International Electronic Journal of Mathematics Education
2021, 16(3), em0650
e-ISSN: 1306-3030
https://www.iejme.com

Research Article

OPEN ACCESS

Reconstruction of the Interpretation of Geometric Diagrams of Primary School Children Based on Actions on Various Materials - A Semiotic Perspective on Actions

Lara Kristina Billion [1]*

[1] Goethe University Frankfurt, Frankfurt am Main, GERMANY
***Corresponding Author:** billion@math.uni-frankfurt.de

Citation: Billion, L. K. (2021). Reconstruction of the Interpretation of Geometric Diagrams of Primary School Children Based on Actions on Various Materials - A Semiotic Perspective on Actions. *International Electronic Journal of Mathematics Education, 16*(3), em0650. https://doi.org/10.29333/iejme/11068

ARTICLE INFO

Received: 18 Jan. 2021

Accepted: 17 May 2021

ABSTRACT

This paper adopts a semiotic perspective on mathematical learning according to Peirce, in which the actions on material arrangements are considered the bases for diagrammatic work. The focus is on the learner's actions using digital and analogue material arrangements which are the starting point for the reconstruction of the learner's mathematical interpretations. In this paper a qualitative interpretative paradigm is adopted for the reconstruction. Specifically, a semiotic specification of the context analysis according to Mayring and an adoption of Vogel, is carried out. The reconstruction of the mathematical interpretation is presented with a focus on a geometrical problem that third graders are working on. Finally, the results of several cases are compared to identify possible differences between the analysed actions when using digital and analogue material arrangements.

Keywords: semiotics, mathematics education, actions on material, mathematical interpretation of diagrams, primary school

INTRODUCTION

In mathematics education, a cognitive psychological perspective largely determines the perception of actions on the learning material and the focus is on their function as means of illustration. Following this perspective, actions on the materials will develop action-based schemes which can be transferred to new elements in recurring situations (Aebli, 1980; Floer, 1993; Lorenz, 1993; Piaget, 1998; Radatz, 1993; Wittmann, 1981). For action-based schemes to emerge, materials with certain basic mathematical patterns which act as carriers of relationships – so-called "prototypes" (Rosch, 1978; Seiler, 2001) – are of central importance. Lorenz (1993) and Aebli (1980) emphasise that by acting on such materials, the function and, thus, the relationship between elements is discovered. The intensive action on the material and the resulting "objectified" (Aebli, 1980, p. 23, translated by the author) hand-based schemes enable the learners to recognise structurally identical or advanced tasks more easily (Aebli, 1980; Lorenz, 1993).

In this paper, a semiotic perspective of actions on the material is taken in order to analyse the actions made on different materials in a more effective way. From a semiotic point of view, mathematical rules and relationships can be constructed by actions on the material itself and the actions are not merely tools for establishing action-based schemes. The focus of this paper is on the actions themselves and, thus, in a semiotic sense, on the visible actions made on the material. Based on these actions, the relationships recognised by the learners will be reconstructed. For this purpose, the context analysis according to Mayring (2014) and an adaption by Vogel (2017), is further adapted based on the semiotic perspective. The aim of this paper is to show which possibilities of reconstruction become available by the semiotic adaptation of context analysis for the investigation of mathematical interpretations. In addition, the paper aims to answer the question of whether differences in the reconstruction of diagram interpretations can be traced back to the use of different materials. The comparison of the results of the analysis will eventually lead to the formulation of implications for the use of digital and analogue materials in primary school.

Firstly, a semiotic perspective on mathematical learning with a focus on actions, according to Charles Sanders Peirce, will be presented. In this context, implications of the semiotic perspective for the view of actions on the material are determined and logical conclusions for the reconstruction of the diagram interpretation are drawn. Secondly, the *MatheMat – Mathematical Learning with materials* study is briefly introduced in which the data for the analyses in this paper were generated. The data gathering focusses on working with prompts; these are used in data processing to identify comparable scenes in the videotaped processing of different child tandems. The transcript form developed for the analysis, as well as the specification of the context analysis according to Mayring (2014) in an adaptation for educational mathematics research made by Vogel (2017), are described. Thirdly, using the learning situation *Relationship between perimeter and area* as an example, a scene for transcription is selected and the prompt whose processing is transcribed is described in more detail. Finally, one of four analyses is presented in detail and compared with the results of another three analyses. The results of the analyses are compiled and discussed in the last section.

THEORETICAL BACKGROUND – ACTIONS ON MATERIAL FROM A SEMIOTIC PERSPECTIVE

The theoretical introduction focusses on the actions made when using the materials. From a semiotic perspective, it becomes clear that actions play a central role in learning mathematics and that they can be used as a starting point for the interpretation of diagrams.

Actions from a Semiotic Point of View

From a semiotic point of view, mathematical rules and their properties arise from thinking about acting with representations[1] (Dörfler, 2015). Concerning mathematical learning, this means that learning mathematics is the activity itself. Inscriptions can be the general result of activities and, thus, they can be lines on paper, on the screen of a tablet, on the board, in the sand, or can be tactilely experienced as a material (Dörfler, 2006a; Gravemeijer, 2002). Due to the nature of the inscriptions, inscriptions can be passed on, i.e., they are mobile without losing their properties, are readable and can be combined (Latour, 2012; Schreiber, 2013).

According to Peirce, diagrams are based on a well-defined structure of inscriptions (Dörfler, 2006a, 2006b) whose purpose is "[…] to represent certain relations in such a form that it can be transformed into another form representing other relations involved in those first represented and this transformed icon can be interpreted in a symbolic statement […]" (Peirce MS [R] 339:286r).

Representations or representation systems are, therefore, not mathematical rules themselves, but rather become mathematical rules only through their use and the actions taken with these representations. Dörfler (1988) understands mathematical rules as relationships that become clear through acting with the diagrams[2]. Some operations can be translated from one representation system into another representation system during the mathematical activity (Dörfler, 2015). If this translation of operations or relationships enables further mathematical reasoning and the recognition of further relationships, according to Gravemeijer (1999) this can be referred to as a model *for*. Nevertheless, each representation system is independent since the rules of the representation system are explained within the system itself (Dörfler, 2015).

As shown later in the empirical data, the change in perimeter and area of similar squares can be represented, for example, geometrically by squares of different sizes or by arithmetic patterns. In both diagrams, the same relationships can be revealed through the actions and, thus, the same mathematical rules can be recognised. The length of the perimeter of a square determined by actions on the geometric representation can be translated into an arithmetic sign. In this way, the arithmetic sign can be seen as a model *for* geometric representation. If the side length is described by 'a' and the perimeter by '4a', the algebraic representation shows fewer arithmetic signs, but more variables and, in Peirce's sense, more indices.

The appearance of the representations, i.e. the concrete fixations or inscriptions, are interchangeable (Dörfler, 2015) since they only gain their significance through rule-based action. Following the example focussed within the empirical data, it is, therefore, irrelevant whether similar squares are drawn or materialised. Dörfler (2015) introduces here a comparison to Wittgenstein's chess game: the appearance of the chess pieces has no meaning, only by the position on the board and the rules of moving do the pieces gain significance. Thus, the operation of a mathematician according to certain rules can be compared to the moving of a piece in chess, since both the role of the signs and the rules for their use are in the foreground (Dörfler, 2015; Shapiro, 1997; Thomas, 2009).

Actions and Signs

So far, Peirce's diagrammatic activity has been the focus of the semiotic perspective. For the analysis of actions on the materials, Peirce's types of signs, especially indices, are also of interest. In the semiotic sense, indices can provide indications of something absent in space or time, for example, a fingerprint can indicate the perpetrator (Krämer, 2007). They can also indicate something that is simultaneous and invisible, such as the weathervane as an indicator to the wind direction (Krämer, 2007). In Peirce's sense, the concept of indices can cover a wide variety of examples (Pape, 2007). According to Peirce, an index is a sign "[…] which refers to its object not so much because of any similarity or analogy with it, nor because it is associated with general characters which that object happens to possess, as because it is in dynamical (including spatial) connection […]" (CP 2.305).

At the same time there are also signs which are not clearly assignable. A proposal by Krämer (2007) draws attention to the distinction between indices and "tracks" (Krämer, 2007, translated by the author). Thus, in contrast to indices, tracks are always characterised by a time break and an unintentional nature, however, it becomes apparent in Krämer's explanations that a clear separation of indices and tracks is not possible (Krämer, 2007). Indices are characterised by temporal and spatial simultaneity, such as fever is an indication of an infection (Krämer, 2007). Krämer introduces a further distinction in that artificial indices are intentional and conscious; Krämer provides the example of a pointing gesture (Krämer, 2007). Natural indices, on the other hand, share their unintentional nature with tracks, such as smoke and fire (Krämer, 2007). Besides, the track of an animal can be read by

[1] According to Gravemeijer (1999, 2002), this paper assumes representations *for* and not representations *of*. This leads to a shift in thinking about modelled situations towards a focus on mathematical relationships (Gravemeijer, 1999). The changed perspective makes mathematics more accessible to learners (Gravemeijer, 2002).

[2] In this paper, diagrams in the sense of Charles Sanders Peirce are assumed. For more information, see Hoffmann (2005).

the hunter as an index which connects the past with the current presence of an animal, so that it results in a simultaneity of the hunter and the hunted (Krämer, 2007). In the semiotic sense, the sign, as the result of the action, can be seen as an index of the action (see Kadunz, 2015, 2016; Sinclair & de Freitas, 2014). Concerning the example from the empirical data given later, the arithmetic sign describing the length of the perimeter is an index to the previous action on the geometric representation. Thus, the sign, which is produced by the action, is characterised by non-simultaneity, but, as an artificial index, it is intentional and conscious since the actions are guided by objectives of the acting. In this case, the index, which refers to the actions, shows attributes of a track because of its non-simultaneity.

To learn concepts, defined relationships are observed during an action, where the view to the resulting sign reports back, if this relationship is mapped (Kadunz, 2015, 2016). The sign on the paper or screen can be referred to as the kind of action which was made to represent the relationship. However, in some situations there can be a complete separation between the action and the intended relationship; if a perpendicular is constructed by a mouse-click, the perpendicular cannot return to the action with the mouse. Only if the programme is used to vary a straight line, "and the behaviour of the drawn sign has to be interpreted again", can a connection between the signs and the relationship, which is considered in the action, be made (Kadunz, 2016, p. 35).

Actions on Various Materials

The use of digital tools may require a structural change in human activity (Dörfler, 1991). With the activities made possible and changed by the tool (e.g., the programme on the tablet), there is, inevitably, a shift in the attention of interest but also of the problems, difficulties and tasks (Dörfler, 1991). It is necessary to know the functions of the various instructions of the programme, for example, to produce a text on the tablet which can involve a "reorganisation" (Dörfler, 1991, p. 53, translated by the author) of the activity of writing. The reorganisation of activities must be defined and implemented educationally for each programme and each learning situation (Dörfler, 1991). Thus, the act of building with analogue material can mean the superimposition or juxtaposition of materials according to a certain construction plan. For the same action in GeoGebra (Hohenwarter, 2001), dragging a finger across the surface of the screen can result in the adjustment of a scrollbar, which, in turn, can affect a three-dimensional object displayed on the screen. In this context, Villamor, Willis and Wroblewski (2010) have compiled common 'touch gestures' on the tablet and resulting 'user actions'. Depending on the programme used, a movement on the screen can initiate several actions in the programme, but an action can also be initiated by several movements. The compilation by Villamor, Willis and Wroblewski (2010) is used, in a modified form, to describe the hand movements and the actions in the programme in the learning situation which is the focus in this paper.

Similar to the different modes, the Touch Gestures (Sinclair & de Freitas, 2014) are included in the analysis because they are directly linked to the impact of the programme. The touch of the screen and the subsequent manipulation of the inscriptions on the screen can be interpreted together as an action.

ACTIONS AS THE STARTING BASIS FOR THE INTERPRETATION OF DIAGRAMS

“When we perform manual actions with material things, for example in experiments, they are guided by ideas and aims and thus become integrative parts of thinking and observing.” (Dörfler, 2015, p. 43, translated by the author). The citation of Dörfler shows that actions on material arrangements are controlled by the interpretations of the learner. Through the actions, which are the expression of an individual’s interpretation process, the inscriptions can become a diagram in Peirce's sense, thus, the basis of mathematical thinking is always the sign or inscription (Dörfler, 2015). The rules for manipulation are expressed in the actions of the learners. If learners have often made manipulations and are, thus, well-versed in the use of inscriptions, they can interpret more flexibly which manipulations of the material and, thus, manipulations of the diagram are possible (Dörfler, 1988). Acting on the inscriptions consequently indicates which (mathematical) interpretation the learners make and which rules they may follow in their actions. Therefore, in a semiotic sense, the actions of the learners in the analysis can be interpreted in order to reconstruct possible diagram interpretations of the learners.

A semiotic specification of Vogel's (2017) adaptation of the context analysis according to Mayring (2014) should enable one to explicate the actions at the material arrangement by adding further transcript and video passages and, thus, to reconstruct the interpretation of learners based on their actions. The justified addition of occurring actions of the learners in the learning situation should enable one to grasp their interpretations over the course of the learning situation and, thus, ensure a solid reconstruction of the interpretation.

To reconstruct the interpretations of the diagram, modes other than the actions on the material, such as gestures and spoken language, are also used in the analysis. Huth (2018) refers to the relationship between actions and gestures and also describes the relationship between gestures and spoken language while considering mode-specific possibilities of expression. The functions of gestures in mathematical learning situations can be identified as the structuring of discourse, the clarification of mathematical interpretations, the presentation of mathematical ideas and the display of possible and impossible manipulations of diagrams and their inscriptional-diagrammatic use (Huth, 2020; Vogel & Huth, 2020). Thus, gestures, as well as actions, can indicate the interpretation of the diagram by the learners. Gestures that are similar to the actions or that clearly show manipulations on the diagram are also considered in the analysis due to their proximity to the actions on the material. Even spoken language that is expressed simultaneously with the action on the material, or which directly refers to the actions, is included.

METHODS AND METHODOLOGY

In the following, the data gathering and preparation for transcription are described. For the reconstruction of the interpretations of the diagrams, based on the actions made by the learners on the diagrams, the context analysis according to Mayring (2014) and the adaptation made by Vogel (2017) are specified.

Methods for Data Gathering

The research focus of the *MatheMat – Mathematical learning with materials* study is on the actions of the learners on different materials (digital and analogue materials). For this purpose of qualitative research, learning situations will be designed whose mathematical content and tasks are identical, but which will be realised once with digital and once with analogue material (Billion, 2018). In terms of the qualitative approach, these developed learning situations form the basis for the data collection and analysis. In four of the designed learning situations, primary school children deal with geometrical problems and, in four others, with stochastic problems. In the cross-sectional study a total of 32 third and fourth-grade students work on learning situations with different materials (digital and analogue). The learners always work together in tandem in a learning situation. Care is taken to ensure that a stronger student works together with a weaker one in mathematics. The assessment of achievement in mathematics is made by the respective teacher. Each tandem works, on the one hand, in a learning situation with digital material and, on the other hand, in one with analogue material. Besides, they each work on a stochastic and a geometric problem. The learners work on the problem in the time frame of one school lesson. The work is videotaped so that video excerpts can be transcribed for analysis.

All learning situations are initiated by so-called prompts; these are presented on paper cards on the table in front of the learners. "[…] Prompts are defined as recall and/or performance aids, which vary from general questions (e.g., “what is your plan?”) to explicit execution instructions […]” (Bannert, 2009). An example of a prompt from the field of mathematical learning could be *Calculate first and then check*. Prompts can stimulate cognitive, metacognitive, motivational and cooperative activities (Bannert, 2009). The prompts used in the learning situations of the *MatheMat* study mainly include the stimulation of cognitive and metacognitive activities: supporting information, questions to stimulate the processing and requests to stimulate the organisation of the learning activities. The learners can freely choose the order in which the prompts are processed. If the learners do not understand one prompt, they can put the processing of this prompt at the back and select another one. The learning situation on which this paper focusses is the relationship between area and perimeter for similar squares. Different thematic aspects are spotlighted when processing the following prompts: determining the area, determining the perimeter and the relationship between area and perimeter. The prompts for a thematic aspect have different linguistic levels in the description and are aimed at a different scope of results. The linguistic level is based on the different language registers of everyday language, educational language and technical language (Meyer & Prediger, 2012), but also on word, sentence and text levels (Wessel, Büchter, & Prediger, 2018). The different range of results is reflected in the varying number of squares to be examined. Based on these two characteristics of the prompts (linguistic level and mathematical thematic aspect), all prompts can be described in a cross table (**Table 1**).

Table 1. Description of the prompts in a cross table

	Level 1: Formulations in a table	**Level 2: Formulation of examples**	**Level 3: General formulations**
Perimeter	In the first column, squares with side lengths 1-5 are given. In the second column, the number of unit lengths on one side and in the third column the number of unit lengths on all sides should be entered.	Only those squares which are twice and three times as large as the unit square shall be considered. The question is, how many unit lengths are needed to be added to all sides of the squares.	The question is, how does the number of unit lengths on all sides change when the side length of the squares increase.
Area	In the first column, squares with side lengths 1-5 are given. In the second column, the number of unit squares which are needed to lay out the respective area of the square, should be entered.	Only those squares which are twice and three times as large as the unit square shall be considered. The question is, how many unit squares are needed to lay out these squares.	The question is, how does the number of unit squares change, when the side length of the squares increase.
Relationship between perimeter and area	In the first column, squares with side lengths 1-6 are given. In the second column, the number of unit squares (area) and, in the third column, the number of unit lengths on all sides (perimeter) should be entered.	___________	The question is about a pattern of changing the number of unit squares and unit lengths when the side length of the squares increases.

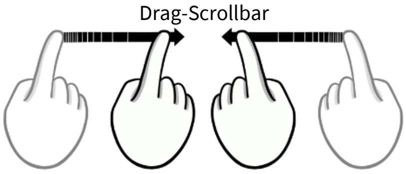
Horizontally, the linguistic level of the prompts is recorded, which increases from left to right. The thematic aspects of the prompts are arranged vertically. Due to the free choice of the processing of the prompts, there is a different processing sequence for each tandem. These sequences can be displayed within this cross table. In this way, each prompt can be assigned a processing number, which indicates at which point in the processing the prompt is processed. By connecting the processing numbers, the path of the processing sequence of a tandem can be displayed graphically (see **Table 3a**, **3b** and **Table 4a**, **4b**). With the help of the graphical representation, it is possible to find places in the processing where two or more tandems have the same processing number at the same prompt. To be able to compare the learner's interpretation of the diagram when working with digital and analogue diagrams, it is suitable to start the analysis at such a point in the process and then extend it to the entire process. It is possible that the order in which the children process the prompts can influence the actions on the diagram and, thus, the learner's interpretation of the diagrams to be reconstructed from their actions.

Methods for Data Preparation

In the transcript, all multimodal utterances of the learners are reproduced in detail. The focus is primarily on the actions on the material and, thus, on the manipulations of the diagram. As already discussed, other modes, such as spoken language and gestures, can refer to previous actions and are, therefore, also relevant for the reconstruction of the interpretations of the diagrams. For this purpose, a transcript form is developed that divides the videotaped processing into scenes. In this way, the multimodal utterances within a scene can be displayed in a coordinated manner. Every spoken utterance is marked with letters, while the actions and gestures that take place at the same time are also marked with the same letter. To distinguish gestures and actions from spoken language, they are numbered. The numbering is started anew in each scene.

The transcription of the actions on the digital and analogue diagrams differ. During the actions on the analogue diagram, all movements of the arms and hands are shown in detail, with or without material. When working with the digital diagram, the movements of the fingers on the tablet and the resulting manipulation in the GeoGebra scenario are described. The *Touch Gesture Reference Guide* compiled by Villamor, Willis and Wroblewski (2010) is adapted for the transcription of movements on the screen and their effects in the programme. **Table 2** shows the relevant movements on the screen and the resulting manipulations in the programme for the GeoGebra scenario presented in the paper. In contrast to other transcripts (Billion & Vogel, 2020a), only one movement on the screen is relevant for the transcription presented in the paper.

Table 2. Actions on the digital diagram[3]

Movement on the screen	Description of the movement	Manipulation in the programme
Drag-Scrollbar 	"Move fingertip over scrollbar without losing contact."	Adjusting the scrollbar

Methods for Data Analysis

For the reconstruction of the interpretations of the diagrams, the adaption of the context analysis according to Mayring (2014) made by Vogel (2017), and Vogel and Huth (2020) must be further specified (Billion & Vogel, 2020a, 2020b). In the context analysis according to Mayring (2014), the meaning of a term is worked out from its use by the participants and, thus, by the further addition of text passages. The adaptation of Vogel (2017) also includes the addition of transcript and video passages to contrast the multimodal expressions of learners with scientific mathematical concepts. Thus, in the context of the Conceptual Change approach (Carey, 1988; Posner et al., 1982), the individual mathematical concepts of the learners can be reconstructed.

To reconstruct the interpretation of the diagrams, the theoretical background of Conceptual Change used by Vogel (2017) is replaced by the diagrammaticity according to Peirce. Thus, the actions on the material, the interpretation of the diagram expressed by the action and the importance of the use through which a material arrangement becomes a diagram, come to the fore in the analysis. The quotation from Dörfler (2015, p. 43) above shows that the actions of learners express rules that allow conclusions to be drawn about the learner's interpretations. By analysing the rule-directed actions, diagrammatic interpretations of the learners can be reconstructed. The usage of the inscriptions can be employed to indicate which rules the learners take into account and, thus, how they interpret the inscriptions. Consequently, the usage of the inscriptions or signs determines the properties of the diagram. Furthermore, in context analysis, the addition of further passages, i.e., usage in context, renders the explication of a term, the reconstruction of a mathematical concept or a mathematical diagram interpretation possible. For these reasons, context analysis can be profitably linked to the semiotic theory presented above, since the usage has a special meaning in both. In this way a relationship between theory and method becomes clear.

As already pointed out in the theoretical background, further modes, such as gestures or spoken language, which refer to previous actions are taken into account in the analysis. For the reconstruction of the interpretations of the diagrams, the actions of the learners are contrasted with the relationships of the diagram resulting from rule-guided actions. Actions that can be derived from the relationships underlying the diagram can be described as the actions of persons experienced in the usage of the inscriptions. These descriptions are feasible, partly creative manipulations that can be used to determine the properties of the diagram. When contrasting the rule-directed actions and the actions of the learners, a discrepancy may become describable; this can also provide information on the interpretation of the learners.

Based on the described specifications, the following rules for the reconstruction of the learner's interpretations of the diagrams can be established (Billion & Vogel, 2020a, 2020b):

Step 1 - Determination of the evaluation unit – a passage of the transcript to be explicated: The search is for a passage in the transcript in which a mathematical (diagrammatic) action is described that is significant in this situation and which is expected to provide an answer to the research question. In this case, the passage in the transcript must be interesting for the reconstruction of the individual diagram interpretation made by a learner.

---

[3] The illustrations of the hands were created by Petra Tanopoulou. The movements on the screen, the description and the manipulations in the programme are partly documented literally, or phrased in the style of Villamor, Willis and Wroblewski (2010).

Step 2 - Explication 1 - Description of rule-guided actions based on the mathematical content of the passage in the transcript: (E1.1) Description of rule-driven mathematical actions or manipulations appropriate to the explicative passage in the transcript by persons experienced in usage, used to describe the properties of the diagram. (E1.2) Analysis of the selected passage in the transcript concerning the research question by contrasting the rule-directed actions with the actions exhibited by the learner. In the identified discrepancy, the interpretations of the focussed learner become clear and describable. (E1.3) Summary of the learner's previous mathematical interpretations from Explication 1.

Step 3 - Explication 2 - close context analysis: (E2.1) All passages in the transcript in which actions are directly related to the explicating passage in the transcript are compiled. (E2.2) Analysis of the identified passages in the transcript, that are directly related to the explicative passage in the transcript, by contrasting the rule-directed actions with the actions actions exhibited by the learner. (E2.3) Similar actions are searched for in the transcript which give further information on the interpretation by the learner. (E2.4) The description of the mathematically rule-guided actions of persons experienced in usage from Explication 1 may have to be extended at this point. (E2.5) The similar passages found in the transcript form the starting points for in-depth reconstructions of the mathematical interpretations. (E2.6) Summary of the learner's previous mathematical interpretations of the diagrams from Explication 2.

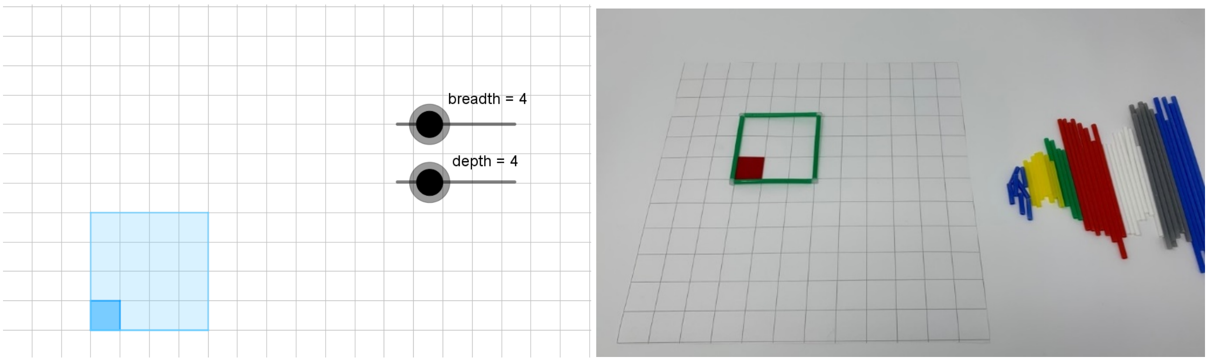
Step 4 - Explication 3 - broad context analysis: (E3.1) Further explicative material (e.g., non-transcribed excerpts from the videotaped learning situation) directly related to the explicative passage in the transcript is compiled and the relevance of this material is checked. (E3.2) Analysis of the compiled videotaped passages, which are directly related to the passage in the transcript, is carried out. The analysis may allow a more in-depth continuation of the reconstruction of the learner's interpretations of the diagrams. (E3.3) Further similar actions are searched for in the video, that are directly related to the actions from E2.3. (E3.4) The further passages from the video are the starting points for more in-depth reconstructions of the learner's interpretations of the diagrams. (E3.5) Summary of the learner's interpretations of the diagrams from Explication 3.

Step 5 - Summary: The aspects of the diagram interpretations reconstructed in the individual steps of the analysis are now described in summary.

A GEOMETRIC EXAMPLE

In this paper, a geometric example from the *MatheMat* study is presented in which learners deal with the relationship between area and perimeter of similar squares. The learning situation presented here was worked on by four student tandems from two third-grader classes of different schools. Two tandems worked with a scenario in GeoGebra (Hohenwarter, 2001), while the other two tandems worked with the material OrbiMath (Huber, 1972). In processing the prompts, the learners should make similar squares with the given materials.

To create a side model[4] of a square in the GeoGebra scenario (**Figure 1a**), learners can use their fingers to control the scrollbar in order to adjust the sides of the rectangular square to the same length. By dragging the scrollbar breadth, both sides of the dimension breadth are changed in the same way. By varying the length of the scrollbar, a connection between the visible signs and the corresponding relationship can be made (Kadunz, 2015). In the lower left-hand corner of the side model, a square with side length 1 remains visible, even during manipulations on the diagram. On the whole surface of the screen there is a flat square grid; this is divided into the size of the unit square.



(a) (b)

Figure 1. Digital (a) and analogue (b) materials for processing the described learning situation

To create a side model of a square using OrbiMath material (**Figure 1b**), learners can choose four rods of equal length from a larger number of rods of different lengths and assemble them with right-angled corner joints. Only right-angled corner joints are available. A right angle is given by the corner joint and, as in the GeoGebra scenario, the focus is exclusively on the side lengths of the two dimensions of breadth and depth. The manipulations of the OrbiMath material take place directly on the side model, unlike the digital material where the manipulation takes place on the scrollbar and the programme transfers this manipulation to the side model on the screen. In the case of the analogue material, learners also have a unit square and a square grid (which is divided into the size of the unit square) at their disposal. After the learners have created a side model of a square with the

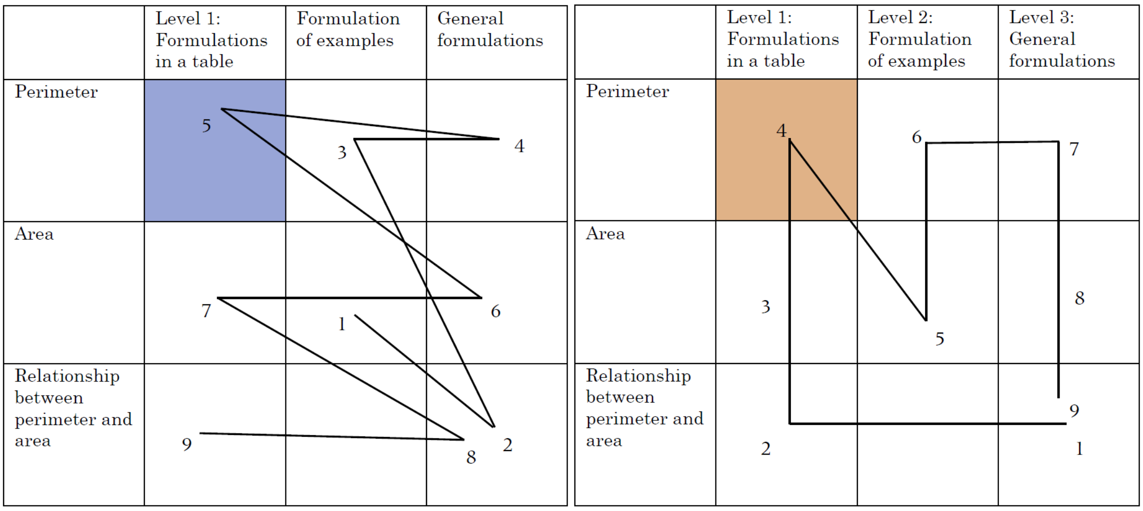
[4] In this case, a side model is understood as a flat figure in which only the sides are reproduced. The term is based on the edge model of geometric bodies.

respective material (larger than the unit square), they can use the unit square and the square grid to specify the area and perimeter in unit squares and unit lengths, respectively.

Selection of the Passages to be Transcribed

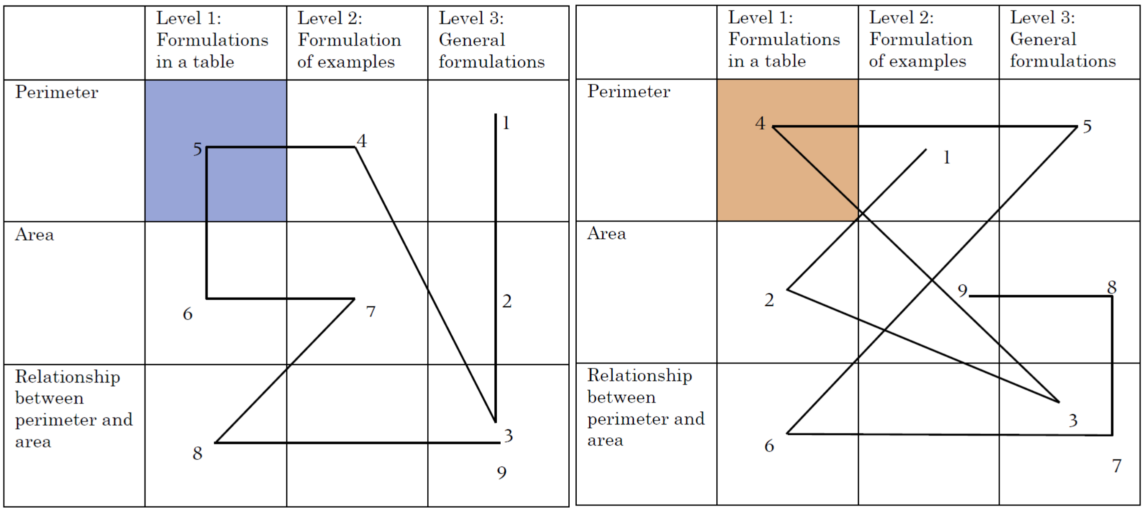
To identify comparable passages for transcription from the videotaped processing of the learners, the sequences of the processing of the prompts were graphically displayed. Tandems that process the same prompt at the same point in the processing have the same processing number in the table for the respective prompt. These passages, where two tandems (one with digital and one with analogue material) have the same processing number, are selected for transcription. In the analysis, these passages can be compared particularly well because the learners work at the same prompt at the same point in the processing and, thus, have the same experience in working with the material. It cannot be ruled out that the sequence of processing and in a semiotic sense, the experience in using the material has an influence on the actions made using the digital and analogue materials and, consequently, the reconstruction of the learner's interpretation of the diagram.

Figures 2a and **2b** show graphically the processing of two tandems when working with analogue material. It can be seen that the first tandem processes the prompt with the mathematical aspect *perimeter* and the language level 1 (*formulations in a table*) at the fifth, while the second tandem at the fourth position. In **Figures 3a** and **3b** the processing of two tandems with the digital GeoGebra scenario are shown graphically. Here it can also be seen that one tandem solves the prompt with the mathematical aspect *perimeter* and the language level 1 at the fifth position, while the other tandem solves the prompt at the fourth position of their processing. One tandem working with the digital material and one tandem working with the analogue material work on the prompt with the mathematical aspect of *perimeter* and language level 1 at the same position in the processing. The processing sequences show that it is useful to transcribe the processing of the prompt with the mathematical aspect of *perimeter* and the language level 1 of the four tandems and to analyse them comparatively. The results of this comparison of the reconstructed interpretations of the diagrams will then be used in a further step to generate a general statement.



(a) (b)

Figure 2. Processing sequences of the tandems when working with the OrbiMath material



(a) (b)

Figure 3. Processing sequence of the tandems when working with the GeoGebra scenario

Prompt with the Mathematical Aspect Perimeter and the Language Level 1

Based on the progressions of the editing process, it has been found that the processing of this prompt by all four tandems is suitable for transcription. The question on the prompt is generally aimed at changing the perimeter of similar squares. The question does not focus on special squares with a given side length. In the second step, concrete information about the side lengths of the squares is given in the table shown below the question. In the second column of the table, the learners should determine the unit lengths on one side length. In the third column, the unit lengths on all sides of the square (perimeter) should be determined (see **Table 1**). The table structures the processing of the prompt by systematically determining the unit lengths, firstly on one side and then on all sides of the square, for squares of different sizes. This prompt is intended to encourage learners to make side models of squares of different sizes and to use the side length of the unit square or square grid to determine the unit lengths on one or all sides of the square. Once the learners have determined the number of unit lengths by comparative actions on the geometric diagram, they can write the numerical value in the table using an arithmetic sign.

Perimeter – Prompt

A square with side length 1 is called a unit square. A square with side length 1 can be laid out exactly with 1 unit square. The area of the unit square is, therefore, 1 unit square. The unit square has 4 sides. Each side has the length 1. This length is called the unit length. If you put all unit lengths of a unit square together, you get a length of 4. This length is called the perimeter.

The side lengths of the squares get larger and the squares still look like squares.
How does the number of unit lengths change on one and all sides of these squares?

Fill out the table:

Square with	Number of unit lengths on one side of the square	Number of unit lengths on all sides of the square (perimeter)
side length 1 (unit square)	1	4
side length 2		
side length 3		
side length 4		
side length 5		

What do you notice?

Figure 4. Prompt with the mathematical aspect *perimeter* and the language level 1 (*formulations in the table*)

In this way, the students record the relationships of the geometric material arrangement or diagram with arithmetic signs. These inscriptions can be interpreted as a new arithmetic diagram and, thus, themselves become the focus of action and thought. The learners can discover the connections and rules of manipulation of the arithmetic pattern sequences that have been created. If the learners refer back to the geometric diagram, it can be seen that the increase of one side length by one unit length has a fourfold effect on the perimeter of the square. The interpretation of the arithmetic diagram can be reconstructed through the gestures and spoken language of the learners. Both modi refer to the act of writing down the signs necessary to create the diagram.

RECONSTRUCTION OF THE INTERPRETATION OF THE DIAGRAMS

In the following, the analysis of the actions of Nils is presented in excerpts to reconstruct his interpretations of the diagrams. Together with his partner, he works on the prompt with the mathematical aspect *perimeter* and the language level 1 at the fifth position with digital material. Subsequently, reference is made to the results of the analysis for Marleen's actions, a student who also processes this prompt as a fifth prompt but using analogue material. Finally, the results of the analysis of two learners (Emre and Li) who have worked on the same prompt at the fourth position are presented and compared.

Reconstruction of Interpretations of Diagrams when Working with Digital Material

In the first step of the context analysis, the reconstruction of Nils' mathematical interpretations of the diagrams refers to an action sequence in the created transcript (see **Table 3**). Subsequently, further passages in the transcript followed by the entire videotaped processing are included in the analysis. The focus of this analysis is the research question:

Which mathematical interpretations of the diagrams of Nils can be reconstructed based on his actions on the digital material while working on the geometrical problem?

Table 3. Transcript excerpt from processing with digital material

22:25	**Nils**	a	We already had that\
(5)	<	c	Two (*silent)*
		d	Actually we can
		e	there
		f	actually copy again \
		01	Looks at the tablet.
		02	Places the left forearm orthogonal to the edge of the table in front of him on the table top.
		03	Leads the right hand with outstretched index finger towards the tablet.
		04	Uses the right index finger to make a drag-scrollbar-movement to the left over the scrollbar *breadth*.
		05	The slider of the scrollbar takes the position of a unit length.
		06	The breadth of the square changes from five unit lengths to one unit length.
		07	Uses the right index finger to make a drag-scrollbar-movement to the left over the scrollbar *depth.*
		08	The scrollbar's slider takes the position of one unit length.
		09	The depth of the rectangle changes from five unit lengths to one unit length.
		10a	Now a square with the unit length one is visible.
		11b	Uses the right index finger to make a drag-scrollbar-movement to the right over the scrollbar *depth.*
		12b	The scrollbar's slider takes the position three unit length.
		13b	The depth of the square changes from one unit length to three unit lengths.
		14c	Uses the right index finger to make a quick drag-scrollbar-movement to the left over the scrollbar *depth*.
		15c	The scrollbar's slider takes the position of two unit lengths.
		16c	The depth of the rectangle changes from three to two unit lengths.
		17	Uses the right index finger to make a drag-scrollbar-movement to the right over the scrollbar *breadth*.
		18	The scrollbar's slider takes the position of three unit lengths.
		19	The breadth of the rectangle changes from one unit length to three unit lengths.
		20	Uses the right index finger to make a quick drag-scrollbar-movement to the left over the scrollbar *breadth*.
		21	The scrollbar's slider takes the position of two unit lengths.
		22	The breadth of the rectangle changes from three to two unit lengths.
		23	Now a square with the unit length two is visible.
		24	Looks at the prompt with the mathematical aspect *perimeter* and the language level 1 (*formulations in a table*).
		25	Removes the index finger of the right hand from the tablet.
		26d	Leads the right hand with outstretched index finger to this prompt.
		27	Touches the prompt with his index finger.
		28	Removes the left forearm from the table top.
		29	Leads the left hand with outstretched index finger to the prompt with the mathematical aspect *perimeter* and the language level 1.
		30e	Moves the index finger of his right hand up and down a few times.
		31	Touches the lower left corner of the prompt with the index finger of the left hand.
		32	Removes the right hand from the prompt with the mathematical aspect perimeter and the language level 1.
		33	Turns his gaze towards the prompt with the mathematical aspect *area* and the language level 1.
		34f	Leads the right hand to the prompt to this prompt.
		35f	Touches this prompt with his fingers.
		36	Leads the right hand back to himself.
		37	Places the right hand flat on the table top to the right of the prompt.
		38	Looks at the accompanying person.
		39	Looks at the prompt with the mathematical aspect *area* and the language level 1.
		40	Turns his gaze towards the prompt with the mathematical aspect *perimeter* and the language level 1.
		41	Bends the arms,
		42	so that the hands are facing each other.

In the transcribed processing (scenes 1-16) of the prompt with the mathematical aspect *perimeter* and the language level 1, the learners should determine the perimeter of similar squares using a scenario in GeoGebra and record their results in a table. The learners have read through the prompt and discussed with the accompanying person what the prompt is about. To start the analysis, the transcribed action of Nils in scene 5, lines 11b-16c is selected. The transcript excerpt from scene 5 displays, exclusively, the utterances of Nils so that the spoken utterance b "e x actly\" of the accompanying person is not performed. The analysis focusses on Nils' actions in combination with the phonetic utterances and gestures that match his actions.

Step 1: In the selected transcript passage (scene 5, lines 11b-16c), Nils changes the length of the square in the dimension depth at the scrollbar. He performs a drag-scrollbar-movement (see **Table 3**) to the right over the scrollbar *depth* and sets the length to 3. Shortly afterwards he makes a drag-scrollbar-movement to the left. During the second drag-scrollbar-movement, he quietly utters “two”.

Step 2 – Explication 1: (E1.1) In the learning situation, learners should examine side models of similar squares in terms of perimeter and area and their relationship. The focus of the prompt is to determine the number of unit lengths with which one or all sides of the different sized similar squares can be measured. To determine this, setting squares of different sizes in the given GeoGebra scenario would be a suitable action. In a square, all sides are of equal length and include an angle of 90°. In the GeoGebra scenario, a right angle, which is enclosed by the sides, is already given and cannot be changed. To create a square in the scenario, the sides in the dimension *breadth* and the sides in the dimension *depth* must be set to the same length. Two scrollbars, *breadth* and *depth*, allow to change or adjust the side lengths of the rectangular square. To do this, the slider of the scrollbar must be moved with the help of a finger. If the slider is moved to the left, the length of the corresponding sides of this dimension is reduced and, correspondingly, increased when moving to the right. To create a square, the sliders of both scrollbars must be set to the same number.

(E1.2) Nils moves the slider of the depth scrollbar to the right and, thus, aims at increasing the side length of this dimension. He sets the number 3. The hastily performed drag-scrollbar-movement of Nils to the left suggests that he is dissatisfied with the, possibly, unintended setting and wishes to correct it downwards. With this drag-scrollbar-movement he sets the number 2 and the side length in the dimension *depth* is reduced to 2. The phonetic utterance "two" during the drag-scrollbar-movement to the left supports this assumption. Setting the side lengths to two unit lengths is indicated by the number two in the table at the prompt. It can be assumed that he reads the sign and adjusts the scrollbar on the screen accordingly. Through his actions on the digital material and phonetic utterances, a relationship between the arithmetic diagram (the table on the prompt) and the geometrical diagram (the representations on the screen) is expressed. It can be assumed that Nils interprets both diagrams and translates relationships of the arithmetic diagram into relationships of the geometric diagram. It can also be assumed that Nils interprets the geometric diagram in such a way that the setting of the scrollbar *depth* affects the corresponding side lengths of the rectangular square and he uses this to lengthen the sides in the dimension *depth*.

(E1.3) The two actions that Nils takes on the scrollbar show that Nils consciously adjusts the slider, even if it takes him two attempts. If the desired length is not achieved, he tries to change it by another action. His actions and phonetic utterances suggest that he translates the relationships in the arithmetic diagram into relationships in the geometric diagram and recognises the diagrams as models *for* each other. It can be assumed that Nils has interpreted the rules for the lengthening of the sides into the geometrical diagram. In the narrow context analysis (below), it needs to be checked whether he uses these rules for the construction of a square.

Step 3 - Explication 2 - narrow context analysis: (E2.1) In scene 5, lines 17-23, Nils sets the slider of the scrollbar *breadth* to two unit lengths. Again, he first makes a drag-scrollbar-movement over the scrollbar *breadth* to the right to increase the side length. He sets the number 3. Afterwards, he performs another quick corrective action so that the scrollbar is set to the number 2. His manipulations on the diagram result in the construction of a square. In the same scene in lines 4-10, Nils sets the scrollbars *breadth* and *depth* to the number 1, so that, by doing this, setting a square with side length 1 can be seen on the screen. He states that they already had this square. In the following scenes (scenes 9, 11 and 14), he sets a square with side lengths 3, 4 and 5, one after the other.

(E2.2) The reconstructed diagram interpretation from Explication 1 can be supported by the further passages and extended by the transcript passages in scene 5 (lines 17-23) and the further passages in which Nils sets a square. The manipulations of the scrollbars suggest that Nils uses the geometric diagram to construct a square because he sets the depth and breadth to number 2 by his actions. When constructing the squares with a side length of 1-5, it becomes clear that he uses the signs on the prompt. The assumption can be supported that Nils translates the relationships of the arithmetic diagram into the relationships of the geometric diagram.

(E2.3) The focus of the prompt is on the relationship between side length and perimeter, so this aspect should be taken into account in similar passages. In scene 7, Nils looks at the tablet on which a square with side length 2 can be seen. When asked by the accompanying person how many unit lengths would fit onto one side, he answers "two". Additionally, in scene 9, Nils answers, after adjusting the square, that three unit lengths would fit onto one side. In scene 14, after setting the square, Nils records the unit lengths that fit onto one side of the square directly at the prompt, without any verbal utterances. In scene 8, in response to the accompanying person asking how many unit lengths fit onto all sides, Nils looks at the square with side length 2, which can be seen on the screen. Scarcely moving his head, twice up and down, he then utters "eight\". In scene 10, Nils looks at the square with side length 3, which can be seen on the screen. He asks himself the question "and (.) on all sides/". He answers his self-posed question with the statement "nine/". Immediately he revises his answer by saying "no\" while looking at the screen, which still shows the square with side length 3. He looks at the screen for 14 seconds without action, gestures or phonetic utterances. Subsequently, he pronounces "twelve/" while not taking his eyes off the screen. In scene 14, after a longer look at the screen, on which a square with a side length of 5 can be seen, Nils says "twenty/". In each scene, Nils notes the number of unit squares on one or all sides in the table on the prompt, following the consent of the accompanying person (scenes 8, 10 and 14). In scene 16, Nils notices that in the first column of the table, at the prompt, there is always one unit length added and in the second column there are always four unit lengths added.

(E2.4) At this point, the description of rule-guided mathematical actions by a person experienced in usage still needs to be supplemented. Until now, creating a side model of a square by moving the scrollbar was at the centre of the passage in the transcript to be explicated. Now the determination of the unit lengths is the focus for one side and for all sides of the square; this is also required in the prompt. To determine the perimeter, it can first be determined how many unit lengths can be used to measure one side of the side model and then how many are needed for all sides of the side model. To determine the number on one side, the lengths of the squares on the square grid bordering the sides of the square can be counted. Thus, the square grid can be interpreted as a measuring instrument. Comparative actions must be performed by the learners to determine a numerical value for the side length. The comparative actions, or the counting of the unit lengths, can be carried out gesturally. At this point, the gestural comparative actions are very closely connected with the measuring actions. In order to determine the unit lengths on all sides, the number of one side length can be added up four times, multiplied by four or counted from one side length.

The actions required to determine the number of unit lengths can be recorded as arithmetic signs on the prompt. If the numbers of unit lengths at one or all sides are expressed as arithmetic signs, they can be interpreted as an arithmetic diagram and the following relationships can be recognised: the number of unit lengths on one side always increases by one, whereas the number of unit lengths on all sides increases by four. If a reference to the geometric diagram is made, it can be seen that the enlargement of one side length affects the perimeter four times because the perimeter includes all four sides of the square. To formulate the relationships between the diagrams, the learners must recognise that the arithmetic diagram is a model *for* the geometric diagram. The interpretation of the relationships in the arithmetic diagram can be done by gestural or phonetic utterances; both modes refer to the action that was necessary to write down the arithmetic signs.

(E2.5) When determining the number of unit lengths on all sides, it can be seen that in scene 8 Nils scarcely notices his head moving up and down twice while looking at the square with side length 2 on the screen. According to de Ruiter (2000), gestures can also be performed with parts of the body other than the hands or arms, for example, when the learner sits on the hands or holds something in the hand. In this case, Nils only has a pen in his hand. It can be assumed that the head movement refers to a counting process. This suspicion can be supported by the longer periods in scenes 8, 10 and 14 (sometimes up to 17 seconds) that Nils looks at the screen. A counting process takes longer than, for example, quadrupling the unit lengths on one side of the square. However, it cannot be excluded that Nils might have added the number of unit lengths on all sides or multiplied it by four as he would need some time for this. With reference to the theories of Kadunz (2015, 2016) and Krämer (2007), the nodding of the head can be interpreted as an index or trace for the counting process. Since the counting process is an invisible sign, it can be a natural index if Nils does not want to make the counting process public. If he wishes to show the counting process to other people, this could be an artificial index. If the nodding of the head occurs after the counting process, this could be called a trace. In scene 16, Nils tries to interpret the relationships between the arithmetic signs on the prompt. It can be assumed from his phonetic utterances that he can interpret the relationships between the inscriptions of the arithmetic diagram as he notices that for the unit lengths on one side of the square, there is always one unit length added and, for the perimeter, there are always four. At this point he uses the arithmetic signs as indices to his previous actions to establish a relationship between the arithmetic and geometrical diagrams.

(E2.6) In the narrow contextual analysis, the reconstruction of Nils' interpretation, i.e., that he can interpret the rules of the geometric diagram for the extension of the sides, can be confirmed and extended. It can be assumed from his actions that he uses the rules of the geometric diagram interpreted by him to construct squares of different sizes. The further manipulations of the scrollbar's breadth and depth settings, in the same and subsequent scenes, support Nils' reconstructed diagram interpretation. In the similar passages to the explicative transcript passage, in comparison to the rule-guided mathematical actions or gestures of a person experienced in usage, it becomes clear that Nils does not use counting gestures with his hands to determine the number of unit lengths on one or all sides of the square. He just looks at the differently sized squares on the screen and, after a period of time, expresses a numerical value for the number of unit squares. It is noticeable that he looks at the screen for a shorter time for the number of unit lengths on one side than for the unit lengths on all sides. Due to the head movements and the longer time he looks at the screen, it can be assumed that he counts the unit lengths bordering the sides of the square. The head movements can be interpreted as an index or track. Nils can interpret the arithmetic signs on the prompt by naming the manipulation rules for changing the number of unit lengths. It is not evident whether Nils knows the reason for the relationships between the inscriptions of the arithmetic diagram. He recognises that the arithmetic diagram is a model *for* the geometric diagram, but does not express how the length of the perimeter changes in relation to the side length.

Step 4 - Explication 3 - broad context analysis: Due to lack of space, only the summary of the broad context analysis can be presented. For the broad context analysis, further passages from the processing that was videotaped were used. Based on the passages found in the broad context analysis, assumptions from the previous expositions can be confirmed. It can be confirmed from Explication 1 that Nils can interpret the effects of the scrollbar on the side model of the square. He transfers, as already described in Explication 2, the rules for constructing a square to the same setting of the scrollbars. The assumption that Nils counts the number of unit lengths due to the slight head movements up and down and the longer time he looks at the screen, could be supported by two video passages. In minutes 29 and 37, as in Explication 2, it can be assumed that Nils can interpret the related inscriptions on the prompt as an arithmetic diagram. In the broad context analysis, it can also be assumed why the manipulations of the geometric diagram decrease in the course of the learning situation. This can be attributed to the fact that Nils interprets the arithmetic signs on the prompt as indices of his previous actions on the geometric diagram. It can be reconstructed that he, thereby, recognises the arithmetic diagram as the model *for* the geometric diagram. On this basis, he can also interpret another arithmetic diagram at a different prompt as a model *for* the geometric diagram and, thus, establish a relationship between the two arithmetic diagrams. The relationship he recognises between the two arithmetic diagrams allows him to transfer the arithmetic signs from one arithmetic diagram to the other which makes manipulation of the geometric diagram unnecessary.

Step 5 - Summary: It can be reconstructed from Nils' actions at the scrollbar that Nils recognises the rules for lengthening the side length and interprets the relationship between the scrollbar and the square as a geometric diagram. He uses the effects of the scrollbar on the side length of the rectangular square to construct a square. He shows this by many actions on the scrollbars. In Explication 2, it can be seen that he establishes a relationship between the side length (which can be seen at the prompt in the table) and the geometric diagram. To determine the number of unit lengths on one or all sides of the square, gestures and spoken language close to the action were included. It becomes clear that Nils does not use finger-generated counting gestures to determine the number of side lengths. Based on the longer time Nils looks at the square on the screen and the barely noticeable head movements, it could be assumed in both Explication 2 and Explication 3 that Nils counts the unit lengths on all side lengths. The head movements can be interpreted as an index on or a track of the counting process. It can be assumed that Nils can interpret the arithmetic diagram after writing down the numbers of the unit squares on one or all sides of the square. He recognises that the arithmetic diagram is a model *for* the geometric diagram and uses it to talk about the geometric diagram. However, it is not clear whether Nils knows the reason for the relationship between the arithmetic signs he has recognised since he does not mention how the length of the perimeter changes in relation to the side length. In the course of the learning situation, it can be seen that the manipulations on the geometric diagram and, thus, the actions on the scrollbar decrease. It can be reconstructed that he interprets the arithmetic signs as indices to his previous actions at the geometric diagram. Therefore, it can be assumed that he recognises the arithmetic diagram as a model *for* the geometric diagram. Similarly, he can also interpret another arithmetic diagram at a different prompt as a model *for* the geometric diagram and establish a relationship between the two arithmetic diagrams. This allows him to transfer the arithmetic signs from one arithmetic diagram to another, thus, making manipulation of the geometric diagram unnecessary.

Reconstruction of Interpretations of Diagrams when Working with Analogue Material

It could be shown that two tandems processed the prompt with the mathematical aspect *perimeter* and the language level 1 in the fifth position of their processing. Firstly, Nils' diagram interpretation using digital materials was reconstructed. In the following, the diagram interpretation of Marleen is reconstructed, who processed the same prompt but with analogue material. Only the last step of the analysis (Step 5 - Summary) is shown.

It can be seen from Marleen's actions that she can interpret the rules for constructing a square (putting the sticks together with corner joints) and uses them to manipulate the geometric diagram. She appears to be aware that she needs four rods of equal length to construct the square; this can be reconstructed by selecting the four rods of equal length. It can be reconstructed from her actions of putting the rods together with the corner joints, or placing them on the lines of the square grid, that her actions are based on the rule that the rods must be at right angles to each other. It becomes clear in Marleen's activity of constructing that she has established a relationship between the arithmetic diagram (the side length in the table on the prompt) and the geometrical diagram (material arrangement). To determine the number of unit lengths, she does not need the unit square, but rather the square grid as a measuring instrument. From the data, it can be seen that Marleen uses gestures to determine the number of unit lengths on all sides of the square. It is striking that the gestures of counting are very much reduced in the narrow context analysis and occur only in two passages. Instead, only her gaze is directed towards the square grid and she expresses the result verbally or relates parts of an arithmetic diagram on another prompt to the arithmetic diagram that she is working on. She tries to transfer arithmetic signs from one arithmetic diagram to the other. In the broad context analysis, it can be seen that the arithmetic signs that Marleen adopts in the narrow context analysis were determined by actions of construction and gestures of counting on the geometric diagram. Therefore, it can be assumed, that she interprets the arithmetic signs as indices to the previous action and recognises that the arithmetic diagrams are models *for* the geometric diagram. She does not always succeed in this interpretation of the arithmetical signs and she uses partial gestures of counting to determine the number of unit lengths again.

Overall, it is noticeable that when constructing the squares at the beginning of the learning situation, the spoken language does not support her actions because Marleen is talking about other topics while she is constructing the squares. Here, Marleen's geometrical diagram interpretation can be reconstructed exclusively based on her actions. As soon as she no longer performs the actions explicitly, since she transfers the arithmetic signs from one arithmetic diagram to the other, she uses her phonetic utterances to refer to the actions on the geometric diagram. It can be observed that as the number of her actions decreases, Marleen's gestures and her spoken language become more mathematical.

Comparison of the Results of the Qualitative Analysis

By comparing the empirical results of the analysis, it can be seen that both learners interpret the material arrangement as a geometric diagram. By constructing the square, both learners establish a relationship between the side length indicated on the prompt (arithmetic diagram) and the geometric diagram. It can also be seen that both can interpret the square grid and use it to determine the number of unit lengths, although Marleen makes the counting gestures public in comparison to Nils. Both learners write down arithmetic signs for the number of unit lengths on one or all sides of the square. Nils expresses an interpretation of the arithmetic signs as a diagram by referring to the change of the number of unit lengths on one or all sides. He, thus, recognises the manipulation rules of the arithmetic diagram. The comparison of the analysis results also shows that the learners interpret the arithmetic diagram as a model *for* the geometric diagram and can correlate arithmetic diagrams on different prompts. Compared to Nils, Marleen does not always succeed in this. Due to the relationship between the arithmetic diagrams recognised by Nils and Marleen, both analyses clearly show that the actions on the diagram of geometry decrease and the importance of gestures and spoken language increases. Overall, the comparison of the analyses shows that although the actions on the digital and analogue materials differ, similar diagram interpretations can be reconstructed for both learners.

Inclusion of Further Analyses

The findings from the comparison of the diagram interpretations of Marleen and Nils will be compared with the results of two other analyses. As shown in **Figures 2a**, **2b** and **Figures 3a**, **3b**, two further tandems dealt with the prompt, focussing on the mathematical aspect *perimeter* and the language level 1, at the fourth position of their processing. As in the previous analyses, the diagram interpretation of one student was reconstructed from the tandems. Thus, the actions of Li and Emre were addressed; Li worked with the analogue material, while Emre worked with the digital material.

In all analyses, it becomes clear that the learners can interpret the rules for the construction of a square in the respective geometric diagram (analogue or digital) and use suitable manipulations for the construction of squares. It can be reconstructed that the learners establish a relationship between the arithmetic and the geometrical diagrams. The learners use counting gestures that make clear a comparative action of the square grid as a measuring instrument with the side of the square, or look at the square grid in order to determine the unit lengths, although here the comparative action is not explicit. Li first makes measurements on the analogue material by measuring the length of the other rods with the rods of length 1. In the course of the learning situation, she transfers this procedure to the square grid so that she moves from the measuring actions to comparative actions. The square grid is used by the other learners as a measuring instrument from the beginning. It becomes clear that the actions during the learning situations are reduced; this can be attributed to the fact that all learners recognise the arithmetic diagram as a model *for* the geometric diagram. In this way, they can establish relationships between arithmetic diagrams and transfer arithmetic signs from one arithmetic diagram to another. This could be reconstructed in the narrow context analysis and also in the broad context analysis. The learners interpret the arithmetic signs as an arithmetic diagram and recognise how the number of unit lengths changes on the sides of similar squares. Exclusively in the reconstruction of Emre's diagram interpretation, it becomes clear that he can speak arithmetically about the geometrical diagram. By establishing a relationship between the arithmetic signs and the geometric diagram, he succeeds in describing that the length of the sides occurs four times in the perimeter. He succeeds in the flexible translations between the arithmetic and geometric diagrams.

A comparison with the results of other analyses also shows that similar diagram interpretations of the learners can be reconstructed, although two learners worked with digital materials and two with analogue materials. It can be seen that the actions are different with using the various materials, but the same relationships are expressed with the actions. Thus, as a result, the same diagram interpretations can be reconstructed.

DISCUSSION

The paper aimed to carry out a semiotic adaptation of the context analysis and to illustrate it with an example. Also, by comparing the results of the analysis, it should be shown whether different interpretations of the learners can be reconstructed when qualitatively analysing their actions on the digital or analogue material arrangement. The results presented in the comparison are to be discussed in the following and, as a result, a perspective for the use of digital and analogue materials in the mathematics lessons of primary school children will be presented. Furthermore, the limitations of the results will be addressed and a perspective will be given.

Major Findings

When reconstructing the learner's diagram interpretations, many parallels can be seen. It can be reconstructed in all analyses that the learners can interpret the relationships of the geometric diagram, can design an arithmetic model *for* the geometric diagram and establish the relationships between the arithmetic and geometry. It can be shown by the reconstruction, that the learners can understand the arithmetic diagram as a model *for* the geometric diagram and translate the relationships from one diagram to the other. This allows the learners to connect the two diagrams and also enables the learners to speak arithmetically about the geometric diagram, or the other way round. Although the different materials result in a structural change of the actions, the same diagram interpretations can be reconstructed from the different actions. Despite the different actions, the same relationships are expressed and there is a reciprocal process between actions and signs (Kadunz, 2015, 2016). Even if two learners use the scrollbar and have two more plastic rods with corner joints at their disposal, they both use rule-based manipulations where the same relationships are observed to create a square and to determine the number of unit lengths. This shows that the appearance of the inscriptions, in this case the material arrangements in this learning situation, have no influence on the children's diagram interpretations. This can be incorporated into the theory, since, according to Dörfler (2015), and Shapiro (1997), the appearance of inscriptions is interchangeable and they only acquire their significance through the relationship between the inscriptions and the activity of the people.

In all analyses, the square grid is used by the learners as a measuring instrument by performing comparative actions between the sides of the square and the square grid. Through these comparative actions, learners can record a numerical value for the number of unit lengths (Dörfler, 1988). In the reconstructions of the diagram interpretations, it becomes clear that the learners use the square grid and, therefore, counting gestures, or may simply look at the square grid to determine the number of unit lengths.

Furthermore, in all analyses, it can be reconstructed that the learners record the number of unit lengths in arithmetic signs. By using arithmetic signs, it is possible to think arithmetically about the change of the perimeter of similar squares and it is also possible to speak arithmetically about the geometrical diagram. In this way, the relationship of the geometric diagram need no longer be constructed by actions, rather the arithmetic signs can be used for arithmetic considerations in the further course of the learning situation. It also makes it possible to recognise the relationships between arithmetic diagrams on multiple prompts.

All in all, concerning the aim of the paper formulated at the beginning, it can be stated that the diagram interpretations can be reconstructed by the actions of the learners and that the analysis with the presented semiotic specifications is suitable for reconstructing diagram interpretations. It can be shown that the appearance of the inscriptions, or more precisely the design of the material, has no effect on the diagram interpretation of the learners in this learning situation. For learning mathematics with different material arrangements, decisive new insights could be gained with the help of the analyses. Digital and analogue material arrangements, or diagrams, can evoke the same mathematical interpretations in learners if one material arrangement can be interpreted as a model *for* the other and the same relationships are expressed despite different actions. For learning mathematics in primary school this means that the same learning goal can be achieved through learning situations realised with digital or analogue materials. In order to achieve the same learning goal, it is important that the learners observe the same mathematical relationships when acting on the material, even if the movements differ. This is because if the learners recognise the same mathematical relationships between the inscriptions and observe them in their actions, the appearance of the inscriptions is interchangeable.

Limitations and Outlook

In this paper, only a small amount of data from geometry and arithmetic was considered. In the future, further analyses should provide a deeper insight into the diagram interpretations of learners who are dealing with a similar mathematical task. Besides, diagram interpretations of learners working on tasks on statistical topics will be reconstructed. Further analysis will show whether the design of the material has any influence on the diagram interpretation of the learners in the statistical field.

Author contributions: All authors have sufficiently contributed to the study, and agreed with the results and conclusions.

Funding: No funding source is reported for this study.

Declaration of interest: No conflict of interest is declared by authors.

REFERENCES

Aebli, H. (1980). *Denken, das ordnen des tuns. Band 1 Kognitive aspekte der handlungstheorie* [Thinking, ordering what is done. Volume 1 Cognitive Aspects of Action Theory]. Klett.

Bannert, M. (2009). Prompting self-regulated learning through prompts. *Zeitschrift für Psychologie*, *23*(2), 139-145. https://doi.org/10.1024/1010-0652.23.2.139

Billion, L., & Vogel, R. (2020a). Grundschulkinder arbeiten digital an einem geometrischen problem – Rekonstruktion mathematischer deutungen [Elementary school children work digitally on a geometric problem - reconstruction of mathematical interpretations]. In S. Ladel, C. Schreiber, R. Rink, & D. Walter (Eds.), *Aktuelle forschungsprojekte zu digitalen medien in der primarstufe* (pp. 135-150). WTM-Verlag. https://doi.org/10.37626/GA9783959871747.0

Billion, L., & Vogel, R. (2020b). Material as an impulse for mathematical activities in primary school – A semiotic perspective on a geometric example. In M. Inprasitha, N. Changsri, & N. Boonsena (Eds), *Interim Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education* (pp. 28-36). PME.

Billion, L. (2018) Mathematical learning processes with varying types of material conditioning. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 207). PME.

Carey, S. (1988). Conceptual differences between children and adults. *Mind and Language*, *3*(3), 167-181. https://doi.org/10.1111/j.1468-0017.1988.tb00141.x

De Ruiter, J. P. (2000). The production of gesture and speech. In D. McNeill (Ed.), *Language and Gesture* (pp. 284-311). University Press. https://doi.org/10.1017/CBO9780511620850

Dörfler, W. (1988). Die genese mathematischer objekte und operationen aus handlungen als kognitive konstruktion [The genesis of mathematical objects and operations from actions as a cognitive construction]. In W. Dörfler (Ed.), *Schriftreihe Didaktik der Mathematik Band 16. Kognitive Aspekte mathematischer Begriffsentwicklung* (pp. 55-126). Verlag Hölder-Pichler-Tempsky.

Dörfler, W. (1991). Der computer als kognitives werkzeug und kognitives medium [The computer as a cognitive tool and medium]. In W. Dörfler, W. Peschek, E. Schneider, K. Wegenkittl (Eds.), *Computer – Mensch – Mathematik* (pp. 51-75). Verlag Hölder-Pichler-Tempsky.

Dörfler, W. (2006a). Diagramme und mathematikunterricht [Diagrams and math lessons]. *Journal der Mathematikdidaktik*, 3/4, 200-219. https://doi.org/10.1007/BF03339039

Dörfler, W. (2006b). Inscriptions as objects of mathematical activities. In J. Maaz & W. Schlögelmann (Eds.), *New mathematics education research and practice* (pp. 97-111). Sense Publisher. https://doi.org/10.1163/9789087903510_011

Dörfler, W. (2015). Abstrakte objekte in der mathematik [Abstract objects in mathematics]. In G. Kadunz (Ed.), *Semiotische perspektiven auf das lernen von mathematik* (pp. 33-49). Springer-Verlag. https://doi.org/10.1007/978-3-642-55177-2

Floer, J. (1993). Lernmaterialien als stützen der anschauung im arithmetischen anfangsunterricht [Learning materials as support for the intuition in arithmetic beginners lessons]. In J.-H. Lorenz (Ed.), *Anschauung und mathematik* (Band 18) (pp. 106-121). Aulis Verlag Deubner & Co KG.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155-177. https://doi.org/10.1207/s15327833mtl0102_4

Gravemeijer, K. (2002). Preamble: from models to modelling. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 7-22). Kluwer Academic Publisher. https://doi.org/10.1007/978-94-017-3194-2

Hoffmann, M. (2005). *Erkenntnisentwicklung. Ein semiotisch-pragmatischer ansatz* [Knowledge development. A semiotic-pragmatic approach]. Vittorio Klostermann.

Hohenwarter, M. (2001). *GeoGebra – Dynamic mathematics for everyone.* Austria & USA.

Huber, H. (1972). *OrbiMath. Mathematik konstruktiv* [OrbiMath. Math constructive]. Herder Verlag.

Huth, M. (2018). Die bedeutung von gestik bei der konstruktion von fachlichkeit in mathematischen gesprächen junger lernender [The importance of gesture bi the construction of technicality in mathematical conversations of young learners]. In M. Martens, K. Rabenstein, K. Bräu, M. Fetzer, H. Gersch, I. Hardy, & C. Schelle (Eds.), *Konstruktion von fachlichkeit. Ansätze, erträge und diskussionen in der empirischen unterrichtsforschung* (pp. 219-231). Verlag Julius Klinkhardt.

Huth, M. (2020). Gestische darstellungen in mathematischen interaktionen [Gestural representations in mathematical interactions]. In H.-S. Siller, W. Weigel, & J. F. Wörler (Eds.), *Beiträge zum mathematikunterricht* (pp. 1381-1384). WTM-Verlag. https://doi.org/10.37626/GA9783959871402.0

Kadunz, G. (2015). Zum verhältnis von geometrischen zeichen und argumentationen [On the relationship between geometric signs and arguments]. In G. Kadunz (Ed.), *Semiotische perspektiven auf das lernen von mathematik* (pp. 71-88). Springer-Verlag. https://doi.org/10.1007/978-3-642-55177-2

Kadunz, G. (2016). Geometry, A means of argumentation. In A. Sáenz-Ludloy & G. Kadunz (Eds.), *Semiotics as a tool for learning mathematics. How to describe the construction, visualisation, and communication of mathematical concepts* (pp. 25-42). Sense Publishers. https://doi.org/10.1007/978-94-6300-337-7

Krämer, S. (2007). Immanenz und transzendenz der spur: Über das epistemologische doppelleben der spur [Immanence and transcendence of the trace: About the epistemological double life of the trace]. In S. Krämer, W. Kogge & G. Grube (Eds.), *Spur. Spurenlesen als orientierungstechnik und wissenskunst* (pp. 155-181). Suhrkamp.

Latour, B. (2012). *Visualisation and cognition: Drawing things together*. http://worrydream.com/refs/Latour%20-%20Visualisation%20and%20Cognition.pdf

Lorenz, J.-H. (1993). Veranschaulichungsmittel in arithmetischen anfangsunterricht [Illustrative means in beginning arithmetic lessons]. In J.-H. Lorenz (Ed.), *Anschauung und mathematik* (Band 18) (pp. 122-146). Aulis Verlag Deubner & Co KG.

Mayring, Ph. (2014). *Qualitative content analysis: theoretical foundation, basic procedures and software solutions.* Klagenfurt. https://nbn-resolving.org/urn:nbn:de:0168-ssoar-395173

Meyer, M., & Prediger, S. (2012). Sprachenvielfalt im mathematikunterricht – Herausforderung, chancen und förderansätze [Language diversity in mathematics lessons - challenges, opportunities and support approaches]. *Praxis der mathematik in der schule*, *PM 54*(45), 2-9.

Pape, H. (2007). Fußabdrücke und eigennamen: Peirces theorie des relationalen kerns der bedeutung indexikalischer zeichen [Footprints and proper names: Peirce's relational core theory of the meaning of indexical signs]. In S. Krämer, W. Kogge, & G. Grube (Eds.) *Spur. Spurenlesen als orientierungstechnik und wissenskunst* (pp. 37-54). Suhrkamp.

Peirce, C. S. (1865-1909). *The logic notebook*. MS [R] 339.

Peirce, C. S. (1901). Index (in exact logic). In J. M. Baldwin (Ed.), *Dictionary of philosophy and psychology* (Vol. I, pp. 531-532). Macmillan and Co.

Piaget, J. (1998). *Der aufbau der wirklichkeit beim kinde* [The construction of reality in the child] (2nd Ed.). Klett.

Posner, G. J., Strike, K. A., Hewson, P. W., & Gertzog, W. A. (1982). Accommodation of a scientific conception: Toward a theory of conceptual change. *Science Education*, *66*(2), 211-227. https://doi.org/10.1002/sce.3730660207

Radatz, H. (1993). MARC bearbeitet aufgaben wie 72 - 59. Anmerkungen zu anschauung und verständnis im arithmetikunterricht [MARC handles tasks like 72 - 59. Notes on visualization and understanding in arithmetic lessons]. In J.-H. Lorenz (Ed.), *Anschauung und mathematik* (Band 18) (pp. 14-24). Aulis Verlag Deubner & Co KG.

Rosch, E. (1978). Principles of categorization. In E. Rosch & B. Lloyd (Eds.), *Cognition and categorization* (pp. 27-48). Erlbaum.

Schreiber, C. (2013). Semiotic processes in chat-based problem-solving situations. *Educational Studies Mathematics*, *82*(1), 51-73. https://doi.org/10.1007/s10649-012-9417-7

Seiler, T. (2001). *Begreifen und verstehen. Ein buch über begriffe und bedeutungen* [Comprehend and understand. A book about terms and meanings] (pp. 129-154). Allgemeine Wissenschaft.

Shapiro, S. (1997). *Philosophy of mathematics. Structure and ontology.* Oxford University Press.

Sinclair, N., & de Freitas, E. (2014). The haptic nature of gestures. Rethinking gesture with new multitouch digital technologies. *Gesture*, *14*(3), 351-374. https://doi.org/10.1075/gest.14.3.04sin

Thomas, R. (2009). Mathematics is not a game but… *The Mathematical Intelligencer*, *31*(1), 4-8. https://doi.org/10.1007/s00283-008-9015-9

Villamor, G., Willis, D., & Wroblewski, L. (2010). *Touch gesture reference guide*. https://www.lukew.com/ff/entry.asp?1071

Vogel, R. (2017). "wenn man da von oben guckt sieht das aus als ob …“ – Die 'dimensionslücke‘ zwischen zweidimensionaler darstellung dreidimensionaler objekte im multimodalen austausch ["When you look from above, it looks like ..." - The 'dimension gap' between two-dimensional representation of three-dimensional objects in multimodal exchange]. In M. Beck & R. Vogel (Eds.), *Geometrische aktivitäten und gespräche von kindern im blick qualitativen forschens. Mehrperspektivische ergebnisse aus den projekten erStMaL und MaKreKi* (pp. 61-75). Waxmann-Verlag.

Vogel, R., & Huth, M. (2020). Modusschnittstellen in mathematischen lernprozessen. Handlungen am material und gesten als diagrammatische tätigkeit [Mode interfaces in mathematical learning processes. Actions on material and gestures as diagrammatic activity]. In G. Kadunz (Ed.), *Zeichen und sprache im mathematikunterricht – Semiotik in theorie und praxis* (pp. 215-255). Springer Spektrum. https://doi.org/10.1007/978-3-662-61194-4

Wessel, L., Büchter, A., & Prediger, S. (2018). Weil sprache zählt – Sprachsensibel mathematikunterricht planen, durchführen und auswerten [Because language counts - language-sensitive planning, implementation and evaluation of math lessons]. *Mathematik Lehren*, *206*, 2-7.

Wittmann, E. Ch. (1981). *Grundfragen des mathematikunterrichts* [Basic questions of math class] (6th Ed.). Vieweg. https://doi.org/10.1007/978-3-322-91539-9