphing a function/curve by transforming the prototype graph of a class of functions/curves (T<sub>G</sub>/τ<sub>G</sub>) implements graphical ostensives and supports the *Approaching line* discourse on the asymptote (θ<sub>AL</sub>). Evaluating the formula of a function/curve for arguments close to a particular point or arguments of large magnitude (T<sub>E</sub>/τ<sub>E</sub>) implements symbolic ostensives and supports the *Approximation* discourse on the asymptote (θ<sub>AP</sub>). The Graph and Evaluate praxeologies are suitable for any class of elementary functions/curves, and support the recognition of an asymptote using different ostensives. Deliberate use and connection of these techniques and technologies provide grounds for:

- an analogy—the task to determine the asymptotes of a function/curve in a class of functions/curves (T<sub>A</sub>) employs the same techniques and technologies as in the Graph and Evaluate praxeologies, and
- a generalisation—the task to develop a formula for the asymptotes of a class of functions/curves (T<sub>D</sub>) engages generalised techniques justified by known technologies and results with the corresponding *Property of the class of functions/curves* discourse (θ<sub>AF</sub>).

Praxeologies in Calculus (labelled with \_C) are given in **Table 2**. Graphing a function by applying tools from Calculus (T<sub>C</sub>/τ<sub>C</sub>) includes determining and interpreting graphically the following 'tools': the domain of the function, its limiting behaviour, parity and symmetry properties, monotonicity observed from the values of the first derivative, and concavity from the values of the second derivative. Evaluating a limit of a function with asymptotes at particular points or at infinity by applying tools from Calculus (T<sub>L</sub>/τ<sub>L</sub>) results in an infinite limit at a point or finite limit at infinity. Connecting the graphical and verbal ostensives related to the relationship between a function graph and its asymptote with the symbolic ostensives introduced in Calculus, supports the *Limiting behaviour* discourse (θ<sub>LB</sub>). Rational functions are an appropriate class of functions to develop and connect different discourses on asymptotes. The *Limiting behaviour* discourse provides grounds for an analogy in the task to determine the asymptotes of a function (T<sub>A</sub>) by evaluating the appropriate limits of an algebraic expression, and a generalisation in the task to develop (T<sub>D</sub>) the *Euclidean distance definition* (θ<sub>ED</sub>) and the *Limit characterisation* discourse (θ<sub>LF</sub>).

**Table 2.** Reference epistemological model for the praxeological organisation in Calculus, example of a rational function

Praxeology	Task	Technique	Technology
Graph_C	$T_G$ Graph a function $f(x) = \frac{x}{x-2}$ .	$\tau_C$ Applying 'tools' from calculus	$\theta_{LB}$ Limiting behaviour Graphical ostensive:  Symbolic ostensive: $f(x) \rightarrow \infty$ as $x \rightarrow 2$ and $f(x) \rightarrow 1$ as $x \rightarrow \infty$
Evaluate_C	$T_L$ Evaluate a limit of a function $f(x) = \frac{x}{x-2}$ for selected arguments.	$\tau_L$ Calculating the limit of a function for $x = 2$ and at infinity	$\theta_{LB}$ Limiting behaviour Verbal ostensive: The values of function $f$ increase without bound as $x$ approaches 2, and they tend to 1 as $x$ increases without bound. Symbolic ostensive: $\lim_{x \rightarrow 2} f(x) = \infty$ hence $x = 2$ is vertical asymptote, and $\lim_{x \rightarrow \infty} f(x) = 1$ hence $y = 1$ is horizontal asymptote
Determine_C	$T_A$ Determine the asymptotes of a function $f(x) = \frac{x}{x^2+1}$ .	$\tau_L$ Evaluating limit of an algebraic expression	$\theta_{LB}$ Limiting behaviour $\lim_{x \rightarrow 1} f(x) = \infty$ and $\lim_{x \rightarrow -1} f(x) = \infty$ hence $x = 1$ and $x = -1$ are vertical asymptotes, and $\lim_{x \rightarrow \infty} f(x) = 0$ hence $y = 0$ is horizontal asymptote
Develop_C	$T_D$ Develop formulas with a limit for different types of asymptotes of a function $f$ .	$\tau_L$ Evaluating limit of an algebraic expression	$\theta_{LF}$ Limit characterisation The formulas with a limit for vertical, horizontal and slant asymptote of a real function $f$



**Figure 2.** Connecting discourses in Algebra and Calculus to support formulas with a limit for different types of asymptotes (Source: Authors' own elaboration)

The Limit characterisation is the epitome of discursive knowledge of asymptotes in Calculus referencing formulas, with a limit for different types of asymptotes of real functions. Consider the task to develop a formula for the limit characterisation of a slant asymptote of a function ( $T_D$ ) using generalised techniques and constructed discourses. By graphically interpreting ( $\tau_G$ ) the relationship between the graph of a function  $f$  and the line  $y = kx + l$ , it follows that the Euclidean distance between the graph and the line is less than or equal to the difference between their values for the same argument. If the line  $y = kx + l$  is an asymptote, then the Euclidean distance tends to zero following the Approaching line discourse ( $\theta_{AL}$ ). By evaluating the limit of the algebraic expression ( $\tau_L$ ) referencing the difference  $f(x) - kx - l$ , it follows that  $\lim_{x \rightarrow \pm \infty} (f(x) - kx - l) = 0$ . This interplay between verbal, graphical, and symbolic ostensives supports the connection between Euclidean distance definition and Limit characterisation of an asymptote (Figure 2).

The process of study of asymptotes proposed in the REM has a similar structure in Algebra and Calculus (Table 3). The Graph and Evaluate praxeologies correspond to the exploration of the type of task moment in the study process. Both praxeologies are appropriate for the first encounter in Algebra and Calculus, and the re-encounter moment in Calculus, and provide grounds for other moments in the study process. The working on technique moment involves the Determine praxeology executed by the analogy to the exploration moment; while the construction of the discourse moment involves the Develop praxeology executed as a generalisation of the working moment. Within Algebra, the institutionalisation moment involves asymptotes as a part of the Property of a class of functions/curves discourse, while within Calculus, it involves the Limit characterisation of asymptotes—the formulas with a limit for different types of asymptotes. Institutionalised knowledge develops through exploration, working, and

**Table 3.** Reference epistemological model for the process of study of asymptotes in Algebra and Calculus

Theory	Moment in the study process	Praxeology	Ostensives
Algebra	First encounter /	Graph_A	Graphical, Verbal
	Exploration of a type of task	Evaluate_A	Symbolic, Verbal
	Working on technique	Determine_A	Graphical, Symbolic, Verbal
	Construction of discourse	Develop_A	Graphical, Symbolic, Verbal
	Institutionalisation	Property of functions/curves ( $\theta_{AF}$ )	Symbolic
Calculus	Re-encounter /	Graph_C	Graphical, Symbolic
	Exploration of a type of task	Evaluate_C	Symbolic, Verbal
	Working on technique	Determine_C	Graphical, Symbolic, Verbal
	Construction of discourse	Develop_C	Graphical, Symbolic, Verbal
	Institutionalisation	Limit characterisation ( $\theta_{LF}$ )	Symbolic

construction moments, hence the evaluation moment references the connection of asymptotes to function graph/curves and limit of a function, and application of those connections.

The praxeological and study process structure of the proposed REM does not depend on a particular class of functions/curves, and hence the REM is comprehensive. The praxeologies corresponding to the working on technique and construction of discourses moments relate to the praxeologies corresponding to the exploration moment in the study process, and the praxeologies in Calculus build on the corresponding praxeologies in Algebra. Hence, the praxeologies increase in complexity within and across theories, and the REM is connected. The proposed discourses purposefully avoid context-dependent ones, such as identifying axes or breakpoint at function domain as asymptotes and interpreting the asymptote as a line the graph monotonically approaches and never reaches. The technologies, supported with different corresponding ostensives, emphasise the asymptotic behaviour of a function, hence the REM is consistent.

### Research Questions

Textbook authors present their proposal for the study organisation by selecting notions, tasks, and discourses, representing them through verbal, graphical or symbolic ostensives, and ordering the work in a particular sequence. The knowledge taught in the classroom likely aligns with the knowledge proposed in the textbooks, as seen in the literature (e.g., Raman, (2002); Kajander and Lovric, (2009)). By identifying the praxeologies, ostensives, and moments in the study process related to asymptotes in upper secondary textbooks, we expose a detailed analysis of the knowledge available to students. Revealing the underlying structure of this knowledge allows a better understanding of the potential origins of student issues in their knowledge of asymptotes at different educational levels.

Previous studies on asymptotes focused on student or teacher knowledge of particular aspects of the notion of asymptote (e.g., the definition of horizontal and vertical asymptotes in Dahl (2017), asymptotes of a hyperbola in Mpofu and Pournara (2018), asymptotes of rational functions in Nair (2010) and Yerushalmy (1997), neglecting its connection to the knowledge available in textbooks), while praxeological analyses of textbooks considered only other notions (e.g., functions in Utami et al. (2025), real numbers in González-Martín et al. (2013), limit of a function in Barbé et al. (2005) and Corica and Otero (2012), and derivative of a function in Chandra et al. (2025)). This study aims to fill this gap by examining the study of asymptotes in upper secondary mathematics textbooks in Croatia.

For that purpose, we propose the following research questions:

- What praxeologies and ostensive objects related to asymptotes are available in the upper secondary mathematical textbooks in Croatia? How do they align with the proposed REM?
- How is the process of study of asymptotes organised in the upper secondary mathematical textbooks in Croatia? How does it align with the proposed REM?

## METHODOLOGY

### Context of the Study

The pre-tertiary education system in Croatia consists of eight compulsory years of primary education for children aged 7 to 14, followed by optional secondary education that includes four-year general programmes (gymnasiums) and three- or four-year vocational or artistic programmes. Gymnasiums provide general upper secondary education, mainly preparing for university education, for students aged 15 to 18 years. Subjects cover different areas, such as modern and classical languages, humanities, social sciences, arts, natural sciences, mathematics, and technology. Commercial publishers offer textbooks aligned with the curriculum requirements and approved by the Ministry of Education. The general upper secondary mathematics curriculum covers the topics of real numbers, algebraic expressions, linear equations, and plane geometry in the first grade; complex numbers, quadratic equations and functions, exponential and logarithmic functions, and solid geometry in the second grade; trigonometric functions, coordinate geometry, and conics in the third grade; and probability, sequences, elementary functions, limits, derivatives, and integrals in the fourth grade.

This study is a part of a larger study concerning the knowledge of asymptotes across upper secondary and university mathematics education in Croatia. We analysed the textbooks that university students participating in the larger study used during their upper secondary education. Two textbook sets from different commercial publishers were available for the four-year general

upper secondary education in Croatia, labelled A and B. The present study focused on the chapters in the corresponding textbooks covering topics related to the knowledge of asymptotes, including chapters on elementary (exponential, logarithmic, tangent and cotangent) functions in the second and third grade, asymptotes of a hyperbola in the third grade, and calculating limits and graphing real functions, particularly rational functions, in the fourth grade.

### Textbook Analysis

A textbook block physically separated from other content, such as a solved example, explanation, or framed content, formed a single unit of analysis. We identified all blocks containing ostensives related to the asymptote across all featured chapters in each textbook set. This means that, within a particular textbook block, the asymptote is either mentioned in the task, used in a technique, or discussed in the solution, and generally, is represented with a verbal, graphical or symbolic ostensive.

We examined each selected textbook block from three standpoints: 1. identifying the elements  $[T/\tau/\theta/\Theta]$  of a praxeology; 2. the ostensive manifestation of the asymptote, and 3. the moment in the process of study of asymptotes. Each separate textbook block contains a practical element, e.g., graphing a function with an asymptote, determining an asymptote of a function, or discursive element, e.g., describing asymptotic behaviour, providing formulas for the equation of an asymptote, or stating the definition of an asymptote. The praxeological organisation proposed in the REM provided the foundation for the praxeological analysis of the textbooks.

In each textbook block, we recorded the type of ostensive (verbal, graphical or symbolic), and the characteristics of the ostensive manifestation of an asymptote. Considering previous studies on asymptotes, we focused on the relationship between a curve and its asymptote in the graphical ostensive, expressions and formulas applied in the techniques used to determine different types of asymptotes in the symbolic ostensive, and descriptions, properties, and definition of asymptote and asymptotic behaviour in the verbal ostensive.

Finally, the praxeology identified in a textbook block represents a moment in the process of study of asymptotes. The particular moment of the six moments in the process of study was determined from the textbook organisation. There were three types of textbook blocks: free text, solved example, and framed text. A free text block corresponds to the exploration of a type of task moment if the focus was on solving a particular task, or the construction of discourse moment if the focus was on obtaining general knowledge. A solved example block corresponds to the working on technique moment, and framed text block to the institutionalisation moment.

Examination of the utilised praxeologies, ostensives, and moments in the study process enabled a detailed analysis of the activities and knowledge available in the textbooks related to a particular mathematical notion.

## RESULTS

### Praxeological Organisation Related to Asymptotes in the Textbooks

Praxeologies related to asymptotes appeared in the textbook chapters covering the knowledge of algebra of elementary functions (exponential, logarithmic, tangent and cotangent functions), coordinate geometry of second-order curves (particularly hyperbola), and application of Calculus for graphing functions. Neither textbook included any reference to the notion of asymptote in chapters related to limits of a function. An additional chapter in the fourth-grade textbook A allocated the application of Calculus for graphing rational functions with an emphasis on asymptotes of rational functions. The textbooks mainly included tasks, techniques, and technologies similar to those proposed in the REM (**Table 4**). We labelled the observed praxeologies with additional enumeration to differentiate praxeological structures, e.g., Graph\_A1 praxeology found in the textbooks was similar to Graph\_A praxeology proposed in the REM. The main discrepancies in the praxeological organisation of the REM and textbooks were related to techniques and technologies in the Graph and Determine praxeologies in Algebra and the Determine praxeology in Calculus, and the inclusion of truncated logos blocks (TLB) containing only discursive elements.

**Table 4.** Praxeological organisation observed in the textbooks

Praxeology	Task	Technique	Technology
Graph_A1	T <sub>G</sub> Graph a function	τ <sub>G</sub> Plotting points*	θ <sub>AL</sub> Approaching line θ <sub>AP</sub> Approximation Relational characterisation*
Graph_A2	T <sub>G</sub> Graph a function	τ <sub>G</sub> Plotting points*	θ <sub>AF</sub> Property of a class of functions/curves
Evaluate_A	T <sub>E</sub> Evaluate an equation of a curve	τ <sub>E</sub> Evaluating an algebraic expression	θ <sub>AD</sub> The Euclidean distance definition of an asymptote
Determine_A1	T <sub>A</sub> Determine the asymptotes of a function	τ <sub>G</sub> Recognising prototype graph*	θ <sub>AF</sub> Property of a class of functions/curves
Develop_A	T <sub>D</sub> Develop a formula for an asymptote	τ <sub>E</sub> Evaluating algebraic expression	θ <sub>AP</sub> Approximation
	T <sub>D</sub> Develop a formula for an asymptote	τ <sub>G</sub> Transforming prototype graph	θ <sub>AF</sub> Property of a class of functions/curves
Graph_C	T <sub>G</sub> Graph a function	τ <sub>C</sub> Applying `tools´ from calculus	Calculus, including θ <sub>LF</sub>
Graph_C1	T <sub>G</sub> Graph a function	τ <sub>C</sub> Applying `tools´ from calculus	Calculus, including Determine_C1
Determine_C1	T <sub>A</sub> Determine the asymptotes of a function	τ <sub>L</sub> Evaluating limit of an algebraic expression	θ <sub>LF</sub> Limit characterisation
		τ <sub>E</sub> Evaluating algebraic expression; τ <sub>D</sub> Performing polynomial division*	Algebraic characterisation*
Develop_C	T <sub>D</sub> Develop a formula for an asymptote	τ <sub>L</sub> Evaluating limit of an algebraic expression	θ <sub>LF</sub> Limit characterisation
Other	Determine the tangent lines on a hyperbola through given point*	τ <sub>E</sub> Evaluating an algebraic expression	Relationship with a tangent line*

\* The task, technique or technology was not among those proposed in the REM

- a. Determine\_A1 Verbal and graphical ostensive Exploration of the task moment [...] *The exponential function has a horizontal asymptote  $y = 0$ , the logarithmic function has a vertical asymptote  $x = 0$ . The hyperbola has two slant asymptotes  $y = \pm \frac{b}{a}x$ .*
- b. Algebraic char. Verbal ostensive Construction of discourse moment *Vertical asymptotes are most often obtained by observing the zeros of the denominator.*
- c. Determine\_A1 Verbal, graphical, and symbolic ostensive Exploration of the task moment *The function  $f(x) = \frac{1}{x}$  has the vertical asymptote  $x = 0$  since function values increase rapidly for values of the variable  $x$  near zero. [...] Function  $f(x) = \frac{1}{x}$  has got horizontal asymptote.*
- d. Limit char. Symbolic ostensive Institutionalisation moment *Slant asymptotes are determined using limits. The line  $y = kx + l$  is the right-side slant asymptote if  $k = \lim_{x \rightarrow \infty} \frac{f(x)}{x}$ ,  $l = \lim_{x \rightarrow \infty} (f(x) - kx)$ . If these limits do not exist, function does not have slant asymptotes. If the coefficient  $k$  is zero and  $\lim_{x \rightarrow \infty} f(x) = l$  exists, function has got a horizontal asymptote  $y = l$ .*

**Asimptote**

S pojmom asimptota upoznali smo se u drugom i trećem razredu kada smo proučavali logaritamsku i eksponencijalnu funkciju te hiperbolu. Prijetimo se grafova eksponencijalne i logaritamske funkcije. Eksponencijalna funkcija ima horizontalnu asimptotu  $y = 0$ , logaritamska funkcija ima vertikalnu asimptotu  $x = 0$ .

Hiperbola ima dvije kose asimptote  $y = \pm \frac{b}{a}x$ .

Slično i neke druge funkcije mogu imati asimptote. Opišimo postupak njihova određivanja.

**a.** Tako na primjer funkcija  $f(x) = \frac{1}{x}$  ima vertikalnu asimptotu  $x = 0$ , jer za pozitivne vrijednosti varijable  $x$  blizu 0 vrijednosti funkcije  $f$  jako rastu. Graf funkcije  $f(x) = \frac{1}{x}$  prikazan je na slici lijevo dole.

Prijetijmo da osim vertikalne, funkcija  $f(x) = \frac{1}{x}$  ima i horizontalnu asimptotu.

**b.** Vertikalne asimptote najčešće dobivamo promatrajući nultočke nazivnika.

**c.** Tako na primjer funkcija  $f(x) = \frac{1}{x}$  ima vertikalnu asimptotu  $x = 0$ , jer za pozitivne vrijednosti varijable  $x$  blizu 0 vrijednosti funkcije  $f$  jako rastu. Graf funkcije  $f(x) = \frac{1}{x}$  prikazan je na slici lijevo dole.

Prijetijmo da osim vertikalne, funkcija  $f(x) = \frac{1}{x}$  ima i horizontalnu asimptotu.

**d.** Kose asimptote se određuju pomoću limesa. Pravac  $y = kx + l$  je desna kosa asimptota ako je  $k = \lim_{x \rightarrow \infty} \frac{f(x)}{x}$ ,  $l = \lim_{x \rightarrow \infty} (f(x) - kx)$ . Ukoliko ovi limesi ne postoje, funkcija nema desne kose asimptote. Ako je koeficijent  $k$  nula i postoji  $\lim_{x \rightarrow \infty} f(x) = l$ , funkcija ima horizontalnu asimptotu  $y = l$ .

**Figure 3.** Excerpt of the study of asymptotes from the calculus chapter in textbook B (Adapted from textbook B)

The preferred technique to graph a function in Algebra was plotting corresponding points of the function graph. The technologies referenced Approaching line, Approximation, or Property of a class discourses from the REM, while the *Relational characterisation* included discourses about an asymptote of a function as recognised in the literature. Both textbooks mentioned that the function graph has a coordinate axis as an asymptote in the cases of prototype exponential and logarithmic functions, and that the vertical asymptote of a function is related to the breakpoint in the function domain in the case of a prototype tangent function. In textbook B, an asymptote of the prototype exponential function was described as the line the function graph approaches but never touches and as its tangent line at an `infinitely distant point´. The Determine praxeology only referenced asymptotes of the prototype elementary functions, recognising them from the function graph (Figure 3a, 3c).

The task to determine asymptotes of functions in Calculus involved two approaches with different technologies: one related to Limit characterisation and the second to Algebraic characterisation of asymptotes of rational functions. The former implied evaluating the formulas with a limit for different types of asymptotes as proposed in the REM. The latter implied: (1) evaluating

Determine\_C1 [...] When dividing the numerator by the denominator of a rational function the quotient obtained is a first-degree polynomial which is the asymptote of the function. [...] divide the numerator  $-x^2 + 3x$  by the denominator  $x - 4$ : [...] The result of the division can be written  $\frac{3x-x^2}{x-4} = -x - 1 + \frac{-4}{x-4}$ . The quotient  $-x - 1$  is the slant asymptote, because  $\lim_{x \rightarrow \pm\infty} \left[ \frac{3x-x^2}{x-4} - (-x-1) \right] = \lim_{x \rightarrow \pm\infty} \frac{-4}{x-4} = 0$ .

Kose asimptote racionalne funkcije možemo potražiti i na drugi način. Riječ je o tome da se dijeljenjem brojnika nazivnikom racionalne funkcije za količnik dobiva polinom prvog stupnja koji je upravo asimptota te funkcije. Pogledajmo na primjeru prethodne, podijelimo brojnik  $-x^2 + 3x$  s nazivnikom  $x - 4$ :

$$\begin{array}{r} -x^2 + 3x \quad : x - 4 = -x - 1 \\ \underline{-x^2 + 4x} \phantom{0} \\ \phantom{-x^2 + 4x} -x + 4 \\ \phantom{-x^2 + 4x} \underline{-x + 4} \\ \phantom{-x^2 + 4x} \phantom{-x + 4} 0 \end{array}$$

Rezultat dijeljenja možemo zapisati u obliku:

$$\frac{3x-x^2}{x-4} = -x - 1 + \frac{-4}{x-4}$$

Količnik  $-x - 1$  predstavlja kosu asimptotu, jer je

$$\lim_{x \rightarrow \pm\infty} \left[ \frac{3x-x^2}{x-4} - (-x-1) \right] = \lim_{x \rightarrow \pm\infty} \frac{-4}{x-4} = 0.$$

**Figure 4.** Excerpt of the Determine praxeology with Algebraic characterisation from the calculus chapter in textbook A (Adapted from textbook A)

**Table 5.** Number of items corresponding to praxeologies across textbook chapters

Praxeology	Textbook A				Textbook B			Total
	EF	GC	CF	RF	EF	GC	CF	
Graph_A1	2				3			5
Graph_A2	1	2			1			4
Evaluate_A		1						1
Determine_A1			6				4	10
Develop_A		3				1		4
Other		1				1		2
TLB in Algebra	4	5	1		2	2		14
Graph_C			2					2
Graph_C1				2			1	3
Determine_C1			1	2			1	4
Develop_C			1					1
TLB in Calculus			4	3			3	10
Total	7	12	15	7	6	4	9	60

Textbook chapters: EF – Algebra of elementary functions, GC – Coordinate geometry of second order curves; CF – Calculus for graphing functions, RF – Calculus for graphing rational functions; TLB – Truncated logos block

the denominator of the formula of a rational function to determine its vertical asymptotes; (2) evaluating the ratio between the leading terms of the numerator and denominator to determine the existence of a horizontal asymptote; or (3) performing the polynomial division of the numerator and denominator to determine its slant asymptote (Figure 4). Consequently, to graph a function included applying 'tools' from Calculus, as proposed in the REM. In the case of a non-rational function, it included evaluating the formulas related to Limit characterisation (Graph\_C) and in the case of a rational function, evaluating the expressions related to Algebraic and Limit characterisation (Graph\_C1).

Textbooks A and B had a similar organisation in the algebra and calculus chapters (Table 5). The algebra chapter included Graph praxeology and TLB referencing Property of functions, Approaching line, Approximation, or Relational characterisation. The calculus chapter included the Determine praxeology with prototype elementary functions, Determine and Graph praxeologies in Calculus, and TLB with Limit characterisation (Figure 3d). Textbooks differed in the praxeological organisation of the geometry chapter and in the construction of discourse on Algebraic and Limit characterisation in the calculus chapter. Textbooks had approximately the same number of praxeologies in the algebra chapter, while textbook B had fewer praxeologies in other chapters compared to textbook A.

In the geometry chapter, textbook A included the Graph, Evaluate, and Develop praxeologies, and TLB referencing Property of a hyperbola, Euclidean distance definition, and the discourse on tangent lines of a hyperbola (Figure 5). Textbook B included the Develop praxeology, TLB referencing Approaching line and Euclidean distance definition, and a praxeology to determine a tangent line to a hyperbola. The Develop praxeology involved the task to develop the formula for the equation of an asymptote of a hyperbola ( $T_D$ ), though each textbook applied different techniques and technologies. In textbook A, praxeology included evaluating the algebraic expression referencing the ordinate value of a point of a hyperbola ( $T_E$ ) resulting with the Approximation technology ( $\theta_{AP}$ ), whereas in textbook B, it included graphing the hyperbola and the lines ( $\tau_G$ ) resulting in Approaching line technology ( $\theta_{AL}$ ).

- a. Develop\_A  
Verbal and symbolic ostensive  
Construction of discourse moment  
*If we draw the hyperbola precisely, we notice that distant points of hyperbola approach a line. The equation of the line is  $y = \frac{b}{a}x$  or  $y = -\frac{b}{a}x$ . We assure this is true from the equation of hyperbola  $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$ . [...] If the value of the abscissa of the point  $x$  increases, the number  $a$  becomes negligible compared to  $x$  and it is  $\sqrt{x^2 - a^2} \approx \sqrt{x^2} = x$ . [...] For points on the hyperbola it is  $y = \pm \frac{b}{a}\sqrt{x^2 - a^2} \approx \pm \frac{b}{a}x$ .*
- b. Eucl. distance def.  
Verbal and symbolic ostensive  
Institutionalisation moment  
*Asymptotes of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  are the lines  $y = \frac{b}{a}x$  and  $y = -\frac{b}{a}x$ . The distance of points on the hyperbola to one of the asymptotes tends to zero as the point moves away from the origin.*
- c. Graph\_A2  
Verbal, graphical and symbolic ostensive  
Working on technique moment  
*Graph the hyperbola  $\frac{x^2}{9} - \frac{y^2}{4} = 1$ . The real semi-axis is  $a = 3$ , imaginary is  $b = 2$  and asymptotes are  $y = \frac{2}{3}x$  and  $y = -\frac{2}{3}x$ . [...]*
- d. Evaluate\_A  
Verbal and symbolic ostensive  
Exploration of the task moment  
*Calculate the distance of points on the hyperbola  $x^2 - \frac{y^2}{4} = 1$  to the asymptote  $y = 2x$ . We calculate for the points with the abscissas  $x_1 = 10, x_2 = 100, x_3 = 1000$ . [...] We see the distances are decreasing and tend to zero as the point  $T$  moves away along the branch of the hyperbola.*
- e. Other  
Verbal and graphic ostensive  
Construction of discourse moment  
*[...] From the point on hyperbola one can draw one tangent line to the hyperbola. From the point outside hyperbola (which is not on its asymptote) one can draw two tangent lines to the hyperbola.*

Elipsa je omeđena krivulja: pravokutnik sa stranicama duljina  $2a$  i  $2b$  paralelnim s osima i središtem u središtu elipse potpuno je obuhvaća. Hiperbola, međutim, nije omeđena krivulja, jer se za ma kako velik  $x$  može odrediti ordinata točke na hiperboli. Crtamo li precizno hiperbolu, primijetiti ćemo da se udaljene točke hiperbole približavaju jednom pravcu. Jednadžba tog pravca je  $y = \frac{b}{a}x$ , odnosno  $y = -\frac{b}{a}x$ . Da je to istina, naslućujemo iz same jednadžbe hiperbole

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$$

Promotrimo, jednostavnosti radi, točke s desne grane hiperbole, za koje je  $x > 0$ . Ako se vrijednost apscise točke  $x$  povećava, tada broj  $a$  postaje zanemariv prema  $x$ , pa vrijedi  $\sqrt{x^2 - a^2} \approx \sqrt{x^2} = x$ . (Usporedite, npr., za  $a = 2$  i  $x = 1000$ .) Zato za točke na hiperboli vrijedi

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2} \approx \pm \frac{b}{a}x$$

**Asimptote hiperbole**  
Asimptote hiperbole  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  su pravci  $y = \frac{b}{a}x$  i  $y = -\frac{b}{a}x$ . Udaljenost točaka s hiperbole do jednog od tih pravaca teži k nuli kad se točka udaljava od ishodišta.

Nacrtajmo hiperbolu  $\frac{x^2}{9} - \frac{y^2}{4} = 1$ .

Realna poluos je  $a = 3$ , imaginarna  $b = 2$  pa su asimptote  $y = \frac{2}{3}x$  i  $y = -\frac{2}{3}x$ . Linearni ekscentricitet je  $e = \sqrt{a^2 + b^2} = \sqrt{13}$ . To je upravo duljina polovine dijagonale pravokutnika koji prolazi tjemena hiperbole. Zato žarišta konstruiramo tako da iz središta hiperbole šestarom kojim je polumjer  $e$  zasijecemo pravac s realnom osi.

Izračunajmo udaljenost točaka hiperbole  $x^2 - \frac{y^2}{4} = 1$  do asimptote  $y = 2x$ . Računat ćemo za točke s apscisama  $x_1 = 10, x_2 = 100, x_3 = 1000$ .

Udaljenost točke  $T(x, y)$  do pravca  $p \dots 2x - y = 0$  je

$$d = \frac{|2x - y|}{\sqrt{2^2 + 1}} = \frac{|2x - y|}{\sqrt{5}}$$

Koordinate točaka (uzimamo točke s pozitivnom ordinatom, jer su one bliže asimptoti) kojima tražimo udaljenost su:

$T_1(10, 2\sqrt{99}) \Rightarrow d_1 = \frac{|20 - 2\sqrt{99}|}{\sqrt{5}} = 0.0447$ ,

$T_2(100, 2\sqrt{9999}) \Rightarrow d_2 = \frac{|200 - 2\sqrt{9999}|}{\sqrt{5}} = 0.00447$ ,

$T_3(1000, 2\sqrt{999999}) \Rightarrow d_3 = \frac{|2000 - 2\sqrt{999999}|}{\sqrt{5}} = 0.000447$ .

Vidimo da se te udaljenosti smanjuju i teže k nuli kad se točka  $T$  udaljava po grani hiperbole.

Prikazan je položaj pravca prema hiperboli. Iz točke na hiperboli može se povući jedna tangenta na tu hiperbolu. Iz točke izvan hiperbole (koja ne leži na asimptotama) mogu se povući dvije tangente na tu hiperbolu.

Figure 5. Excerpt of the study of asymptotes from the geometry chapter in textbook A (Adapted from textbook A)

The praxeology related to the tangent line in the exercise section of textbook B involved a task to determine the tangent lines to a hyperbola that pass through a given point lying on its asymptote. Solving the task requires manipulating algebraic expressions obtained by inserting the data into the general tangency condition formula  $k^2a^2 - b^2 = l^2$  for the hyperbola  $b^2x^2 - a^2y^2 = a^2b^2$ , and posing and solving a system of equations for the coefficients  $k$  and  $l$  of the equation of the tangent line  $y = kx + l$  through the point  $(x_0, y_0)$  on the hyperbola. Justification of the solution—one of the obtained tangent lines is actually the asymptote—relies on a relevant property of second-order curves: through any point outside the curve there are two tangent lines to the curve, and particularly, that the asymptote of a hyperbola is its tangent line at the infinitely distant point of a hyperbola. The corresponding praxeology can be summarised as follows:

- Type of task: determine tangent lines to the hyperbola through given point;
- Technique: manipulate algebraic expressions related to tangency condition formula;

- a. Euc. distance def. *Asymptotes*  
Verbal ostensive *Let point  $T$  move continuously on the graph  $\Gamma_f$  of a function  $f$  so that at least one of its coordinates tends to  $+\infty$  or  $-\infty$ . If its distance to the line  $p$  tends to zero, then the line  $p$  is called an asymptote of the function  $f$ .*  
Institutionalisation moment
- b. Limit charact. *Vertical asymptotes*  
Verbal, graphical and symbolic ostensive *If for a function  $f$  it holds*  
Construction of discourse moment 
$$\lim_{x \rightarrow c} f(x) = \pm\infty$$
  
*then the line  $x = c$  is its vertical asymptote.*  
*The image shows vertical asymptotes. The distance of the points from the graph of the function to the vertical asymptote tends to zero as  $x \rightarrow c$ .*  
*\*distance to the asymptote\**  
*\*vertical asymptote\**
- c. Develop\_C *The right slant asymptote is a line  $y = kx + l$  such that*  
Symbolic ostensive 
$$\lim_{x \rightarrow \infty} (f(x) - kx - l) = 0. (1)$$
  
Construction of discourse moment *[...] If the limit (1) exists, then it must be*  
$$\lim_{x \rightarrow \infty} \frac{f(x) - kx - l}{x} = \lim_{x \rightarrow \infty} \frac{f(x)}{x} - k - \frac{l}{x} = 0$$
  
*and the expression for the coefficient  $k$  derives:  $k = \lim_{x \rightarrow \infty} \frac{f(x)}{x}$ .*  
*When the  $k$  is determined, calculate the coefficient  $l$  from (1).*

**Asimptote**

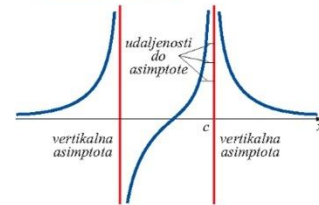
Neka se točka  $T$  neprekidno giba po grafu  $\Gamma_f$  funkcije  $f$  tako da barem jedna od njezinih koordinata teži u  $\infty$  ili  $-\infty$ . Ako pri tom njezina udaljenost do pravca  $p$  teži k nuli, onda se taj pravac naziva **asimptota** funkcije.

### 1. Vertikalne asimptote

Ako za funkciju  $f$  vrijedi

$$\lim_{x \rightarrow c} f(x) = \pm\infty,$$

onda je pravac  $x = c$  njezina **vertikalna asimptota**.



Na slici su prikazane vertikalne asimptote. Udaljenost točaka s grafa funkcije do vertikalne asimptote teži u nulu kad  $x \rightarrow c$ .

**Desna kosa asimptota** je pravac  $y = kx + l$  za koji vrijedi

$$\lim_{x \rightarrow \infty} [f(x) - kx - l] = 0. (1)$$

Ako ovakav limes postoji kad  $x \rightarrow -\infty$ , onda je pravac **lijeva kosa asimptota**.

Ako kosa asimptota postoji, kako se računaju njezini koeficijenti  $k$  i  $l$ ? Postoji li limes (1), onda pogotovo mora biti

$$\lim_{x \rightarrow \infty} \frac{f(x) - kx - l}{x} = \lim_{x \rightarrow \infty} \frac{f(x)}{x} - k - \lim_{x \rightarrow \infty} \frac{l}{x} = 0$$

pa se dobiva izraz za koeficijent  $k$ :

$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x}.$$

Sad, kad je  $k$  određen, koeficijent  $l$  računamo iz (1). Ponovimo:

**Figure 6.** Excerpt of the study of asymptotes from the calculus chapter in textbook A (Adapted from textbook A)

- Technology: be familiar with the ‘two tangent lines’ property of the second-order curves, and characterise the asymptote of a hyperbola as a tangent line at its ‘infinitely distant point’; and
- Theory: coordinate geometry, projective geometry.

The praxis part of the praxeology was a familiar activity in both textbooks, mainly limited to the case of tangent lines to a hyperbola through a point on the hyperbola, but the logos part was not included in the study of asymptotes in either textbook. Textbook A explained that the “two tangent lines” property applies to points not lying on an asymptote as part of a TLB, while textbook B employed the “tangent at an infinitely distant point” discourse within Relational characterisation in the Graph praxeology for the exponential function, as previously mentioned. Textbook A adopted a more extensive and formal approach to the organisation of the calculus chapter (Figure 6) than textbook B (Figure 3). The Develop praxeology in textbook A referenced how the formulas for the coefficients of the slant asymptote derive from the limit of a difference of a function and its asymptote, but no connection was made between Limit characterisation and Euclidean distance definition. Textbook B provided only the formulas with a limit for the slant and horizontal asymptotes (Figure 3d). In textbook A, the Algebraic characterisation included ready-made valid inferences related to polynomials in the numerator and denominator of the rational functions, e.g., that the vertical asymptotes occur at the zeros of the denominator that are not also zeros of the numerator. Textbook B only referenced obtaining the vertical asymptotes of a function from the zeros of the denominator, without further restrictions (Figure 3b).

### Ostensives Related to Asymptotes in the Textbooks

The praxeologies in the analysed textbooks engaged verbal, graphical, and symbolic ostensives related to asymptotes. We explored patterns related to the occurrence of different ostensives across chapters and textbooks and their integration in praxeologies. Verbally expressed discourses on asymptotes observed in the textbooks, in order from more to less frequent, were:

- reference to the type of an asymptote, i.e., vertical, horizontal or slant asymptote;
- potentially conflicting discourse as suggested in the literature, e.g., curve approaches but never intersects its asymptote, vertical asymptotes are related to the breakpoint in the function domain, coordinate axis is asymptote of a curve;
- appropriate discourse as proposed in the REM, e.g., Approaching line or Approximation discourse, and Euclidean distance definition of an asymptote;
- reference to the position of an asymptote in relation to the function graph, e.g., one- or two-sided asymptote, left- or right-side asymptote;

**Table 6.** Number of items corresponding to ostensives across textbook chapters

Ostensive	Discourse	Textbook A				Textbook B			Total
		EF	GC	CF	RF	EF	GC	CF	
Verbal	Type of an asymptote	5		8	2	2		4	21
	Potentially conflicting discourse	4	2	3	2	5		3	19
	Appropriate discourse	2	5	2		2	3	2	16
	Position of an asymptote			6	4			4	14
	Property related to the algebraic characterisation				6				6
	None or another verbal			3		1	1		5
Symbolic	Equation of an asymptote of a class of functions/curves	7	11	6		6	2	4	36
	Formulas with a limit for different types of asymptotes			7	3			3	13
	Breakpoint in a function domain	1		2	2	2		2	9
	Values of a function or an expression near a point or at infinity	1	3	1		1			6
	Expression related to the algebraic characterisation				3				3
	No symbolic		1	1	2		2	2	8
Graphical	Hyperbola / Linear rational function graph		6	3			2	2	13
	Exponential or logarithmic function graph	3		1		4		2	10
	Tangent or cotangent function graph	2		1		2			5
	Other			3	4			2	9
	No graph	2	6	7	3		2	3	23
Total		7	12	15	7	6	4	9	60

Textbook chapters: EF – Algebra of elementary functions, GC – Coordinate geometry of second order curves; CF – Calculus for graphing functions, RF – Calculus for graphing rational functions

- property related to the algebraic characterisation of asymptote of a rational function, e.g., given a function  $f(x) = \frac{x}{x^2-1}$ , the degree of the numerator is smaller than the degree of the denominator, hence line  $y = 0$  is its horizontal asymptote; and
- discourse related to the tangent line of a hyperbola.

Symbolic, numeric or algebraic expressions related to asymptotes observed in the textbooks, in order from more to less frequent, were:

- equation of an asymptote of a class of functions/curves, e.g., asymptotes of a hyperbola given with the formula  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  are the lines  $y = \frac{b}{a}x$  and  $y = -\frac{b}{a}x$ ;
- formulas with a limit for different types of asymptotes as proposed in the REM (Figure 2);
- breakpoint in a function domain, e.g., function  $f(x) = \log_4(x - 5)$  is defined for  $x > 5$ ;
- values of a function or an expression near a point or at infinity, e.g., for the function  $f(x) = e^{-\frac{x^2}{2}}$  it is  $f(x) \rightarrow 0$  as  $x \rightarrow \infty$ ; and
- algebraic characterisation, e.g., polynomial quotient in Figure 4.

Some verbal and symbolic ostensives for asymptotes were typically integrated within specific content, while others appeared across chapters and praxeologies (Table 6). The type and position of an asymptote were emphasised in the calculus chapters in both textbooks. Only vertical asymptotes of tangent and cotangent functions were mentioned as a type of asymptote in other chapters. The equation of an asymptote, as a part of Property of a class, was integrated in the Graph and Determine praxeologies, and TLB in the algebra chapter in both textbooks, and in the Develop and Evaluate praxeologies in the geometry chapter in textbook A (Figure 5). Formulas with a limit for different types of asymptotes contributed in the Graph and Determine praxeologies, and TLB in the calculus chapter in both textbooks. Verbal and symbolic ostensives related to algebraic characterisation of rational functions were a part of the Graph and Determine praxeologies, and TLB in the rational function chapter in textbook A.

We examined when appropriate and potentially conflicting discourses emerged within praxeologies (Table 7). The Graph and Determine praxeologies, and TLB in Algebra included both appropriate and potentially conflicting discourses related to asymptotes; mainly related to prototype elementary functions. The Evaluate and Develop praxeologies in Algebra included appropriate discourses, e.g., Approximation technology. The potentially conflicting discourse in the Determine and Graph praxeologies in Calculus mainly related to the vertical asymptotes in the breakpoint in function domain.



of a hyperbola, as a part of Property of a class (**Figure 5b**), was applied in the Graph (**Figure 5c**) and Evaluate praxeologies (**Figure 5d**). The process of study in textbook B remained at the construction and institutionalisation moments. The exception was the praxeology of determining a tangent line to a hyperbola through a point on its asymptote, discussed previously and presented at the working on technique moment.

In the calculus chapter in both textbooks, the re-encounter of the asymptote was at the working on technique moment in the Determine praxeology in Algebra via the graphs of prototype elementary functions (**Figure 3a**). The process of study in both textbooks revolved around construction and institutionalisation of the discourse followed by working on technique moments. Namely, TLB provided Limit and Algebraic characterisation, which were hence applied in the Determine and Graph praxeologies. Textbook A adopted a more formal approach in this chapter than Textbook B, and included, in the following order, the institutionalisation of the Euclidean distance definition (**Figure 6a**), the construction of discourse on the formula with a limit for a vertical asymptote supported by verbal, graphical, and symbolic ostensives (**Figure 6b**), and construction of discourse through the Develop praxeology on the formula with a limit for a slant asymptote using only the symbolic ostensive (**Figure 6c**).

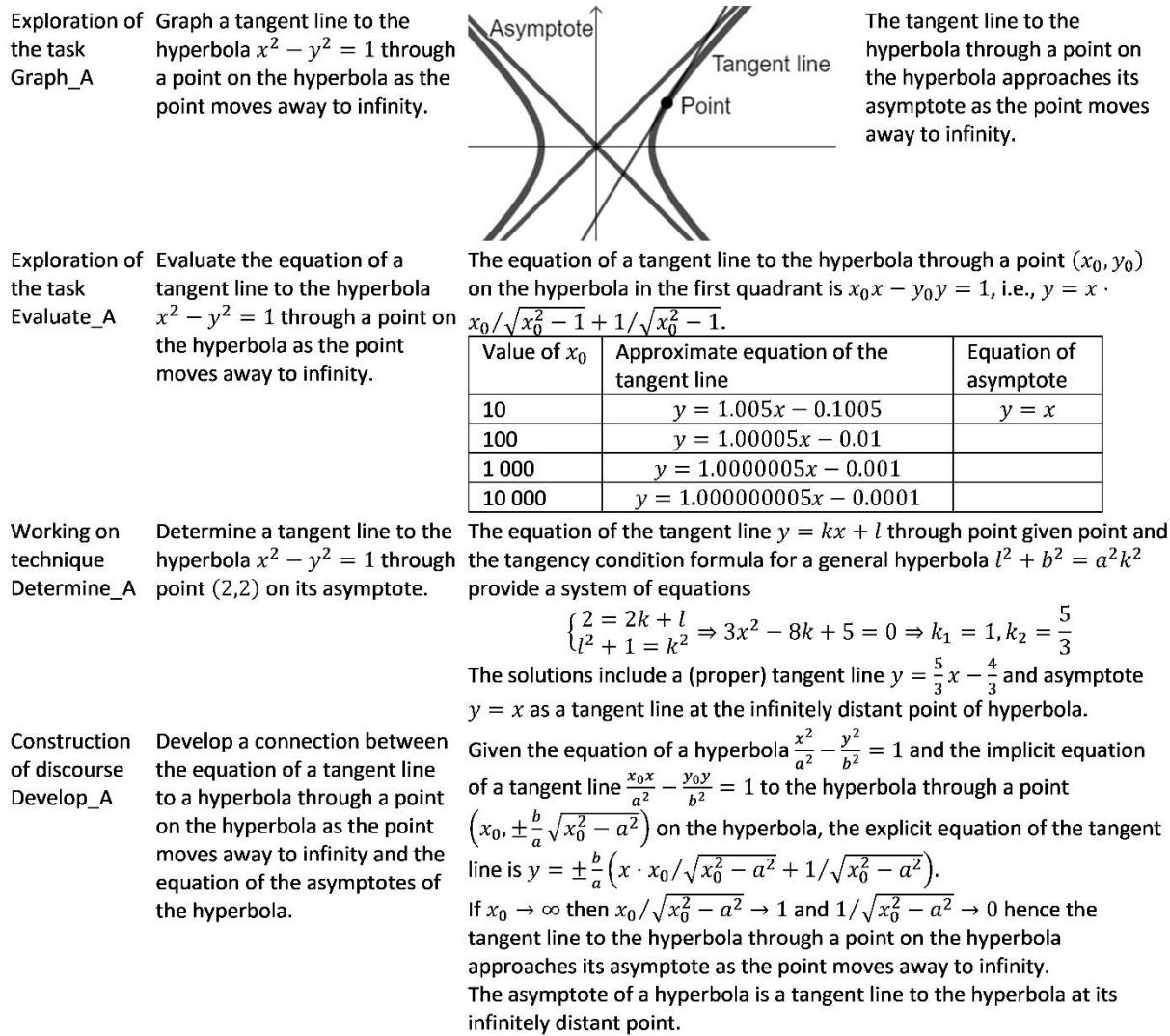
## DISCUSSION

The study of asymptotes in the analysed textbooks differed from the proposed REM for the study of asymptotes in upper secondary education (**Tables 1** and **2**) regarding the available praxeologies, implemented ostensives, and the organisation of the process of study. The textbooks focused on: (1) recognising an asymptote as a property of a graph of prototype elementary functions, namely, functions  $a^x$ ,  $\log_a x$ ,  $\tan x$  and  $\cot x$ , (2) knowing the equation of the asymptotes  $y = \pm \frac{b}{a}x$  of a hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , and (3) knowing and applying the provided ready-made rules or formulas to determine different types of asymptotes of a (rational) function.

The praxeologies observed in the analysed textbooks corresponded, to some extent, to those proposed in the REM. There were items in both textbooks corresponding to graphing an elementary function and discussing the existence of its asymptotes, and determining the asymptotes of the graph of an elementary function. Evaluating the formulas with a limit to determine different types of asymptotes and applying 'tools' from Calculus to graph a function were significant contributions in both textbooks. Items corresponding to evaluating an algebraic expression related to asymptotes, developing the formula for an asymptote of a particular class of curves or formulas with a limit were rare, while items corresponding to evaluating function values or limits of a function related to asymptotes were missing. *There was no evidence of connecting praxeologies through technologies and through increasing their complexity in the analysed textbooks.* The restricted focus on recognising an asymptote as a property of a graph of prototype elementary functions in Algebra and the provision of ready-made rules or formulas to determine different types of asymptotes in Calculus reduced the opportunities to develop knowledge increasing in complexity through analogies and generalisations as proposed in the REM. The omission of evaluating limits of elementary functions and connecting the resulting infinite limit at a particular point to a vertical asymptote or finite limit at infinity to a horizontal asymptote reduced the opportunity to connect familiar knowledge of asymptotes of elementary functions and the growing knowledge of limiting behaviour of functions.

The analysed textbooks included knowledge related to the tangent line of a hyperbola through a given point and the asymptotes of a general rational function, which were not explicitly included in the proposed REM. The rules to determine different types of asymptotes of general rational functions are beneficial within the proposed REM as the part of the Property of a class of functions technology in Algebra (**Table 1**). Rational functions, as a class of functions, provide a rich environment to explore asymptotes and limiting behaviour (Katalenić et al., 2023; Nair, 2010; Yerushalmy, 1997). Establishing grounded knowledge of asymptotes of rational functions in Algebra provides the means for a stronger connection to the knowledge of asymptotes in Calculus, e.g., justifying the first-degree polynomial quotient of the numerator and denominator of a rational function as its slant asymptote by evaluating the corresponding limit of difference as in **Figure 4**.

The connection between a tangent line and asymptote of a hyperbola originates from projective geometry, which is beyond the scope of upper secondary level mathematics. An informal reference to the asymptote as the tangent line at an infinitely distant point of the curve is attainable through the praxeological organisation proposed in the REM (**Figure 9**). Exploration of the tangent line to a hyperbola through a point on the hyperbola as it moves away to infinity by graphing the curve and the line or evaluating the equations of tangent lines and asymptotes motivates the appropriate discussion, i.e., the tangent line approaches the asymptote or the equation of the tangent line approximates the equation of the asymptote of hyperbola. Working on determining the tangent lines to the hyperbola through a point on its asymptote motivates the construction of generalised discourse, that the asymptote is a tangent line to the hyperbola at its infinitely distant point. Furthermore, the discourse can be justified by evaluating the limit of the explicit equation of a tangent line at infinity. *We propose that the underlying structure of our REM, described as exploring different ostensives, determining by analogy, and developing by generalisation, can be transposed to the study of other properties of functions and curves.*



**Figure 9.** Study of the “tangent line at infinity” aligned with the exploration-analogous work-generalised discourse structure (Source: Authors’ own elaboration)

The REM and the analysed textbooks differed in the produced and utilised ostensives. When discussing the asymptotes of graphs of prototype elementary functions, textbooks equally emphasised appropriate discourses, e.g., “the curve approaches its asymptote”, and context-dependent discourses, e.g., “the abscissa axis is the asymptote of the prototype exponential function” or “the breakpoint in the tangent function domain is its vertical asymptote”. The latter discourses were deemed potentially conflicting according to the literature. Graphical ostensives for asymptotes almost always showed a curve approaching and never intersecting its asymptote, and the connection between the vertical asymptote and zero of the denominator or breakpoint in the function domain manifested often in verbal and symbolic ostensives, further strengthening the potentially conflicting discourses. *The ostensives used in the textbooks contribute to the known issues in the knowledge of asymptotes, thus hindering the development of comprehensive knowledge of asymptotes.* The emphasis on the prototype elementary functions and context-dependent discourses presented with different ostensives could explain the transference of the potentially conflicting discourses to the general knowledge of asymptotes.

In the textbooks, ostensives were mainly used in the following manner: graphical ostensive as the result of graphing functions, symbolic ostensive as the delivery of the recognised equations of asymptotes of prototype elementary functions and hyperbola and ready-made formulas with a limit, and verbal ostensive as the description of the asymptotes of graphs of prototype elementary functions and hyperbola, and as the rules to determine asymptotes of rational functions. *There were few opportunities to functionally utilise and link ostensives to develop connected and consistent knowledge on asymptotes.* The graphical ostensive could support the connection between the definition and the limit characterisation of an asymptote, as proposed for different types of asymptotes in the REM (Figure 2), which was observed only for the vertical asymptote in one of the textbooks (Figure 6b). The symbolic ostensive could support evaluating the function at a particular point or at infinity and developing the discourse on approximation. Kidron (2011) found that focusing on the numerical ostensive aided a student to move away from the conflicting discourse about asymptotes, i.e., that the curve approaches but never reaches its asymptote. Implementing the symbolic ostensive when evaluating a function and discussing approximation related to the asymptotes at the upper secondary level provides grounds for university level knowledge of asymptotic equivalence in computational mathematics (Katalenić et al., 2023). There were no items related to the suggested ostensive in the analysed textbooks.

Two patterns were discerned in the organisation of the study process in the analysed textbooks: providing examples then exposing discourse, or providing discourse then applying it in tasks. The example-discourse pattern revolved around exploration of the type of task to graph a prototype function or curve, and recognising and describing an asymptote as its property in Algebra. The discourse-application pattern revolved around presenting formulas or rules related to different types of asymptotes and applying them to determine asymptotes when working on graphing functions in Calculus. *The organisation of the process of study in the textbooks did not align with the process of study in mathematics and mathematics education.* In mathematics, examples provide grounds for making a mathematical assumption regarding a particular situation. Working on analogous situations supports the construction of a generalised statement. Upon justification of the statement by appropriate argumentation or proof, it becomes part of the institutionalised mathematical knowledge. Contemporary mathematics education emphasises the problem-solving and modelling approach in the study of mathematical knowledge, as opposed to the *applicationism* approach, described as exposing mathematical knowledge expected to be used as given (Hochmuth & Peters, 2022; Winsløw et al., 2014). The applicationism approach revolves around institutionalisation and working on technique moments. Omitting or downplaying the exploration of a type of task and construction of discourse moments in the study process determines the role of students as passive receivers of knowledge rather than active participants in the construction of knowledge.

Considering the results of the textbook analysis, the proposed REM can be considered a feasible model for the comprehensive, connected, and consistent study of asymptotes in upper secondary education. Though the focus in the textbooks skewed towards asymptotes of prototype elementary functions and curves and calculation of formulas with a limit for different types of asymptotes, we traced evidence of the praxeologies, discourses, and ostensives suggested in the REM appearing in the textbooks. This was particularly the case for the praxeological organisation in textbook A, which included different praxeologies and discourses related to asymptotes, and provided opportunities for constructing discursive elements related to the formulas for asymptotes. Further, textbook A supported discourses and ostensives more cautiously against raising issues in the knowledge of asymptotes than textbook B. The organisation of the study process observed in the textbooks can be complemented and rearranged to align with the logical-deductive structure of study in mathematics, as proposed in the REM. Different ostensives for the Approaching line and Approximation discourses, and Euclidean distance definition could be implemented and connected to construct generalised rules or formulas for different types of asymptotes of classes of elementary functions or any real function.

### Limitations and Recommendations for Further Study

This study was limited to the knowledge related to asymptotes, textbook analysis, and the Croatian upper secondary education context. The conclusions should be stated cautiously with regard to generalising the results, e.g., as to whether knowledge related to other notions fails at connecting and increasing in complexity, whether textbooks as a whole share the same organisation of the study process referred to as *applicationism*, and whether similar issues arise in other educational contexts. Though no generalisations can be made, the results of the study are relevant for further consideration. Textbook analysis related to the notion of the asymptote covered a range of textbook chapters and school years; hence, a similar organisation of knowledge and study process could be expected across other textbook chapters related to algebra and calculus. The discrepancies between the proposed REM and the study in the textbooks could be related to the recognised issues in the knowledge of asymptotes, and therefore similar discrepancies could be expected in other educational contexts, and in student issues concerning their knowledge of asymptotes.

The proposed REM for the study of asymptotes in upper secondary education is an original contribution of the study. The textbook analysis supported only the feasibility of the constructed REM, i.e., its alignment with mathematical, curriculum, and didactical requirements on knowledge related to asymptotes at the upper secondary level. Further research is required to support its viability, such as an experimental study in the upper secondary mathematics classroom. Additional studies could be aimed at accommodating the proposed study process based on exploration of the type of task using different ostensives, working on analogous tasks, and constructing a generalised discourse to the study of different mathematical notions.

The results of the textbook analysis suggest two topics suitable for study and research in upper secondary mathematics education. The first is the connection between the tangent line and the asymptote of a hyperbola. The present study connects knowledge within different theories, i.e., graphing and evaluating equations of lines and curves in coordinate geometry, solving a system of linear and quadratic equations in algebra, and evaluating limits of algebraic expressions in Calculus. The second is the positional relationship between a graph and its asymptote in the case of a rational function. Reconstructing an algebraic formula of a function given the graph of the function and its asymptotes requires studying how the change in coefficients and their relation affects the function graph, beyond the rule that “vertical asymptotes are in the zeros of the denominator”. The proposed topics could be implemented as study and research paths, an inquiry-based approach proposed by ATD, wherein the teacher poses an open question, understandable but challenging, requiring students to consult media, derive answers, and ask new questions (Chevallard, 2007; García et al., 2006; Jessen, 2022). Such an approach promotes mathematical skills relevant in modern society, such as problem-solving, creativity and innovation, and communication and collaboration, as proposed in Taja-on et al. (2025).

## CONCLUSION

This study examined how asymptotes are presented in upper secondary mathematics textbooks in Croatia. The ATD provided a framework for a detailed and systematic analysis of the organisation of knowledge, the study process, and representations related to asymptotes in these textbooks. The findings indicated that the textbooks failed to connect knowledge and representations of asymptotes across different contexts, to organise the study in a mathematically and pedagogically coherent manner, and to address recognised issues related to asymptotes.

We proposed a reference model for the study of asymptotes in upper secondary education, aimed at fostering the development of connected knowledge that is increasing in complexity. This model is grounded in exploration using graphical and symbolic representations, developed through work on analogous tasks bridging multiple representations, and culminating in the construction of a generalised mathematical discourse. Prior research supports the idea that engaging with various representations and gradually building related discourses helps reduce conflicts and promotes the development of coherent, connected knowledge.

The research findings can be generalised for future studies in two distinct directions. The first involves evaluating and refining the proposed reference model for the study of asymptotes in upper secondary education. The second entails applying the underlying principles of this model—exploration using different representations, analogous work, and construction of the generalised discourse—to the development of reference models for other mathematical notions. This approach is particularly well-suited for notions that span different domains, increase in complexity, and lend themselves to representation in various forms, as is the case with asymptotes.

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