International Electronic Journal of Mathematics Education
2023, 18(4), em0758
e-ISSN: 1306-3030
https://www.iejme.com

MODESTUM

Research Article **OPEN ACCESS**

Real objects as a reason for mathematical reasoning – A comparison of different task settings

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Citation: Jablonski, S. (2023). Real objects as a reason for mathematical reasoning – A comparison of different task settings. *International Electronic Journal of Mathematics Education, 18*(4), em0758. https://doi.org/10.29333/iejme/13859

ARTICLE INFO

Received: 31 Jul. 2023

Accepted: 27 Oct. 2023

ABSTRACT

In this article, the goal is to describe students' mathematical reasoning in the context of different settings of problem-solving tasks. The core of the tasks are real objects, which are presented to the students with the help of photos, a 3D model or in the environment itself. With reference to the experiential learning theory and relations to problem-solving and modelling, theoretical potentials for mathematical reasoning emerge. In a qualitative study with 19 secondary school students these are empirically tested. The evaluation of the video recordings of the students' solution processes are coded with the help of qualitative content analysis, among others with references to problem-solving and linguistic categories of conclusive speech acts. The results show that mathematical reasoning can be observed especially in the work with photos and that the work with real objects generally evokes reasoning activities in the area of planning and exploration of strategies.

Keywords: argumentation, mathematical reasoning, outdoor mathematics, problem-solving, task settings

INTRODUCTION

Mathematical reasoning shows high relevance in the teaching and learning of mathematics. It is not only a central goal of mathematics teaching, but a basis for a deeper mathematics understanding of relations, too (Hanna, 2000). Still, researchers have observed a gap between this high relevance and the actual formulation of reasons and arguments on the side of the students (e.g., Lipowsky et al., 2007; Schwarzkopf, 2000). Therefore, recent initiatives focus on the question of how task design can foster reasoning processes (e.g., Stylianides et al., 2019). The article takes up this gap by aiming at the creation of task settings that involve mathematical reasoning on the side of the students.

One possible way to foster mathematical reasoning might be the involvement of the environment and real objects that can be found there. The theoretical basis for this hypothesis is *experiential learning theory* (ELT) by Kolb (1984). It emphasizes the "central role that experience plays in the learning process" (Kolb et al., 2000, p. 1). Hattie et al. (1997), in addition, highlight the importance of experiences that are gained directly in the environment, so called *out-of-class experiences*. In particular, it is not only the gaining of experiences, but also their reflections that support learning according to ELT (Kolb, 1984)–a process that might foster mathematical reasoning on the side of the students that gain and reflect experiences at real objects.

In the article, the intention to create task settings to foster mathematical reasoning is combined with the potential to gain and reflect experiences argumentatively in the work with real objects. Hereby, different representations of real objects in tasks (in the following "task settings") are possible, e.g., working with photos, three-dimensional models or outside at the actual object. Therefore, students are observed while they solve mathematical tasks in different task settings that involve different representations of reality. Hereby, their mathematical reasoning activities during the work with the real objects and their representations are analyzed. In doing so, the article targets to answer the question whether mathematical reasoning can be observed in task settings that involve different representations of real objects taken from the environment and how each of the settings can be characterized.

In the article, firstly, the theoretical basis is presented. Here, mathematical reasoning is understood following the definition of Klein (2008). In addition, a literature review on task settings involving real objects is outlined. From a theoretical point of view, an analysis of different task settings' potentials for fostering mathematical reasoning prepares the empirical observation of students

A preliminary analysis of data concerning mathematical modelling was published in "Educational Studies in Mathematics" in March 2023. This paper extends this focus by providing a detailed analysis in context of argumentation & problem-solving. Theoretical consideration as well as data analysis from this different contextual point of view is being reported for the first time here.

working in the different task settings that involve real objects. Also the introduced considerations of ELT (Kolb, 1984) are examined in more detail. Afterwards, the method and results of the empirical qualitative study with 19 secondary school students are reported. In the methodology, the sample, instruments and analysis–following a qualitative content analysis (Mayring, 2000) with reference to the argumentation categories according to Klein (2008) and phases of problem-solving according to Rott et al. (2021)–are presented. The findings give an insight in the reasoning activities of the students in different task settings, aiming at a general overview of mathematical reasoning activities being linked to tasks that involve real objects and at a comparison of the task settings.

THEORETICAL BACKGROUND

Mathematical Reasoning

Reasons and arguments are a fundamental part of human language. Following different definitions (e.g., Habermas, 1981; van Eemeren et al., 1996), reasoning can be described as a social activity, more specifically in a conversation, which mostly starts from a disagreement concerning a point of view. The participants aim at speaking for or against the point of view, trying to justify or delegitimize their statements in order to find a shared conclusion (cf. Jablonski & Ludwig, 2022).

Taking a linguistic perspective, reasoning can be seen as a part of conclusive speech acts, i.e., speech acts that aim at realizing a conclusion. In his work about argumentation, the linguist Klein (2008) provides an overview of different conclusive speech acts and their relation to each other. **Figure 1** summarizes his categorization in the context of *spiral* conclusions. In this context, spiral means that the speech acts are developed circular, involving different directions during the realization of a conclusion.

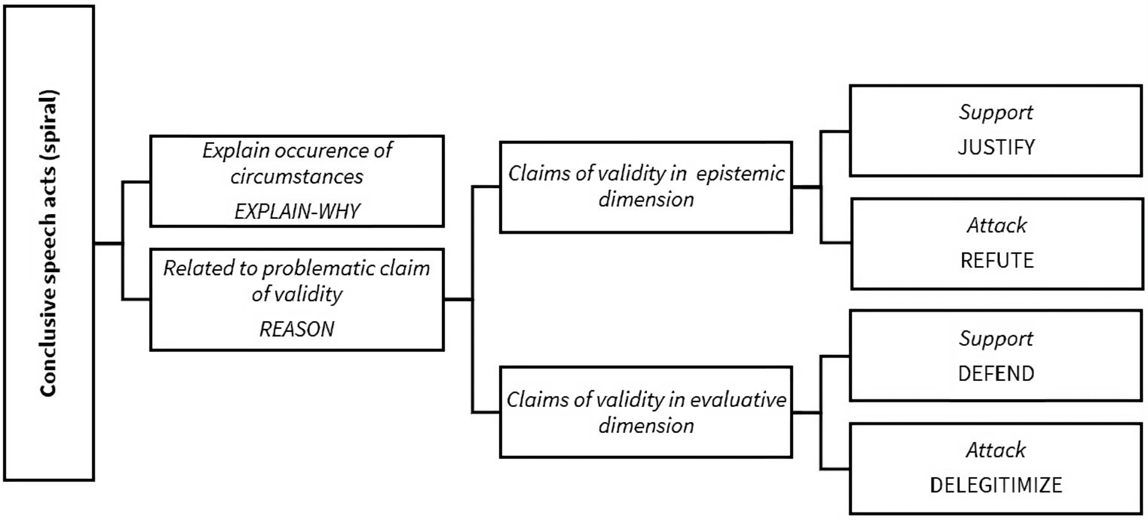

Figure 1. Categorization of conclusive speech acts (cf. Klein, 2008, p. 1316; translated by the author)

All mentioned conclusive speech acts have in common the aim of realizing a conclusion. Still, their pragmatic function in the realization of a conclusion differs (Klein, 2008). The spiral conclusions are divided into speech acts that explain the occurrence of circumstances (EXPLAIN-WHY) and speech acts that are related to a problematic claim of validity in the sense of a disagreement concerning a point of view (REASON). REASON is divided in the epistemic dimension (development of a claim of truth) and the evaluative dimension (assessment of a claim of truth). Each has a supporting and an attacking form of speech act: On the epistemic dimension, it is distinguished between JUSTIFY (supporting validity claims, holding for true/probable/possible) and REFUTE (argumentative attack on truth claims). On the evaluative dimension, the distinction contains DEFEND (support of validity claims for positive or at least non-negative evaluations) and DELEGITIMIZE (argumentative attack on positive or at least non-negative evaluations).

Whereas a problematic claim of validity can often be found in everyday reasoning about politics, work and society, it is rare in mathematics education. Often, it is not a disagreement that is the starting point of a mathematical reasoning activity (Budke & Meyer, 2015). For example, Schwarzkopf (2000) reports a lack of a need for mathematical arguments when students work mathematically, i.e., students do not initially ask themselves why a mathematical statement or discovery could be true or false. This observation legitimates the assumption that mathematical reasoning mainly involves the speech acts EXPLAIN-WHY. In addition to this potential predominance, Lipowsky et al. (2007) report a general imbalance in the turn taking during conversations in the classroom setting. Usually, it is the teacher who speaks mostly during a lesson, whereas students only formulate short answers that rarely include argumentative statements. It is therefore questionable whether conclusive speech acts are a sustainable part of mathematics classes.

Especially through the absence of a critical starting point, mathematical reasoning is strongly linked to the idea of problem-solving, i.e., by aiming at a deeper understanding and explaining of mathematical statements (Baker, 2003). In order to highlight the connection of mathematical reasoning and problem-solving in more detail, the latter is outlined in the following.

Mathematical Problem-Solving

Problem-solving in mathematics education describes the work on tasks for which students do not (yet) know a solution scheme (Schoenfeld, 1985). To nevertheless solve problem tasks, students have to proceed several steps. Many models have been created to describe, analyze and evaluate the problem-solving processes of students. In the example of Pólya's (1945) model (see left part in **Figure 2**), the four steps *understand the problem*, *create a solution plan, carry out the solution plan*, and *validate the result and procedure* divide the problem-solving process of students. Whereas the process is ideally presented as linear and the steps being clearly differentiated, Rott et al. (2021) compare and refine Pólya's (1945), Schoenfeld's (1985) and other models. Their model to describe problem-solving processes is shown in right part in **Figure 2**.

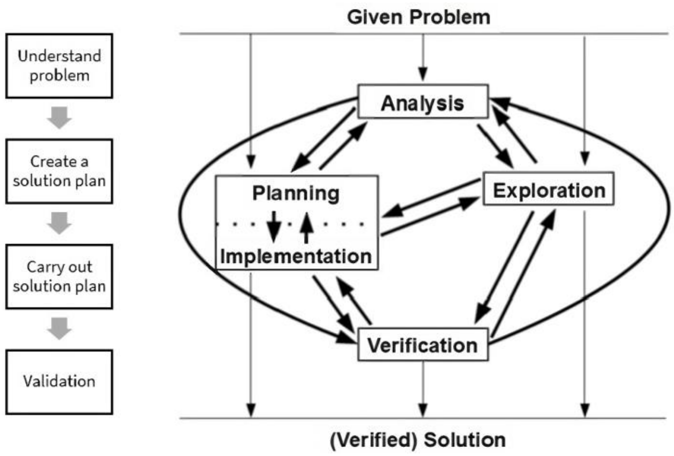

Figure 2. Models of problem-solving by Pólya (1945) (left) & Rott et al. (2021) (right)

Whereas the steps *analysis*, *planning*, *implementation*, and *verification* are directly comparable to the steps defined by Pólya (1945), the authors add an additional stage, which is called *exploration*. Already Schoenfeld (1985) breaks up Pólya's (1945) *planning* step into a structured planning phase and an unstructured exploration phase in the sense that students include heuristic strategies and try out their ideas here (Rott et al., 2021). In addition, the model of Rott et al. (2021) involves relations between the different steps in both directions, showing the non-linear approach that students usually undertake while solving problem tasks.

Mathematical reasoning is not explicitly mentioned as a step or phase in the models of problem-solving. Still the role of reasoning during problem-solving processes is emphasized (Tavsan & Pusmaz, 2021). Vice versa, Bergqvist et al. (2008) describe mathematical problems as a potential to observe mathematical reasoning since they link concepts and operations in a meaningful, logical way (Kilpatrick et al., 2001). Mulis et al. (2005) state that students reason mathematically during the analysis of the problem, the development of ideas for the solution process and in terms of validation. Especially when it comes to the development of solution strategies, mathematical reasoning is applicable in problem-solving (Boesen et al., 2010; Sumpter, 2018). Therefore, mathematical reasoning seems to be a supportive way of mathematical thinking and working in different steps of the problem-solving process.

For several years, it has been the goal of researchers to design tasks that foster mathematical reasoning activities (e.g., Stylianides et al., 2019). With the previously evaluated potentials in the context of problem-solving tasks, a possible focus has been examined. The article takes up this intention in the context of mathematical problem tasks that involve the environment and real objects. This choice is legitimated in the following.

Environment as A Part of Mathematical Tasks

Kolb (1984) claims that learning is mostly narrowed to being "a personal, internal process requiring only the limited environment of books, teacher, and classroom. Indeed, the wider 'real world' environment at times seems to be actively rejected by educational systems at all levels." (Kolb, 1984, p. 34). It is a claim that actively asks for the integration of the real world environment into education in contrast to dividing them artificially into different branches. Kolb's quote stands in the context of ELT that emphasizes the gain and reflection of own experiences as a "central role" in learning (Kolb et al., 2000, p. 1). This acquisition is related to out-of-class and first-hand experiences (Hattie et al., 1997) that link learning to the embodied cognition by which "ideas and modes of reasoning grounded in the sensory-motor system" (Lakoff & Nuñez, 2009, p. 5) are used.

In the context of mathematics education, it is, among others, possible to integrate these kind of experiences through tasks involving reality. To provide an overview for different presentations of reality and real objects in tasks, **Figure 3** summarizes the results from a literature review (cf. Jablonski, 2023). In the following, the term "setting" refers to one of these ways to involve and present a real situation or object in a task.

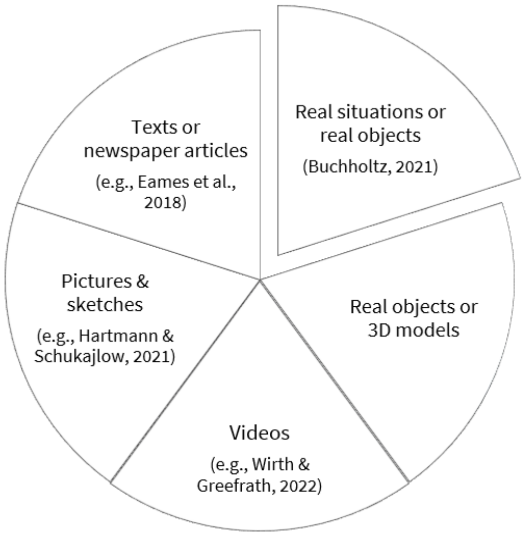

Figure 3. Different settings for a mathematical (modelling) task (Source: Author's own elaboration)

In the classroom, a real situation or object can be represented by texts and newspaper articles. This is done, for example, by Eames et al. (2018) with the volleyball problem as represented in **Figure 4** together with a table containing data from the players in 18 games. In this example, the main information about the real situation used as the task's context comes from textual representation.

The Volleyball Problem

Organizers of the volleyball camp need a way to divide the campers into fair teams. They have decided to get information from the girls' coaches – and to use information from try-out activities that will be given on the first day of the camp. The table below shows a sample of the kind of information that will be gathered from the try-out activities. Your task is to write a letter to the organizers where you: (1) describe a procedure for using information like the kind that is given below to divide more that 200 players into teams that will be fair, and (2) show how your procedures works by using it to divide these 18 girls into three fair teams.

Figure 4. An example for text/newspaper setting (Eames et al., 2018, p. 175)

In addition, photos and sketches or possibly even videos can be used to provide a visual impression of reality. Hartmann and Schukajlow (2021) use a picture of a real object together with a person as an object of reference (cf. **Figure 5**).

The Climbing Frame

How many metres of rope does the climbing frame consist of?

Figure 5. An example for photo setting (Hartmann & Schukajlow, 2021, p. 156)

In order to solve the task, "students must notice the missing information and make realistic assumptions […]. Photographs or videos can be helpful for estimating the missing information and can make the relation between the problem and the real world more obvious." (Hartmann & Schukajlow, 2021, p. 155).

In addition, smaller real objects or 3D models can be brought to the classroom and give students the possibility to touch the object despite only seeing it (Duijzer et al., 2019).

In **Figure 6**, an example of Jablonski et al. (2023) is shown. Here, students recreated Frankfurt's fair tower (cf. left part in **Figure 6**) with 3D print technology. Results (cf. right part in **Figure 6**) can be setting for mathematical tasks focusing on real object inside the classroom. Possible questions deriving from 3D print model of tower would target its original sizes by adding an object of reference.

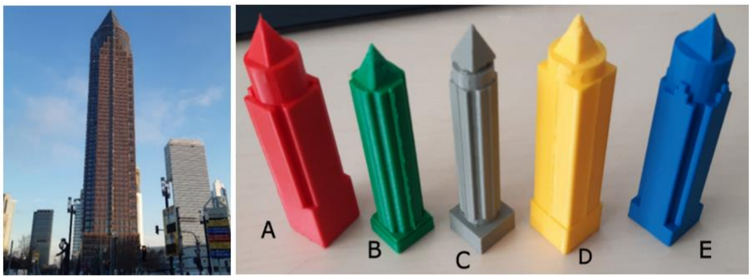

Figure 6. Fair tower in Frankfurt (left) & 3D prints as an example for 3D model setting (right) (Jablonski et al., 2023)

In **Figure 3**, the setting *real situation or real object* are emphasized since–in contrast to the previously introduced settings–the work happens outside the classroom. Students have to leave the classroom and solve the task in the environment at the real object. **Figure 7** shows the example of Buchholtz (2021) in which mathematical tasks are posed at a real object. In contrast to the photo example in **Figure 5**, it is not possible to solve the task by means of estimations from the picture since no references and only an excerpt of the object are given. To solve the task properly, it is mandatory to go to the object, see the whole situation and collect necessary data on side of the object (Buchholtz, 2021; Ludwig & Jesberg, 2015).

The well-known Peacock Fountain in Oslo.
When the fountain is turned on and the pool is filled in the spring, the city council must find out how much water the pool holds.

a) For this purpose, the perimeter of the pool must be determined.
b) Next, find the area of the pool.
c) How much water can the pool hold?

Provide your answer in m³.

Figure 7. An example for real object setting (Buchholtz, 2021, p. 143)

By analyzing the different tasks involving real objects, it becomes obvious that all of them can be linked to mathematical modelling. Its core idea is the relation between mathematics and reality (cf. Blum & Leiss, 2007). In the examples of **Figure 4**, **Figure 5**, **Figure 6**, and **Figure 7**, a real situation is given and used as the starting point of a mathematical question. To solve a modelling task, students usually start by analyzing the given real situation. Based hereon, the students follow several steps, usually arranged in a cycle, in order to proceed the transfer from reality to mathematics and vice versa as well as to validate their achieved results (Blum & Leiss, 2007).

Mathematical Reasoning in Tasks About Real Objects

It is the aim of the article to analyze students' mathematical reasoning activities during the work with real objects and their representations. By means of the former explanations, this subsection evaluates from a theoretical point of view the role of mathematical reasoning in tasks that involve real objects.

Mathematical reasoning in the context of problem-solving processes

The tasks about real objects can be characterized as problem-solving tasks. Relating the definition by Schoenfeld (1985) to the presented examples in **Figure 4**, **Figure 5**, **Figure 6**, and **Figure 7**, it can be seen that they are particularly open in the possible solution strategies. On the same side, it can be assumed that the students usually do not have a solution strategy ready to be used when the task is introduced. The example of the *photo* setting requires to estimate lengths and analyze patterns, whereas in the real object setting, the students have to firstly collect relevant data when being on side of the object. When additionally following the assumption that the students work at the tasks in small groups, the problem-solving steps *analysis* of the given task and object, *planning* of the solving process, *exploration* and *implementation* of ideas, and *verification* of the results might lead to group interactions and reasoning for and against ideas and strategies. In particular, the potential for different possible solution strategies might lead to problematic claims of validity in the groups' conversations. Therefore, with a reference to Klein (2008), especially the conclusive speech acts EXPLAIN-WHY and REASON should be observed in the context of problem-solving.

Mathematical reasoning on experiences in the context of experiential learning

Considering the role of experiences as examined in ELT (Kolb, 1984), the interaction with real objects and their representations can be highlighted for mathematical reasoning. Students gain experiences while analyzing the real object in the given representation. Working with texts, this includes to imagine the object from a given textual representation, whereas the work with pictures requires a transfer from a two-dimensional to a three-dimensional perspective, and the possibility to touch (and if possible manipulate) the 3D models or the real object itself. Still, it is not only the collection of first-hand experiences at the real objects' representations. Additionally, the students have to reflect their experiences gained by formulating mathematical statements and assumptions during the solution process (cf. Ord & Leather, 2011). This raises the potential for Klein's (2008) description of epistemic conclusive speech acts, in particular JUSTIFY and REFUTE.

Mathematical reasoning during mathematical modelling activities

Despite the importance of problem-solving processes, the tasks about real objects emphasize central ideas of mathematical modelling, too (cf. Blum & Leiss, 2007). In particular, the representation of real objects in connection to a mathematical question links mathematics and reality. Working with real objects and their representations, students have to collect the necessary data by distinguishing relevant from all given data. Working in the described settings, the amount of available data might be higher than in a "classic" schoolbook task. This is grounded in the fact that students either have to estimate relevant data from a picture or model or that they have the possibility to measure at the object (or model of the object) (cf. Ludwig & Jablonski, 2021). In the presented tasks, the students have to transfer their mathematical knowledge to (a representation of) reality. This requires reasons about the occurrence of mathematical characteristics in reality. In this reflection, a potential for conclusive speech acts, in particular EXPLAIN-WHY and REASON, arises.

STATE OF ART & RESEARCH QUESTION

In the theoretical background, the mutual influence of mathematical reasoning and mathematical problem-solving has been highlighted. Also empirical results show the importance of mathematical problem-solving for reasoning on the one hand (e.g., Mullis et al., 2005) and mathematical reasoning for solving mathematical problems on the other hand (e.g., Wyndhamn & Säljö, 1997). In the latter, the results show that discussion and collective reasoning helped students in giving more realistic answers to word problems–a finding that fits particularly with the context of the article, namely the involvement of real objects in tasks.

A few selected studies focus on mathematical reasoning in the context of tasks involving real objects and their representations. Trust and Maloy (2017) see the chance to foster mathematical reasoning skills by using and analyzing geometric relations in 3D representations of objects. Here, the usage of 3D models is closely linked to spatial reasoning, i.e., a skill in visualization that allows effective mental manipulations (Medina Herrera et al., 2019). Tall and West (1986) argue that this kind of mental manipulation can more effectively be realized by means of dynamic representations instead of texts and pictures. Also Lowrie (2002) investigated the visual and spatial reasoning of young children (on average six years old) with a real object being introduced in a 3D computer environment firstly and in a photograph secondly. He could observe that the link between two-dimensional and three-dimensional representations was not obvious to all of them. Especially the issue of interpreting perspective when working with photos is raised by Schukajlow (2013).

For tasks that are solved at real objects, the reasoning activities of students were examined in a case study (Jablonski, 2022). Primary school students were observed while they solved several tasks outdoors located at real objects. Their mathematical reasoning was analyzed concerning the related problem-solving step according to Pólya (1945) and its purpose derived from a qualitative content analysis (Mayring, 2000). The results show that the outdoor math tasks lead students to the formulation of reasons without an explicit question for reasoning. Most reasoning activities were categorized to the problem-solving step of *creating a solution plan*. Here, the students mainly reasoned about strategies that could be used to solve the task and the characteristics of the task objects. Also in the steps of *task solution* and *validation*, reasoning was observed, mainly in terms of how to do measurements and calculations. However, the study solely focused on reasoning during outdoor tasks, i.e., a comparison with other settings was not part of the case study.

In contrast, there are already a few studies that focus on the comparison of different settings. Most of these studies are quantitative in nature. The focus lies on constructs such as mathematical performance (Barlovits & Ludwig, 2023) and interest and emotion (Hartmann & Schukajlow, 2021) when comparing tasks that are solved at real objects and tasks that are solved by means of photos of the real object. In terms of a qualitative comparison of the different task settings *real object*, *photo* and *3D model*, similarities and differences concerning selected modelling steps (cf. Blum & Leiss, 2007) were examined (Jablonski, 2023). In this study, the main difference in the step *simplifying and structuring* was the students' claim to work in the most precise way when being outdoors at the real object, e.g., by changing the own perspective to the real object to analyze relevant inaccuracies. With the photo and 3D model, the students accepted not being on side of the object and therefore having to face inaccuracies in their modelling. For *mathematizing*, the students' work processes differed concerning a focus on the collection of data in the real object setting, a focus on the transfer from a two-dimensional to a three-dimensional representation in the *photo* setting, and a focus on scaling in the *3D model* setting. The following statements summarize the state of the art:

1. The role of mathematical reasoning in problem-solving tasks has been investigated and examined. Problem-solving tasks seem appropriate to foster mathematical reasoning on the one hand and mathematical reasoning seems to support problem-solving activities on the other hand.
2. The reasoning activities of students in problem tasks with real objects are rarely examined. For the work at the real object itself, it could be shown that reasoning activities can be observed, mainly when it comes to the choice of a strategy and the analysis of the real object's characteristics. A verification and the extension to the other settings, too, can be motivated through the theoretical considerations explained before.
3. The previous statement can be extended in terms of a comparison concerning the reasoning activities in the different task settings. Some comparisons already exist–still, none of them focuses on mathematical reasoning, especially from a process-oriented perspective. From the findings in the context of mathematical modelling, it can be expected that students work partly differently in the task settings. Therefore, it can be hypothesized that differences might occur in mathematical reasoning, too. Here, an empirical investigation is necessary to either verify or reject this hypothesis.

To address this research desideratum, mathematical reasoning will be understood in a broader sense and considered as an amalgamation of Klein's (2008) spiral conclusive speech acts in the following work. The following research questions aim to close the previously identified research gaps:

RQ1. At what problem-solving steps in the solution of tasks with real objects are conclusive speech acts of importance?

RQ2. Which conclusive speech acts can be observed?

RQ3. Which similarities and differences can be observed when comparing different task settings?

METHOD

The research questions are answered in the context of the study "Modelling, argumentation and problem-solving in the context of outdoor mathematics" (2021-2024; funded by Dr. Hans Messer Stiftung). It took place at Goethe University Frankfurt in Spring and Summer 2021. It aims at investigating the specialness of task settings involving different representations of real objects.

Hereby, the focus is on the task solution processes by students. For the focus *modelling*, the results can be found in Jablonski (2023), whereas the scope of this article comprises *argumentation* (in the article: reasoning) and *problem-solving*. The method of this particular focus on reasoning and problem-solving of the study is presented hereafter. Some methodological considerations (e.g., the sample and the tasks) can be found in Jablonski (2023), too.

Sample

The sample contains 19 secondary school students that volunteered for the data collection. All of them visited the enrichment program Junge Mathe-Adler Frankfurt that supports mathematically gifted and interested students in addition to their regular school education. The students were chosen from this context since it was assumed that gifted and interested students might be more willing to solve new tasks in unknown task settings. As a comparison of the solution processes in the different task settings is only possible if the students work on the task with stamina, the study happened in this context. Still, since this selection might lead to a potential positive selection, it will be discussed in the limitations section. During the data collection, the students were in grades six, seven or eight and between twelve and fourteen years old. In the study, the students were asked to solve tasks involving real objects. For this purpose, especially to foster group interactions and conversations, the students were divided in six groups of two, three or four members, depending on their availability for the study.

Tasks & Task Settings

The tasks were formulated concerning real objects that are located close to Goethe University Frankfurt. **Table 1** gives an overview of the three tasks involved in the study concerning their title, task object, formulation and one possible way of solution.

Table 1. Tasks involved in study (from left to right: *the Stone*, *the Rotanzione Sculpture*, & *the Body of Knowledge*)

	the Stone	***the Rotanzione Sculpture***	***the Body of Knowledge***
Task object & formulation			
	Determine volume of stone shown. Give result in m³.	Determine surface of sculpture (without bottom). Specify result in m².	Sculpture represents a seated person with legs drawn up. Determine height of person if he or she were to stand up. Give result in m.
Example solution	*The Stone* is modelled as a cuboid. Thus, we need to determine average length, width, & height of *the Stone*. Highest measured height is 305 cm, while *the Stone* is 245 cm high at lowest point. So, average value is 275 cm, which corresponds to 2.75 m. Analogously, we get a length of 1.60 m & a width of 1.55 m. $V=2.75\ m\times1.60\ m\times1.55\ m\approx6.8\ m^3$.	*The Rotazione Sculpture* is about 6 m high. We calculate its surface by modeling individual prongs as triangles. Legs & height of a triangle are approximately 6 m long, base about 30 cm. This results in an area of about 0.9 m² per triangle. Figure consists of 12 prongs with two triangles each. So, surface area is 21.6 m².	*The Body of Knowledge* is about 8 m high. With help of a test person, ratio between height of upper body & height of body below hips is calculated (about 0.35). Dividing height of sculpture by this ratio gives desired height of *the Body of Knowledge*. This is approximately 22.9 m.

The tasks are open as different ways of solving them are made possible. **Table 1** contains an example solution in each case, including the quantities measured on site, so that the relations and shapes of the objects can be imagined. In principle, it is quite conceivable that the students use different strategies to solve the tasks. For example, *the Stone* does not have to be assumed to be a cuboid, but could, for example, also be a composite body consisting of a cuboid and a prism. Also the approach with mean values is certainly only one of many ways to deal with unevenness on the object. Furthermore, in *the Rotazione Sculpture*, rectangles can be used instead of triangles, and the basic question here is how to consider the rotation of the prongs and how to determine the height. This question is also relevant for *the Body of Knowledge*. The approach of using a comparison person and working with proportions is also not the only way to solve the task here. For example, the sculpture could be divided into subsections so that their sum results in the total height. In all cases, it can be assumed that the students do not have a solution ready to be used and must first work it out. The starting and finishing points are therefore clear, while the path must be worked out individually. It can be assumed that the students go through the different problem-solving steps, such as *planning* and *exploration*–accordingly, the tasks can be classified as problem-solving tasks.

As shown in **Figure 3**, there are different forms of representation for tasks with real objects. Since the selected objects and questions deal with geometric content, where aspects like shape and sizes are relevant, textual forms of representation are excluded. Videos also seem to have no visible added value for fixed objects compared to photos from different perspectives. Accordingly, three task settings were created for each of the three task objects: *photos*, *3D model*, and *real object*.

In the *photo* setting, students were given three photos showing the task object from different perspectives. Furthermore, a 1.75 m tall person was positioned in front of or next to the object in the photo, so that this person could be approximated and used as a reference value (left part in **Figure 8**). Next to the photos, the students received a ruler to measure sizes in the picture.

Figure 8. *The Stone* represented in photo setting (left) & 3D model setting (right) (Pictures taken by the author)

For *3D model* setting, each of the objects was replicated by 3D printing. 3D printing technology was used at this point, as the production could be implemented in a time-saving manner and as close to reality as possible with help of a scanning app. Objects were scaled so that a LEGO figure, which was also provided, would roughly correspond to a 1.75 m tall person (see right part in **Figure 8**). Also in this setting, students were given a ruler to measure both 3D printed model and LEGO figure. For *real object* setting, students were guided to real object itself. With the help of a folding rule, they were able to solve the task directly on site.

Data Collection

Each group participated in the study for an approximately 90 minutes' session at Goethe University Frankfurt in Spring or Summer 2021. The groups got an introduction to the purpose of the session and their participation. It was especially highlighted that their thoughts and conversations were of high relevance and that the task solutions were not assessed in any way. During a group's session, the students related to this group were asked to solve three tasks with the three different objects and in the three different settings. The order of the settings was arranged systematically according to the Latin Square Design (Field & Hole, 2022). For example, group A solved *the Body of Knowledge* in the *photo* setting, *the Stone* in the *3D model* setting and *the Rotazione Sculpture* at the real object. So, each of the objects and settings was experienced exactly one time per group. To reduce the influence from one setting to another, the order of the settings was changed systematically, i.e., group B solved *the Body of Knowledge* by means of the 3D model, *the Stone* at the real object and *the Rotazione Sculpture* with photos, and so on.

The solution processes of the students were filmed by an accompanying student assistant who in addition provided the material for each task and answered smaller organizational questions. In total, the six groups solved 18 tasks. On average, a solution process took approximately 13-15 minutes independent from the setting. This results in approximately 200 minutes of video recording of the solution processes, which builds the basis for the data analysis.

Data Analysis

In a first step, the solution activities were coded deductively concerning the problem-solving steps according to Rott et al. (2021). **Table 2** gives an overview of the categories and definitions.

Table 2. Coding scheme for problem-solving steps according to Rott et al. (2021)

Problem-solving step	Definition	Coding
Analysis (A)	Understand given task, identify starting point, & goal of problem & analyze given information.	*All activities that involve* -reading & clarification of task formulation -analysis of preconditions & (missing/given) information, & -production of sketches & notes
Planning (P)	Structured phase in which a solution plan is developed, e.g., by choosing a strategy from a known problem.	*All activities that involve* -development, suggestion, & discussion of solution strategies -references to known procedures from other problems
Exploration (E)	Unstructured phase in which a solution plan is developed, e.g., by a heuristic strategy.	*All activities that involve* application of heuristic strategies to develop & test ideas for solution plan
Implementation (I)	Carry out of a solution plan/application of a strategy with aim to solve problem.	*Activities that involve* mathematical activities that aim at carrying out of a solution plan
Verification (V)	Validation of result & look back at used solution strategy with aim to evaluate procedure & result gained.	*Activities that involve* -a check of the gained solution -reflections concerning used strategies &d procedures -ideas for improvements

The coding in this step was done by two independent coders for 20 sequences, reaching a good reliability (Cohens Kappa κ=0.75). This first step results in *activity diagrams* (cf. Ärlebäck & Albarracín, 2019) for each group, as shown in **Figure 9**. In five different shades, occurrence of problem-solving steps is presented, providing information about their chronology and duration.

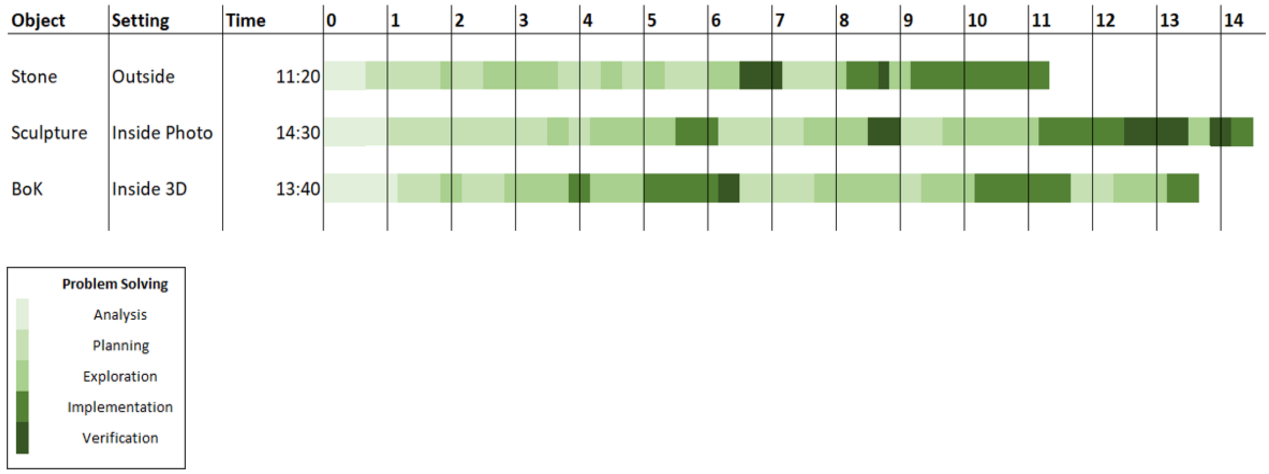

Figure 9. Activity diagram showing problem-solving steps of one group (Figure created by the author)

In the second step, relevant scenes were selected for an analysis of the conclusive speech acts. These scenes were selected, among other things, by linguistic elements and markers of conclusive acts (cf. Klein, 2008). Furthermore, this selection was extended by context-based single scenes. The diagrams were supplemented with shades at corresponding scenes (see **Figure 10**).

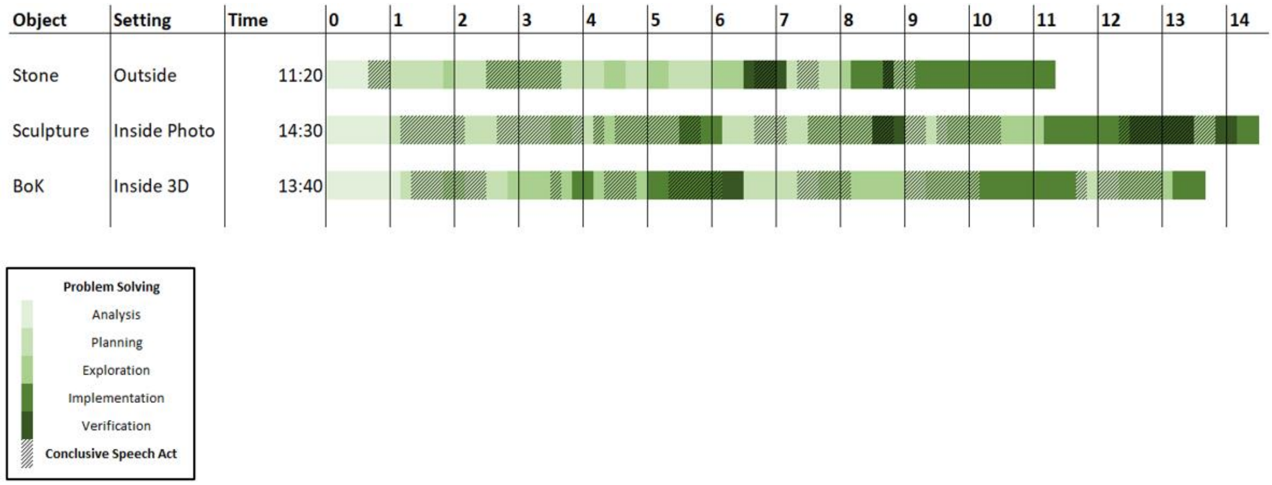

Figure 10. Activity diagram extending problem-solving steps by occurrence of conclusive speech acts of one group (Figure created by the author)

In the third step, these scenes were coded

(a) deductively by means of the speech acts (Klein, 2008) and

(b) inductively concerning the purpose of the formulated speech act (following Mayring's, 2000 qualitative content analysis).

Appendix provides an overview with the categories and examples of (a). The categories of (b) are presented in the result's section. Again, two coders rated 30 sequences and reached a substantial reliability (Cohens Kappa κ=0.73 for (a) and κ=0.7 for (b)).

RESULTS

Overview of Problem-Solving Steps & Conclusive Speech Acts

The results of the first (problem-solving steps) and second (conclusive speech acts) coding steps are presented in **Figure 11**. For a general overview concerning their occurrence, a quantitative presentation of the step's and thereof conclusive speech acts' duration is given. Left part in **Figure 11** shows distribution of individual problem-solving steps' duration for 18 solution processes of the six groups. It can be seen that planning and exploration steps, in particular, take the most time to complete, averaging at 34% and 28% of time. It is followed by the implementation (22%), verification (12%) and analysis (5%) step. If we now look at the occurrence of the conclusive speech acts in right part in **Figure 11** under the condition that a problem-solving step was assigned, the most conclusive speech acts are also produced in the two steps *planning* and *exploration*. More than half of the time students spend in this step involves conclusive speech acts. Since steps are most frequently observed and include the longest parts of conclusive speech acts, combination of both is the highest in comparison to other steps and conclusive speech acts therein.

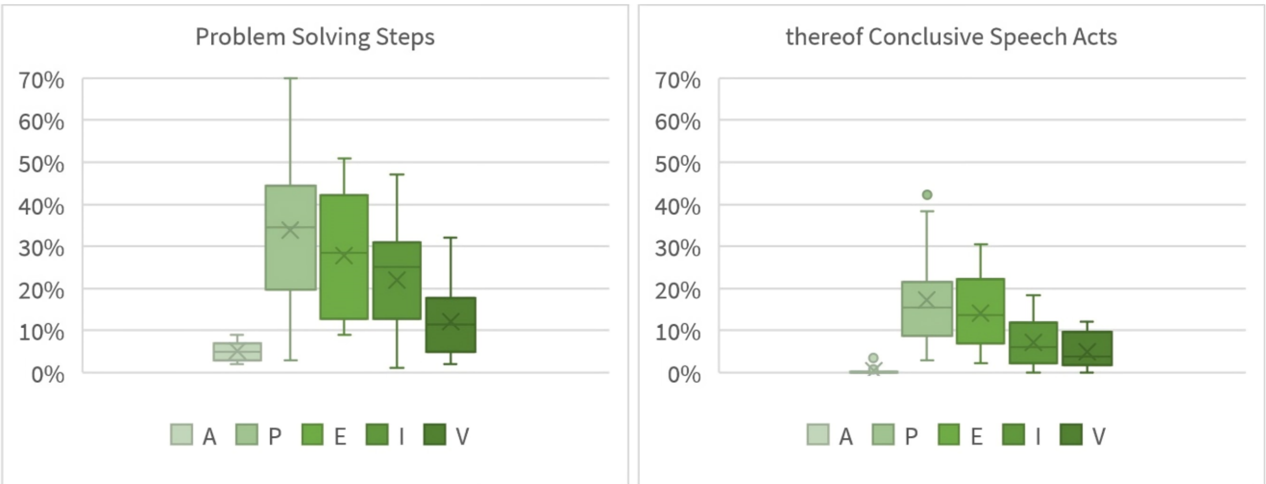

Figure 11. Boxplots visualizing duration of problem-solving steps in % (left) & duration of conclusive speech acts being linked to a problem-solving step in % (right) (Figure created by the author)

Furthermore, it is noticeable that the verification step occurs second least frequently (12%), but by considering the conclusive speech acts thereof (5%), it can be seen that the children formulate speech acts in nearly 50% of the total duration.

In total, 286 scenes including conclusive speech acts were coded. **Table 3** gives a quantitative overview of their assigned categories according to Klein (2008).

Table 3. Overview of conclusive speech acts

Spiral conclusion		
Explain-why		97 (33.9 %)
Reason		
Epistemic	Justify & refute	149 (52.1 %)
Evaluative	Defend & delegitimize	40 (14.0 %)

All spiral conclusive speech acts according to Klein (2008) could be found in the data material. It turns out that more than half of all coded speech acts occur in the epistemic dimension and thus fall into the activities JUSTIFY and REFUTE. One third of the speech acts can be assigned to the activity EXPLAIN-WHY, while the remaining 14% contain conclusive speech acts in the evaluative dimension, namely DEFEND and DELEGITIMIZE. It can further be observed that the activities JUSTIFY and REFUTE as well as DEFEND and DELEGITIMIZE often occur together. Therefore, the activities at the same dimension are not further distinguished in the context of the study.

To get a deeper impression of the occurrence of speech acts in the individual steps of problem-solving, **Figure 12** shows the distribution of the types of conclusive speech acts among the five problem-solving steps. There is a dominance of conclusive speech acts in the exploration and implementation steps. The conclusive speech acts in the epistemic dimension occur primarily in the steps *planning* and *exploration*. The evaluating conclusive speech acts are mainly found in the steps *implementation* and *verification*.

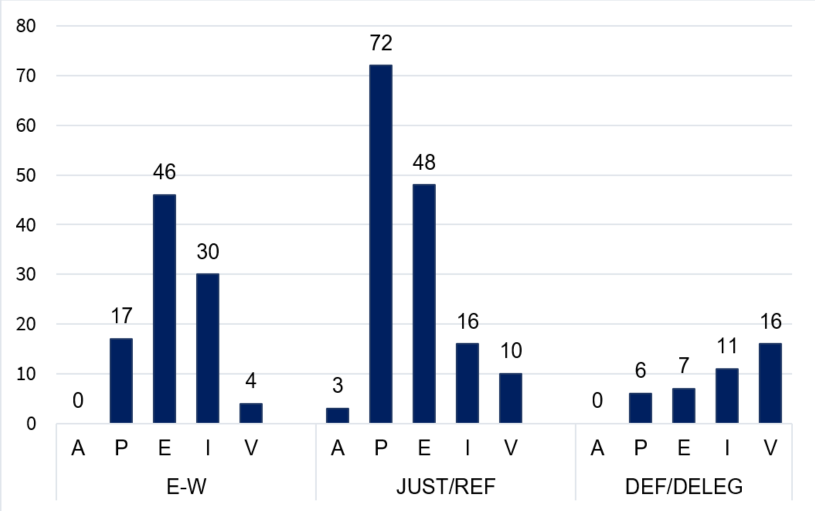

Figure 12. Distribution of categories of conclusive speech acts to problem-solving steps (Figure created by the author)

Comparison of Task Settings

After describing the occurrence of problem-solving steps and conclusive speech acts for all tasks, a differentiation with respect to the three task settings–*photo*, *3D model* and *real object*–shall be made at this point.

Figure 13 shows the absolute frequencies of the 286 coded conclusive speech acts distributed among the problem-solving steps and task settings. First of all, it can be seen that most of the conclusive speech acts occur in the *photo* setting (130 of 286). It is followed by the *3D model* setting with 93 coded sequences and the *real object* setting ranked third with 63 conclusive speech acts. These results coincide not only with the number of scenes but on the associated duration of the conclusive speech acts.

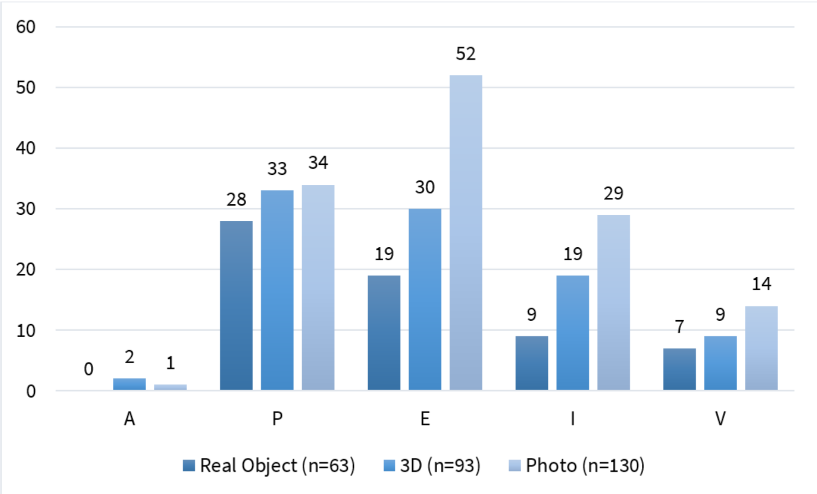

Figure 13. Distribution of amount of conclusive speech acts to problem-solving steps according to different task settings (Figure created by the author)

Figure 11 has already emphasized the dominance of conclusive speech acts in *planning* and *exploration*. This remains the case even when divided into different task settings. In all three settings, these two steps are assigned the most conclusive speech acts. In particular, for the *photo* setting a special importance on exploration is evident. In addition, this setting involves most speech acts in the implementation step.

Figure 14 shows the conclusive speech acts not only according to the task setting, but furthermore according to the categorization by Klein (2008). It shows the distribution of the conclusive speech acts being linked to a problem-solving step according to Rott et al. (2021) and a dimension of conclusive speech act according to Klein (2008). In order to simplify the comparison of the settings, the amount of speech acts is given in percentage.

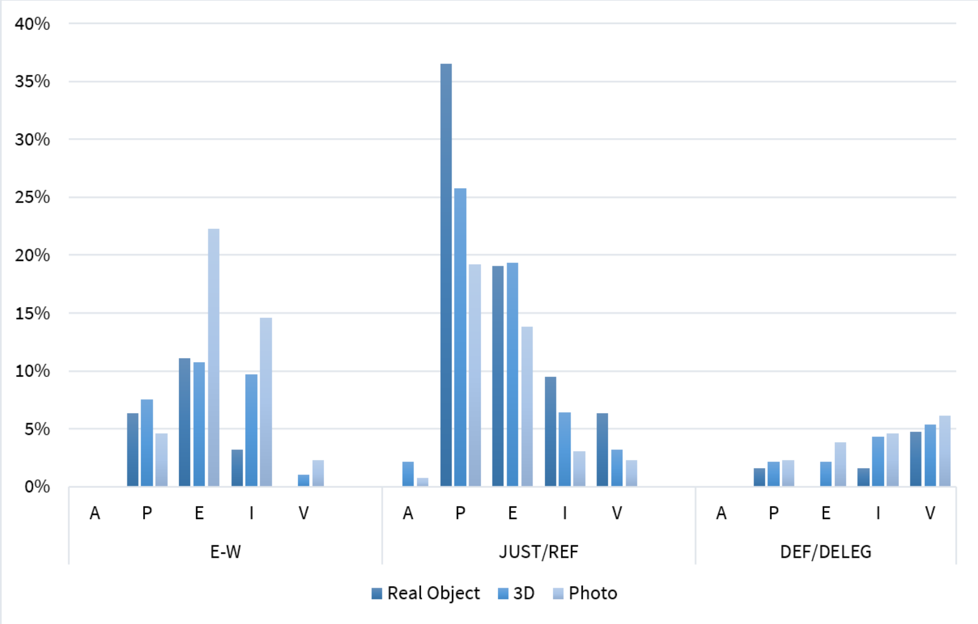

Figure 14. Distribution of categories of conclusive speech acts to problem-solving steps according to different task settings (in %) (Figure created by the author)

First of all, the dominance of conclusive speech acts in the epistemic dimension is confirmed for all settings. Furthermore, individual foci for the *photo* setting in EXPLAIN-WHY in *exploration* and *implementation* become apparent. For the *3D model* and *real object* setting, it is the conclusive speech acts in the epistemic dimension that can mostly be assigned to *planning* and *exploration*. Especially for the *real object* setting, JUSTIFY and REFUTE are dominant activities in the *planning* step.

Results from Qualitative Content Analysis

In the last part of the results' report, the inductively derived categories from the qualitative content analysis are presented. **Table 4** gives an overview of the identified categories containing a definition, relation to the problem-solving steps as well as to the conclusive speech acts. In addition, a differentiation of each category in the three task settings *real object* (RO), *3D model* (3D), and *photos* (P) is given in percentage based on the total amount of 286 coded conclusive speech acts.

Table 4. Results from qualitative content analysis

Category	Definition	Results		
		RO	3D	P
Choice of a strategy	Students formulate a conclusive speech act that aims at choice of a strategy to solve problem. As reasons for or against a strategy choice can be: -aberrations at object, -finesse of strategy, -object properties, -possibilities in approach (e.g., available material), -comparison of strategies, -connection of reality & object representation. Mostly related to problem-solving steps (P) & (E) & conclusive speech act JUST/REF.	31%	18%	12%
Data collection	Students formulate conclusive speech acts to agree on procedures for data collection. Reasons listed amount to: -object properties, -use of reference variables, -accuracy in the procedure, -influence of perspective, -estimations. Mostly related to problem-solving steps (P) & (E) & conclusive speech act JUST/REF.	26%	24%	47%
Calculation procedure	Students use conclusive speech acts to agree in terms of how to proceed with calculation. Their justifications include: -scale calculations, -object properties, -conversion of units. Mostly related to problem-solving steps (E) & (I) & conclusive speech acts E-W & JUST/REF.	20%	37%	18%

Table 4 (continued). Results from qualitative content analysis

Category	Definition	Results		
		RO	3D	P
Use of mathematical rules	In this category, students explain & use mathematical rules & laws in areas such as: -multiplication, -divisibility, -conversion factors for length & volume. Mostly related to problem-solving steps (E) & (I) & conclusive speech act E-W.	7%	10%	9%
Evaluation of result	With help of conclusive speech acts, students evaluate their achieved (intermediate) results. They base their arguments on: -approach and potential errors in previous processing steps, -comparison with alternative results, -estimates. Mostly related to problem-solving steps (V) & conclusive speech act DEF/DELEG.	16%	11%	14%

The conclusive speech acts can be grouped in five different categories: The students used conclusive speech acts to reason on the choice of a strategy, data collection, calculation procedure, the use of mathematical rules and evaluation of their results. From the relation of the problem-solving steps, it can be seen that they are mainly related to the steps *planning* and *exploring* followed by *implementation* (cf. results explained before). Also concerning the relation to the conclusive speech acts, the results from 5.2 can be confirmed through a focus on the epistemic dimension.

By comparing the different settings in the categories, it can be seen that reasoning about the choice of a strategy is mainly relevant when working at a real object, especially in terms of deviations at the object, the finesse of the strategy and comparison of different strategies. The data collection is mostly reasoned in the *photo* setting. This happens mostly in terms of considerations of the perspectives shown in the photos and the use of the reference variables. Reasons about calculation processes are mostly observed in the *3D model* setting, especially when it comes to scale calculations. For the use of mathematical rules and the evaluation of the results, no patterns in the different settings can be observed.

DISCUSSION & OUTLOOK

In the following section, the results are discussed in the context of the three posed research questions. The focus of **RQ1** is to identify the role of conclusive speech acts in the problem-solving steps. Concerning the time spent by the groups per problem-solving step, it can be seen that this duration is usually highest in the steps *planning* and *exploration*. When combining the duration of time spent working on the steps with the duration of conclusive speech acts, the importance of these two steps can be confirmed, i.e., the steps *planning* and *exploration* do not only seem to be highlighted in terms of their duration, but also in terms of the involved conclusive speech acts. For the *verification* step, it can be seen that it generally does not appear as frequently as the other two reported steps. Still, if students go through the *verification* step, it can be assumed that they apply conclusive speech acts here.

The general occurrence of conclusive speech acts in the problem-solving processes can confirm earlier findings concerning the role of reasoning in problem-solving tasks (e.g., Kilpatrick et al., 2001). Also the role of conclusive speech acts in *planning, exploration* and *validation* is in line with earlier research on the connection of reasoning and problem-solving (e.g., Boesen et al., 2010). In contrast to the results of Mullis et al. (2005), a special role of reasoning during the analysis of a problem could not be found in this study. The hypothesis that students formulate mathematical reasons without being forced to when working at a real object outdoors (Jablonski, 2022) can be confirmed and extended to the three different task settings involving representations of real objects.

Concerning **RQ2**, the different kinds of conclusive speech acts are additionally taken into consideration. First, it can be highlighted that the students formulate all spiral conclusive speech acts during their task solutions. The majority of the formulated speech acts belongs to the epistemic dimension. In combination with the problem-solving steps, it can be seen that conclusive speech acts in the epistemic dimension primarily occur in *planning* and *exploration*. It seems that the necessity to formulate epistemic conclusive speech acts goes along with the steps' aim of developing, testing and optimizing a solution plan. This assumption is supported by the results from the qualitative content analysis in the category *choice of a strategy* and *data collection*. The reasons why students formulate conclusive speech acts related to *planning* and *exploration* mostly involve a detailed analysis of the object, considerations concerning possible strategies and different data collection procedures. All of them happen during the actual work on the task and at the real object. A reference to the epistemic dimension in the formulation of conclusive speech acts can therefore be assumed.

Conclusive speech acts that belong to the category EXPLAIN-WHY are mainly relevant in the steps of *exploration* and *implementation*. From the qualitative content analysis, it can be seen that the students discuss their *calculation procedure* by considerations in the context of scale, object properties and the conversion of units. In addition, the category *use of mathematical rules* contains conclusive speech acts about arithmetic operations. Mostly, these considerations are inner-mathematical and non-contextualized.

The conclusive speech acts that are linked to the evaluating dimension are observed the least often. If they occur, they are mostly assigned to the *validation* step. Based on the results from the qualitative content analysis, the relation between the evaluating dimension and the *validation* step becomes clear: Students evaluate their results by assessing their previously made processing steps. Since a result has already been achieved, it is the core of the discussion and conclusive speech acts mainly aim at defending or delegitimizing the result.

In contrast to the previously made assumption based on Schwarzkopf (2000), the main focus of conclusive speech acts is not on EXPLAIN-WHY, but on acts in the epistemic dimension. With most acts being coded as JUSTIFY and REFUTE, it seems that the students formulate reasons based on a critical point of view. Therefore, it can be assumed that the task settings can contribute to mathematical reasoning.

Finally, the focus is on **RQ3**, and the questions, which similarities and differences can be observed when comparing the different task settings *real object*, *3D model*, and *photos*. For all settings, *planning* and *exploration* are the most relevant steps in terms of duration and, in addition, relevant steps when it comes to the involvement of conclusive speech acts. Concerning the different dimensions of the conclusive speech acts, the focus on the epistemic dimension can be confirmed for all three settings. From these similarities, it can be concluded that the involvement of real objects–independent from the representation–lets students put emphasis on the creation and testing of a solution plan and involve epistemic conclusive speech acts in this creation. In addition, conclusive speech acts about the use of mathematical rules and the evaluation of the results seem not to depend on the task settings, but more on the group constellations and dynamics.

Individual differences are seen in the following aspects: the amount of speech acts and their content. From the total amount of conclusive speech act occurrences, it can be seen that they are mostly observed when students work with photos. In addition to *planning* and *exploration*, the steps *implementation* and *validation* can be highlighted for this setting in terms of the occurrence of conclusive speech acts, too. On the content level, the *photo* setting involves more conclusive speech acts concerning EXPLAIN-WHY in *exploration* and *implementation*. The results of the qualitative content analysis, in addition, show that *data collection* is a main reason for conclusive speech acts in the *photo* setting. This happens mostly in terms of considerations of the perspectives shown in the photos (cf. Schukajlow, 2013) and the use of the reference variables.

For the work with the 3D model, it can be seen that not only conclusive speech acts in the epistemic dimension, but also conclusive speech acts concerning EXPLAIN-WHY during *implementation* are overserved. Analyzing the content, it seems that students formulate conclusive speech acts mainly in the context of calculations and scaling, whereas they happen in both *planning* and *exploration* on an epistemic dimension and the implementation through explaining mathematical rules and facts. Hence, students put emphasis on reasoning not only during the actual implementation of calculation and scaling, but also during the planning and testing of the strategies to proceed the calculation and scaling.

In the *real object* setting, the least amount of conclusive speech acts can be observed. For those speech acts, a dominance of the epistemic dimension is related to the *planning* step. For the content, the reasoned *choice of a strategy* is mainly relevant here when working at a real object, especially in terms of deviations at the object, the finesse of the strategy and comparison of different strategies. Probably, this can be explained by the nearly unlimited options in changing perspective and gaining data that is given when being on site of the task's object. Hereby, students have the chance to see irregularities, decide on the finesse of a chosen strategy and choose between different approaches. This assumption would be in line with the findings of Jablonski (2023) in the context of mathematical modelling and the students' claim to work as precisely as possible when simplifying and structuring the object to be modelled.

Interestingly, the students formulate less conclusive speech acts about data collection in the work with the real object than in the *photo* setting. This result is even more remarkable when comparing it to mathematical modelling. For modelling, it was shown that the students put more emphasis on gaining data when being outdoors (Jablonski, 2023). Still, from the point of view of argumentation, it can be highlighted that the students formulate more conclusive speech acts about data collection when working with photos. So when working with photos, it seems that more explanations and critical statements force the students to reason. This might on the one hand be explained by the higher role of interpreting perspective when working with two-dimensional objects (cf. Schukajlow, 2013). On the other hand, the use of embodiment (cf. Lakoff & Nuñez, 2009) might be higher when working with three-dimensional objects: When working at the real object or with a 3D model, students can touch it and explain their ideas non-verbally through pointing and showing gestures. Thus, it can be assumed that in the settings *3D model* and *real object*, students express their ideas also by non-verbal communication. This would potentially explain why, when working with photos, students formulate the most conclusive speech acts.

To summarize the findings, it can be concluded that problem-solving tasks involving real objects can force students to formulate mathematical reasoning without an explicit impulse to reason. From the data, it could be seen that the students mainly formulated conclusive speech acts that are related to their planning and exploration of solution strategies, which goes in line with a focus on conclusive speech acts in the epistemic dimension. It is therefore more a reasoning about strategies, data collection and procedures despite reasoning about mathematical rules and contents that could be observed. The comparison of different representations of the real object shows that most conclusive speech acts were found in the photo setting. It was assumed that especially considerations about perspective as well as the missing possibility to touch and point at the object might lead to this result.

This hypothesis could be the starting point for a follow-up study in which not only the spoken word of the students is analyzed, but in addition, their non-verbal interactions with the objects' representations. In addition, the limitations of the study give cause for further investigations. First, the study is limited in the sample. In the methods, the selection of interested and gifted students was reasoned. Still, it has to be assumed that lower-performing students would solve the tasks differently and probably not formulate mathematical reasons during the problem-solving process if ideas are missing. Second, the results show that an analysis/understanding of the problems was usually of a short duration during the students' working process. Other findings, e.g., Mullis et al. (2005), emphasize this step to be relevant for reasoning. It might therefore be possible that the task formulation did not force this step in a proper manner and that other tasks, e.g., from other mathematical fields, would have caused conclusive speech acts in this step, too. Finally, it is not possible to conclude from the results of the study whether these tasks would lead to more argumentation and higher proportions of conversation in the "classical" classroom setting (cf. Lipowsky et al., 2007). Since the study examined groups in special task settings, it is not possible to make a statement about the normal classroom setting with a teacher being present.

Nevertheless, the results can contribute to the discussion about mathematical reasoning, the implementation in mathematics education and its relation to the problem-solving steps. The usage of real objects and their representations as task settings has shown that reasoning activities can be observed during the students' work without a clear reasoning assignment. Also the different foci among the three different task settings can contribute to a varied realization of reasoning activities. Therefore, it is promising to further explore the integration of real objects and their representations as task settings from a research point of view. Especially, the listed limitations can be taken as a reason to deepen the results. In addition, the results can serve as a suggestion for practical teaching implementations since they contribute to the claim of argumentation-stimulating tasks (e.g., Stylianidies et al., 2019) and thus can make an enriching contribution to reasoning among students in mathematics education.

Funding: The study entitled, "Modelling, argumentation and problem-solving in the context of outdoor mathematics", was funded by Dr. Hans Messer Stiftung.

Ethical statement: The author stated that consent was obtained from the participants before the research was conducted; and they were informed about the research, anonymity and their rights about withdrawing from the study without adverse repercussions.

Declaration of interest: No conflict of interest is declared by the author.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the author.

REFERENCES

Ärlebäck, J., & Albarracín, L. (2019). An extension of the MAD framework and its possible implication for research. In U. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the 11th Congress of the European Society for Research in Mathematics Education* (pp. 1128-1135). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Baker, M. (2003). Computer-mediated argumentative interactions for the co-elaboration of scientific notions. In J. Andriessen, M. Baker, & D. Suthers (Eds.), *Arguing to learn: Confronting cognitions in computer-supported collaborative learning environments* (pp. 47-78). Springer. https://doi.org/10.1007/978-94-017-0781-7_3

Barlovits, S., & Ludwig, M. (2023). *Effective or not? The impact of mobile learning on students' interest, self-efficacy, and performance in outdoor mathematics education* [Paper presentation]. The 13th Congress of the European Society for Research in Mathematics Education.

Bergqvist, T., Lithner, J., & Sumpter, L. (2008). Upper secondary students task reasoning. *International Journal of Mathematical Education in Science and Technology, 39*(1), 1-12. https://doi.org/10.1080/00207390701464675

Blum, W., & Leiss, D. (2007). How do students and teachers deal with mathematical modelling problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling* (pp. 222-231). Woodhead Publishing. https://doi.org/10.1533/9780857099419.5.221

Boesen, J., Lithner, J., & Palm, T. (2010). The relation between types of assessment tasks and the mathematical reasoning students use. *Educational Studies in Mathematics, 105*, 75-89. https://doi.org/10.1007/s10649-010-9242-9

Buchholtz, N. (2021). Students' modelling proceses when working with math trails. *Quadrante, 30*(1), 140-157. https://doi.org/10.48489/quadrante.23699

Budke, A., & Meyer, M. (2015). Fachlich argumentieren lernen–Die Bedeutung der Argumentation in den unterschiedlichen Schulfächern [Learning to argue professionally–The importance of argumentation in different school subjects]. In A. Budke, M. Kuckuck, M. Meyer, F. Schäbitz, K. Schlüter, & G. Weiss (Eds.), *Fachlich argumentieren lernen. Didaktische Forschungen zur argumentation in den Unterrichtsfächern* [*Learn to argue professionally. Didactic research on argumentation in teaching subjects*] (pp. 9-28). Waxmann. https://doi.org/10.25656/01:14021

Duijzer, A. C. G., Van den Heuvel-Panhuizen, M., Veldhuis, M., Doorman, L. M., & Leseman, P. P. M. (2019). Embodied learning environments for graphing motion: A systematic literature review. *Educational Psychology Review, 31*(3), 597-629. https://doi.org/10.1007/s10648-019-09471-7

Eames, C., Brady, C., Jung, H., & Glancy, A. (2018). Designing powerful environments for examining and supporting teachers' models and modeling. In W. Blum, & R. Borromeo Ferri (Eds.), *Lehrerkompetenzen zum Unterrichten mathematischer Modellierung. Realitätsbezüge im Mathematikunterricht* [*Teacher skills for teaching mathematical modeling. Reality references in mathematics lessons*] (pp. 237-266). Springer. https://doi.org/10.1007/978-3-658-22616-9_11

Field, A., & Hole, G. (2002). *How to design and report experiments*. SAGE.

Habermas, J. (1981). *Theorie des kommunikativen Handelns, band 1. Handlungsrationalität und gesellschaftliche Rationalisierung* [*Theory of communicative action, volume 1. Action rationality and social rationalization*]. Suhrkamp.

Hallström, J., & Schönborn, K. J. (2019). Models and modelling for authentic STEM education: Reinforcing the argument. *International Journal of STEM Education, 6*, 22. https://doi.org/10.1186/s40594-019-0178-z

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics, 44*(1/2), 5-23. https://doi.org/10.1023/A:1012737223465

Hartmann, L.-M., & Schukajlow, S. (2021). Interest and emotions while solving real-world problems inside and outside the classroom. In F. K. S. Leung, G. A. Stillman, G. Kaiser, & K. L. Wong (Eds.), *Mathematical modelling education in east and west. International perspectives on the teaching and learning of mathematical modelling* (pp. 153-163). Springer. https://doi.org/10.1007/978-3-030- 66996-6_13

Hattie, J., Marsh, H., Neill, J., & Richards, G. (1997). Adventure education and outward bound: Out-of-class experiences that make a lasting difference. *Review of Educational Research, 67*(1), 43-87. https://doi.org/10.2307/1170619

Jablonski, S. (2022). Mathematical reasoning outside the classroom – A case study with primary school students solving math trail tasks. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education* (pp. 201-208). Free University of Bozen-Bolzano and ERME.

Jablonski, S. (2023). Is it all about the setting? — A comparison of mathematical modelling with real objects and their representation. *Educational Studies in Mathematics, 113*, 307-330. https://doi.org/10.1007/s10649-023-10215-2

Jablonski, S., & Ludwig, M. (2022). Examples and generalizations in mathematical reasoning – A study with potentially mathematically gifted children. *Journal on Mathematics Education*, *13*(4), 605-630. https://doi.org/10.22342/jme.v13i4.pp605-630

Jablonski, S., Läufer, T., & Ludwig, M. (2023). Make it real: Students' mathematical modelling realized with 3D printing [Accepted paper for presentation]. In *13th congress of the European Society for Research in Mathematics Education* (CERME13), Budapest, Hungary.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.

Klein, J. (2008). Erklären und Argumentieren als interaktive Gesprächsstrukturen [Explaining and reasoning as interactive conversational structures]. In K. Brinker, G. Antos, W. Heinemann, & S. F. Sager (Eds.), *Ein internationales Handbuch zeitgenössischer Forschung* [*An international handbook of contemporary research*] (pp. 1309-1329). De Gruyter Mouton. https://doi.org/10.1515/9783110169188.2.18.1309

Kolb, D. (1984). *Experiential learning: Experience as the source of learning and development*. Prentice-Hall, Inc.

Kolb, D., Boyatzis, R., & Mainemelis, C. (2000). Experiential learning theory: Previous research and new directions. In R. Sternberg, & L. Zhang (Eds.), *Perspectives on cognitive, learning, and thinking styles*. Lawrence Erlbaum. https://doi.org/10.4324/9781410605986-9

Lakoff, G., & Nuñez, R. (2009). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books. https://doi.org/10.2307/3072449

Lipowsky, F., Rakoczy, K., Pauli, C., Reusser, K., & Klieme, E. (2007). Gleicher Unterricht–gleiche Chancen für alle? Die Verteilung von Schülerbeiträgen im Klassenunterricht [The same instruction–The same chances for everyone? The distribution of contributions from students in whole-class discussions]. *Unterrichtswissenschaft* [*Teaching Science*]*, 35*(2), 125-147. https://doi.org/10.25656/01:5489

Lowrie, T. (2002). The influence of visual and spatial reasoning in interpreting simulated 3D worlds. *International Journal of Computers for Mathematical Learning, 7*, 301-318. https://doi.org/10.1023/A:1022116221735

Ludwig, M., & Jablonski, S. (2021). Step by step – Simplifying and mathematizing the real world with MathCityMap. *Quadrante, 30*(2), 242-268. https://doi.org/10.48489/quadrante.23604

Ludwig, M., & Jesberg, J. (2015). Using mobile technology to provide outdoor modelling tasks–The MathCityMap-project. *Procedia Social and Behavioral Sciences, 191*, 2776-2781. https://doi.org/10.1016/j.sbspro.2015.04.517

Mayring, P. (2000). Qualitative content analysis. *Forum: Qualitative Social Research, 1*(2), 20. https://doi.org/10.17169/fqs-1.2.1089

Medina Herrera, L., Castro Pérez, J., & Juárez Ordónez, S. (2019). Developing spatial mathematical skills through 3D tools: Augmented reality, virtual environments and 3D printing. *International Journal on Interactive Design and Manufacturing, 13*, 1385-1399. https://doi.org/10.1007/s12008-019-00595-2

Mullis, I. V. S., Martin, M. O., Ruddock, G. J., O'Sullivan, C. Y., Arora, A., & Erberber, E. (2005). *TIMSS 2007 assessment frameworks*. TIMSS & PIRLS International Study Center.

Ord, J., & Leather, M. (2011). The substance beneath the labels of experiential learning: The importance of John Dewey for outdoor educators. *Journal of Outdoor and Environmental Education, 15*, 13-23. https://doi.org/10.1007/BF03400924

Pólya, G. (1945). *How to solve it*. Princeton University Press. https://doi.org/10.1515/9781400828678

Rott, B., Specht, B., & Knipping, C. (2021). A descriptive phase model of problem-solving processes. *ZDM Mathematics Education, 53,* 737-752. https://doi.org/10.1007/s11858-021-01244-3

Schoenfeld, A. H. (1985). *Mathematical problem-solving*. Academic Press.

Schukajlow, S. (2013). Lesekompetenz und mathematisches Modellieren [Reading skills and mathematical modeling]. In R. Borromeo Ferri, G. Greefrath, & G. Kaiser (Eds.), *Mathematisches Modellieren für Schule und Hochschule. Realitätsbezüge im Mathematikunterricht* [*Mathematical modeling for schools and universities. Reality references in mathematics lessons*] (pp. 125-143). Springer. https://doi.org/10.1007/978-3-658-01580-0_6

Schwarzkopf, R. (2000). *Argumentationsprozesse im Mathematikunterricht* [*Argumentation processes in mathematics education*]. Franzbecker.

Stylianides, G., Buchbinder, O., Cramer, J., Durand-Guerrier V., Moutsios-Rentzos A., & Valenta, A. (2019). Introduction to the papers of TWG01: Argumentation and proof. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the 11th Congress of the European Society for Research in Mathematics Education* (pp. 100-103). Freudenthal Group & Institute, Utrecht University and ERME.

Sumpter, L. (2018). Students' and teachers' values, attitudes, feelings and beliefs in mathematics class-rooms. In H. Palmer, & J. Skott (Eds.), *Grade 9 students' reasoning about division of fractions: What are their arguments anchored in?* (pp. 135-143). Springer. https://doi.org/10.1007/978-3-319-70244-5_13

Tall, D. O., & West, B. (1986). Graphic insight into calculus and differential equations. In R. F. Churchhouse, B. Cornu, A. G. Howson, J.-P. Kahane, J. H. van Lint, F. Pluvinage, A. Ralston, & M. Yamaguti (Eds.), *The influence of computers and informatics on mathematics and its teaching* (pp. 107-119). Cambridge University Press. https://doi.org/10.1017/CBO9781139013482.012

Tavsan, S., & Pusmaz, A. (2021). Examination of 7th-grade students' mathematical reasoning and argumentation process during problem-solving. *International Journal of Mathematical Education in Science and Technology*. https://doi.org/10.1080/0020739X.2021.2005168

Trust, T., & Maloy, R. W. (2017). Why 3D print? The 21st-century skills students develop while engaging in 3D printing projects. *Computers in the Schools, 34*(4), 253-266. https://doi.org/10.1080/07380569.2017.1384684

Van Eemeren, F. H., Grootendorst, R., Johnson, R. H., Plantin, C., & Willard, C. A. (1996). *Fundamentals of argumentation theory. A handbook of historical backgrounds and contemporary developments*. Lawrence Erlbaum Associates, Inc.

Wirth, L., & Greefrath, G. (2022). Examining heuristic worked example videos in a collaborative setting: The conception of the project MoVie. In U. T. Jankvist, R. Elicer, A. Clark-Wilson, H.-G. Weigand, & M. Thomsen (Eds.), *Proceedings of the 15th International Conference on Technology in Mathematics Teaching* (pp. 250-251). Danish School of Education, Aarhus University.

Wyndhamn, J., & Säljö, R. (1997). Word problems and mathematical reasoning–A study of children's mastery of reference and meaning in textual realities. *Learning and Instruction, 7*, 361-382. https://doi.org/10.1016/S0959-4752(97)00009-1

APPENDIX

Table A1.

Spiral conclusive speech act (Klein, 2008)	Definition	Coding	Example from material
Explain-why	Explication of occurrence of a state of circumstances.	-*Prerequisite*: Identification of a causal relation. -*Elements*: State of affairs, which is unclear with regard to its constitutional condition, explanation, which explicates its occurrence. -*Differentiation*: Spiral procedure, ambiguity about a state of affairs.	[*Context: Derivation of conversion formula for volume*]. S1: I forgot the conversion number, I think it was 100 or so […] No, it was 1,000. S2: Yeah, I think it was 1,000. S1: Yes, because to power of three & therefore three zeros.
Reason (related to a problematic claim of validity; main term)			
Justify & refute (epistemic)	Supporting of validity claims for epistemic attitudes (holding for true/probable/possible) to propositions or propositional complexes or argumentative attack on truth claims.	-*Prerequisite*: Identification of a causal relation. -*Elements*: Problematic or contentious position, rationale that attempts to make that position uncontroversial -*Differentiation*: Spiral approach, contentious validity claim, epistemic, & supportive/attacking.	[*Context: Deviation of height, width and depth of the Stone*; justify]. S1: There is not really an average value, because that just ... Here, it hardly changes. S2: So, values are somehow thinner here. But not little enough that we can disregard it. S1: Or we subtract a little bit at end.
Defend & delegitimize (evaluative)	Supporting validity claims for evaluative attitudes of positivity or at least non-negativity of states of affairs for which a subject (or group of subjects) is responsible and/or of subjects who are responsible for state of affairs or argumentative attack on positive or at least non-negative evaluations.	-*Prerequisite*: Identification of a causal relation. -*Elements*: Problematic or disputed claim of validity, statement that tries to justify this claim of validity -*Differentiation*: Spiral approach, contentious validity claim, evaluative, & supporting/attacking.	[*Context: Evaluation of an intermediate result*; delegitimize]. S1: And besides, you cannot just round that. Huh, then we are at 420.65, which means we would be at 4 meters. S2: 4 meters looks somehow unrealistic. Besides, it cannot be, because compared to the stone, the guy is not ... he does not fit in there twice as often.