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Prospective mathematics teachers' experiences with an alternative approach to solving quadratic equations

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ABSTRACT

Reliance on formulas and memorization has often been reported, leading to difficulties in understanding quadratic equations. This study investigates how 22 prospective mathematics teachers perform and reflect when solving quadratic equations using an alternative approach proposed by Po-Shen Loh—an intuitive and efficient strategy. Data were collected from worksheets and semi-structured interviews with seven participants. Analysis revealed that while many did not fully grasp the conceptual foundation of the method, they could follow procedural steps to reach solutions. The method was largely understood procedurally rather than conceptually, with participants sometimes reverting to memorized strategies. Misconceptions emerged around ideas such as the arithmetic mean, particularly with negative values. This study presents an alternative solution strategy to prospective teachers and encourages reflection on its purpose and application. The findings suggest that this approach could be incorporated into high school and university level algebra curricula.

Keywords: mathematics education, algebra education, prospective mathematics teachers, quadratic equations, alternative solution methods

INTRODUCTION

Quadratic equations are used in various fields such as physics, engineering, economics and business, where they are often used to model real life phenomena, particularly in optimization problems (e.g., maximizing or minimizing quantities) (Stewart, 2016). Given their wide applicability in solving diverse word problems, modeling real-world situations, and serving as the foundation for addressing more advanced topics in algebra, calculus, and beyond a strong understanding of quadratic equations is essential in mathematics education. Quadratic equations have long been integral not only to secondary mathematics curricula but also to the historical development of algebra (Didis & Erbas, 2015; Güner & Uygun, 2016). Over time, various methods for solving quadratic equations have been developed, utilizing different representations such as arithmetic, algebraic, and geometric approaches (Didis & Erbas, 2015; Katz & Barton, 2007).

In secondary school education, quadratic equations represent a significant and challenging topic within the algebra curriculum. They act as a bridge between basic algebra and the more advanced study of calculus (Sağlam & Alacacı, 2012). In Türkiye, students are introduced to quadratic equations in the 10th grade, where the curriculum focuses on understanding the concept of a quadratic equation, solving such equations, performing operations with complex numbers, and exploring the relationship between the roots and coefficients of quadratic equations (Ministry of National Education [MoNE], 2018).

While the 2018 curriculum treated quadratic equations primarily as algebraic objects solved through symbolic algorithms (Didis & Erbas, 2015), the most recent national curriculum—officially known as the Türkiye Century Education Model (The Maarif Model) (2024)—represents a significant shift in pedagogical emphasis (Ministery of National Education [MoNE], 2024). In this new framework, quadratic equations are no longer taught as isolated procedural topics; instead, they are integrated into the broader theme of "Quantities and Changes" (MoNE, 2024). The 2024 curriculum prioritizes conceptual modeling and graphical reasoning, treating quadratic expressions as transformations of the reference function $f(x) = x^2$ and focusing on their role in representing real-world relationships (MoNE, 2024). This shift away from rote procedural performance (MoNE, 2024) provides the core motivation for this study, as alternative approaches like Po-Shen Loh's method align with this new emphasis on structural understanding and geometric properties of quadratic functions.

Quadratic equations, like many algebraic concepts, often pose significant challenges for students due to the abstract nature of the symbols involved. Studies have consistently identified quadratic equations as one of the most difficult topics for students, both procedurally and conceptually (Carballo et al., 2022; Didis Kabar, 2023; Hu et al., 2022; Natsai et al., 2020). In a comprehensive review of quadratic equation research in mathematics education from 2000 to 2021, Didis (2023) highlighted that many students struggle to solve these equations, facing both cognitive and procedural obstacles (e.g. Tall et al., 2014; Zakaria & Maat, 2010). These challenges are particularly evident in the low success rates of students in solving quadratic equations, with many encountering difficulties across a range of topics, including algebra, fractions, and integers, all of which are fundamental to understanding quadratic equations (Didis Kabar, 2023; Güner, 2017; Sarwadi & Shahrill, 2014; Vaiyavutjamai & Clements, 2006).

While equations (quadratic) are crucial in secondary education (Kotsopoulos, 2007; Sağlam & Alacacı, 2012; Vaiyavutjamai & Clements, 2006) students often struggle with the topic, particularly when it comes to grasping the underlying conceptual meanings (Carballo et al., 2022; Hu et al., 2022; Wilkie, 2021). These difficulties arise primarily because students tend to memorize procedural methods—such as factoring, completing the square, or applying the quadratic formula—without fully understanding the foundational concepts behind these techniques (Didis & Erbas, 2015; Güner, 2017). This aligns with the broader observation in algebra education research that students often struggle to move beyond a “letter-symbolic and symbol-manipulation view” toward a more meaningful and conceptual understanding (Kieran, 2007). As a result, while students may successfully solve equations that fit the standard forms they have memorized, they often encounter difficulties when faced with equations that deviate from these familiar forms. Research has shown that, although students tend to excel in procedural tasks, they often struggle to understand the relationships between the components of quadratic equations (Didis & Erbas, 2015; Sönnerhed, 2021). Furthermore, their inability to connect the symbolic aspects of quadratic equations to their conceptual meaning makes it difficult for them to apply learned procedures to non-standard equations (Didis & Erbas, 2015; Kotsopoulos, 2007; Sönnerhed, 2021). Thus, common difficulties include factoring, applying the zero-product property, and solving equations that are not in standard form (Bosse & Nandakumar, 2005; Didis & Erbas 2015; Kotsopoulos, 2007; Sönnerhed, 2021; Tall et al., 2014; Vaiyavutjamai & Clements, 2006). While many students are familiar with the quadratic formula, and factorization, they lack deeper understanding (O’Connor & Norton, 2024). Despite the frequent use of traditional methods like factorization, completing the square, and applying the quadratic formula, research indicates that students tend to solve quadratic equations by memorizing procedures and students may fail to comprehend the deeper algebraic structures involved and concepts behind these methods (Fonger et al., 2020; Güner & Uygun, 2016; Kotsopoulos, 2007; Lim, 2000; Yahya & Shahrill, 2014).

Many students fail to view a quadratic equation as the product of two first-degree equations, which can obscure their understanding (Martinez-Cruz & Contreras, 2014). Typically, quadratic equations are solved through a memorized factoring process, where the solution is anticipated by factoring the equation and then checking the result. This approach is often limited to specific cases where the solutions are easily identifiable, highlighting the need for more robust conceptual understanding (Loh, 2019). Didis and Erbas (2015) found that students often applied rules for one type of equation incorrectly to other types. In particular, students tended to prefer factoring as a solution method, using it to quickly solve equations without focusing on the conceptual understanding of the equation’s structure. This tendency complicates the process of factoring, especially when the coefficient a in the equation ($ax^2 + bx + c = 0$) is not equal to one or when a and c have multiple factors, resulting in numerous potential factor pairs (Kotsopoulos, 2007). To address these challenges, Bossé and Nandakumar (2005) recommended using the quadratic formula or completing the square more frequently, as these methods can help avoid the frustration of factoring difficult expressions. Sönnerhed (2021) conducted a study in which he observed that Swedish mathematics textbooks encouraged students to use factoring for quick solutions, but without addressing the underlying structure and conceptual meaning of the equations. Despite factoring being a common solution method, it is also considered a significant learning difficulty for students worldwide (Hu et al., 2022). Moreover, formulas can sometimes obscure the importance of the various components of an equation, especially in word problems where the contextual meaning is crucial (Edwards & Chelst, 2019).

Research has shown that while students often struggle with solving quadratic equations (Nielsen, 2015; Fonger et al. 2020). Didis and Erbas (2015) proposed that these challenges might stem from students focusing primarily on symbols to find the roots of quadratic equations, treating them as simple calculations rather than considering their conceptual meanings. This leads students to memorize rules, formulas, and algebraic procedures without grasping the underlying concepts, which creates problems when they attempt to apply these methods to equations with non-standard structures. Given these challenges, it is crucial to shift the focus towards conceptual learning, where students not only learn to solve equations but also understand the relationships and reasoning behind the methods they use.

Given the conceptual and procedural difficulties students face, there is a growing need for alternative methods that can help strengthen students’ understanding of quadratic equations. Several studies have focused on enhancing the conceptual understanding of the quadratic formula (Allaire & Bradley, 2001; Edwards & Chelst, 2019; Kristen, 2021; Loh, 2019). The introduction of multiple methods may be attributed to the historical significance of the quadratic formula, an algebraic tool developed specifically to solve quadratic equations (Allaire & Bradley, 2001). Alternatively, the use of different algebraic methods may aim to emphasize important mathematical ways of thinking. These alternative teaching strategies, which connect algebraic procedures with their geometric interpretations, have the potential to foster a deeper understanding of the underlying concepts. These alternative methods aim to provide students with a deeper understanding of the underlying concepts, making them more flexible and effective problem solvers. Research emphasizes the importance of focusing on conceptual learning, where students not only learn to solve equations but also understand the reasoning behind each method (Lim, 2000; Yahya & Shahrill, 2014). In this regard, alternative approaches that link algebraic techniques with their geometric representations have gained attention as a way to enhance student comprehension (Allaire & Bradley, 2001; Edwards & Chelst, 2019; Loh, 2019; Wilkie, 2021). One such alternative method is Po-Shen Loh’s approach, which connects algebraic procedures to their geometric interpretations.

The Po-Shen Loh’s method establishes a direct bridge between algebraic procedures and the geometric properties of parabolas. At the core of this approach lies the fact that the value $-\frac{b}{2a}$, derived from the equation’s coefficients, is not merely an arithmetic average but represents the axis of symmetry and the midpoint of the roots, expressed algebraicaly as $\frac{x_1+x_2}{2}$ (Edwards &

Chelst 2019; Frank, 2021). The aim was to serve as an instructional vehicle to promote relational understanding rather than just instrumental fluency (O'Conner & Norton, 2024). Instead of replacing one memorized formula with another, this approach fosters conceptual understanding by making the underlying geometric structure of the parabola explicit, requiring learners to engage with the positional relationship between roots and the symmetry center (Edwards & Chelst, 2019; Vaiyavutjamai & Clements, 2006)

Recent studies have shown that this method can significantly improve students' ability to solve quadratic equations, offering a deeper conceptual understanding (Andam et al., 2024). Savage (1989) introduced a simplification in solving quadratic equations, a method that was later expanded upon by Loh (2019). Loh's method has demonstrated positive results, particularly in high school settings, suggesting that it could be a valuable addition to the mathematics curriculum.

These findings suggest that more research is needed to explore students' understanding, interpretations, and perspectives on alternative teaching approaches. Recently, Andam et al. (2024) conducted an action research study with 45 high school students, which is the only research in algebra instruction to apply Po-Shen Loh's new method and assess its impact on students' ability to solve quadratic equations. The study revealed a significant improvement in students' problem-solving skills and concluded that this approach should be incorporated into the high school mathematics curriculum. The current study aims to further contribute to the growing body of knowledge regarding the effectiveness of this method in teaching quadratic equations.

Studies (Ellerton & Clements, 2011; Huang & Kulm, 2012; Lim 2000) also showed that prospective teachers (PTs) and also teachers lacked a deep understanding of the methods for solving quadratic equations. Mathematics teachers, pre-service teachers, and educators must be familiar with alternative approaches to teaching quadratic equations to apply them effectively in the classroom. Additionally, Hu et al. (2022), and Didis (2023) emphasized the need for further research into mathematics teachers' and pre-service teachers' pedagogical content knowledge, specifically regarding quadratic equations. Didis (2023) pointed out that quadratic equations are typically not taught as standalone topics in calculus courses but are instead integrated into broader subjects such as functions, function graphs, or derivatives. As a result, much of the existing research has focused on pre-service teachers' understanding of quadratic equations within the context of teaching functions, as observed in studies like Huang and Kulm (2012), rather than examining their specific content knowledge of quadratic equations. The difficulties in studying quadratics have also been found to affect teachers in the middle secondary years (Huang & Kulm, 2012). Didis (2023) also highlighted a significant gap in research on the teaching of quadratic equations, noting that a comprehensive review revealed insufficient studies focused specifically on this topic. Given the critical role teachers play in teaching these concepts, it is important to conduct studies on alternative methods that can improve students' understanding of quadratic equations. Such research would not only enhance teachers' content knowledge but also improve teaching practices, helping students avoid common errors and misconceptions in solving quadratic equations.

Although quadratic equations have traditionally been treated as algebraic objects solved through symbolic manipulation, the newly introduced Turkish mathematics curriculum—the Maarif Model—positions them under the broader theme of functions (MoNE, 2024). Within this framework, quadratic expressions are understood as transformations of the reference function $f(x) = x^2$, and their solutions are explored through graphical reasoning and contextual modeling. The alternative method examined in this study, based on Po-Shen Loh's approach, aligns with this pedagogical shift by emphasizing symmetry and conceptual understanding over rote procedural execution.

An Alternative Approach to Solving Quadratic Equations

In many school textbooks general form of a quadratic equation is given in the form of $ax^2 + bx + c = 0$. In schools, students try guess and check method to factor this equation. They try to guess two numbers that have a certain sum and a certain product. Before learning the quadratic formula, secondary school students often learn to factor equations (usually with a=1 in general quadratic equation $ax^2 + bx + c = 0$) by looking for two numbers, roots of the equation, whose sum is $-b$ and whose product is c. The sum of the roots is

$$x_1 + x_2 = -\frac{b}{a}$$

And the product is

$$x_1 . x_2 = \frac{c}{a}$$

The two solutions of are the x-intercepts of the equation.

Factoring quadratic equations requires the skill to quickly and accurately identify the factors, which makes factoring these equations challenging. Factoring more complex equations (such as $ax^2 + bx + c = 0$ when $a \neq 1$) is even more difficult. This is because factoring quadratic equations is dependent on the ability to establish relationships between the sum and product of the numbers in the coefficients and constant term (Bosse & Nandakumar, 2005). Although this method is commonly preferred in schools, it is often seen as a rote technique that requires strong estimation and computational skills (Didis & Erbas, 2015).

On the other hand, because sometimes quadratic equations are a lot harder to factor and thus harder to solve, students are usually taught to find the roots x_1 and x_2 of this equation by plugging the coefficients in the formula,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

This formula helps students to find the roots of an equation and thus solve quadratic equations. However, for some complex quadratic equations, ($x^2 + x = 1111111122222222$) using the quadratic formula requires a calculator capable of handling 16-digit numbers, which is often too advanced for many calculators allowed in schools (Elishakoff & Reddy, 2021). On the other hand, Loh’s method allows for finding the roots without the need for any hints or formula for the solution. The idea behind the Loh’s method is that the roots differ from the average of the roots by some value u.

In this study, the method was utilized as an instructional tool to foster representational fluency, as it inherently leverages the principle of parabolic symmetry to bridge algebraic steps with geometric reasoning. In the equation $ax^2 + bx + c = 0$, the line of symmetry $x = -\frac{b}{2a}$ serves as the exact midpoint between the roots (Edwards & Chelst, 2019), which is represented algebraically as the average of roots $\frac{x_1+x_2}{2}$. Because a parabola is symmetric, any real roots must be equidistant from this axis, providing the geometric justification for the $x = \frac{x_1+x_2}{2} \mp u$ representation (Edwards & Chelst, 2019; Frank, 2021). Thus, the algebraic variable u acquires a clear geometric meaning as the symmetric horizontal displacement from the center, transforming the solving process into an exploration of the parabola’s shape rather than a trial-and-error search for factors (Frank, 2021).

The choice to define u as a real number while representing it as a distance on the number line was intentionally designed to create a conceptual tension. This duality encourages learners to confront the relationship between algebraic solutions (where $u^2 = k$ yields $\mp\sqrt{k}$) and geometric properties (where distance is traditionally positive). The goal is to determine if learners recognize that due to the symmetric nature of the $\frac{x_1+x_2}{2} \mp u$ form, the choice of sign for u is algebraically redundant, as both values generate the same pair of roots.

Thus, Loh focused on the desired sum, looking for two numbers (x_1 and x_2) equidistant from the desired average (half of the desired sum). Loh explains that if you find x_1 and x_2 with sum $-b$ and product c, then

$x^2 + bx + c = x^2 - (x_1 + x_2).x + x_1.x_2$, and they are all the roots. (Viete, known hundreds of years ago)

Loh’s method can be briefly explained through an example as follows. Suppose a quadratic equation given by Muhammad Ibn-Musa al-Khwarizmi, $x^2 + 10x - 39 = 0$, the sum of the roots $x_1 + x_2 = -10$ and the average of the roots is -5. The roots must be equdistant from their average. The roots differ from this average by some value u.

$x_1 = -5 + u$ and $x_2 = -5 - u$.

The product is $x_1x_2 = -39$. Thus, $x_1x_2 = (-5 + u)(-5 - u) = -39$. When you multiply, you come up with $(-5)^2 - u^2 = -39$. Solving for u gives, $u^2 = 64 \Rightarrow u = \mp 8$ both value of u work.

Substituding either of these values back in the equations gives the two roots x_1 and x_2 as -13 and 3. This method is quicker and more thoughtful than the quadratic formula, eliminating the need for guess and check in factoring. It is easier to understand than many methods and formulas found in textbooks, improving the learning experience for students. Additionally, it requires minimal background knowledge, mainly the distributive property, and emphasizes the importance of explaining school mathematics concepts in more thoughtful ways.

Although widely referred to as Po-Shen Loh’s method, its underlying principles are deeply rooted in mathematical history, synthesizing Vieta’s classical relationships with the concept of arithmetic averaging—a logic tracing back to the third-century work of Diophantus (Elishakoff & Reddy, 2021). Notably, a similar simplification was presented by Savage (1989), and more recently, Frank (2021) highlighted this approach as a powerful way to leverage symmetry and averaging to simplify the solution process. Loh’s contribution lies in reformulating these ideas into a coherent didactical framework that shifts the focus from procedural performance to structural understanding. By connecting algebraic procedures to geometric reasoning and leveraging the symmetry of the parabola, the method directly supports the vision of the Maarif Model, encouraging conceptual modeling over the rote memorization of the quadratic formula.

The persistent difficulties students face with quadratic equations, combined with the prevalent misconceptions identified among pre-service teachers, underscore the need for research that actively supports conceptual learning (Hu et al., 2022). Despite the insights offered by existing literature, there remains a significant gap in research focusing specifically on how prospective teachers engage with alternative, meaning-centered methods. To address this, the present study investigates what prospective mathematics teachers are able to comprehend and apply after being introduced to the Po-Shen Loh method through a worksheet, while exploring their unique perspectives on its utility in the teaching and learning of quadratic equations.

METHODOLOGY

Research Settings

The research is a qualitative case study conducted as part of the “Algebra Teaching” course during the final class of a 14-week program. The “case” is defined as the conceptual and reflective process of PTs as they engage with an alternative algebraic method. Following the definition of a case as a bounded system (Creswell, 2007), this study is bounded by the specific context of a 14-week “Algebra Teaching” course and is delimited to the final instructional session involving 22 participants. This bounded approach allows for an intensive, in-depth analysis of how teacher candidates articulate conceptual links - a level of detail often overlooked by broader methodological designs (Joshi, 2019). By focusing on this specific instructional window, the study aimed to capture the nuances of their understanding as it arose within their established learning environment.

Throughout the semester, prospective teachers (PTs) engaged with topics such as the history of algebra, algebraic thinking, teaching and learning approaches, algebraic expressions, variables, equality, identity, equations, inequalities, and patterns. Lesson plans addressing misconceptions and student difficulties were also prepared and discussed. Before the study, PTs were already familiar with four traditional methods (factorisation, formula, completing squares, and graphical methods). In the course, these traditional methods were taught through a conceptual lens that prioritized relational understanding over mere procedural execution. Rather than focusing on how to solve equations, the course emphasized why each method works. For instance, the quadratic formula was introduced through its formal derivation from the completing the square process, and completing the square itself was grounded in its historical and geometric roots using area models (Clark, 2012; Vinogradova & Wiest, 2007). Similarly, the graphical method was employed to explicitly link algebraic solutions to the geometric properties of functions, such as the relationship between x-intercepts and roots (Chazan, 1992; Edwards & Chelst, 2019). Furthermore, a significant portion of the course was dedicated to analyzing common student misconceptions, requiring participants to critically unpack the algebraic structures and logic underlying standard algorithms before encountering the Po-Shen Loh's approach.

The researcher was also the course instructor, which allowed for a deeper contextual understanding but also required attention to potential bias. To strengthen trustworthiness, peer debriefing was carried out with a mathematics educator, who reviewed and discussed the coding process and emerging themes to ensure consistency.

Participants

The participants of this study were 22 PTs in a four-year teacher education program in the faculty of Education in Türkiye. The PTs participating in the study are students in the Elementary School Mathematics Teaching program who are taking the third-year "Algebra Teaching" course. All of the PTs have completed the following courses: Foundations of Mathematics I and II, Calculus I, II and III, Teaching of Numbers, Approaches to Mathematics Learning and Teaching. Their completion of these courses ensures that the participants had the necessary mathematical background and initial pedagogical preparation, so the difficulties observed reflect genuine issues of conceptual understanding rather than lack of prior instruction.

Data Collection Tools

The data collection for this study involved two primary methods: a worksheet with structured questions and follow-up interviews. This combination provided data triangulation, which strengthened the validity and reliability of the findings by enabling cross-checking from different perspectives. The worksheet (see **Appendix**) contained three key questions, each designed to gather insights into the teacher candidates' understanding of the new method (Po-Shen Loh's quadratic equation method), as well as their reflections on its application and potential future use in teaching. The interviews were used to further explore and validate the participants' responses from the worksheets.

A worksheet has been prepared as part of the research and is presented in the attachment. The worksheet demonstrates the application of the method through an example. Following this, as the first question, PTs were asked to explain the purpose of the method, the idea behind it, and why the method works. As the second question, PTs were asked to solve a quadratic equation using this method. Finally, PTs were asked whether they would use this method and if they would prefer it when they become teachers. During the interviews, questions were asked about, "what is the aim of the method?" whether they use the method or not?", and general thoughts/comments about the method.

Once participants had completed the worksheet, they were invited to take part in an individual interview. Every participant in the study was asked if s/he was willing to be interviewed; 7 out of 22 participants agreed. Two of the interviewees were male (PT1, PT19) and five were female (PT4, PT5, PT16, PT20, PT22). The interviews were conducted within 2-3 days after the study was administered to avoid retention loss. The interviews were semi-structured; audio recorded with the consent of the participants and lasted from 30 to 40 minutes. During the interviews participants were provided with their worksheet and were given about 5 minutes to think and recall the activity. The records were taken through Blackboard. In the Blackboard program, both the worksheets were displayed as screenshots, audio recordings could be made, and the worksheets could be tracked. Blackboard is an online learning management system (LMS) used for delivering course content, facilitating communication, and managing assessments. Within the scope of the course, we were able to use the Blackboard program for these purposes.

Interviews were conducted with the PTs to understand their general thoughts about the method, how they interpret the purpose of the actions taken within the method, and to gather their impressions. Additionally, the interviews aimed to identify the reasons behind any mistakes they made. Follow-up interviews were conducted to deepen the analysis by addressing any ambiguities and to validate the results from the worksheet responses.

Data Analysis

The data obtained from PTs were analyzed using content analysis. All interviews were audio-recorded and transcribed verbatim. The analysis aimed to understand how students reason about the new method for solving quadratic equations. In the first step, the worksheet responses were reviewed to familiarize the researcher with the data and note initial impressions, providing an overall understanding. For the first research question, the analysis focused on procedural steps, explanations, mistakes, and general tendencies. Beyond procedural accuracy, the analysis specifically sought evidence of whether participants could articulate conceptual links between algebraic manipulation and the visual, symmetric structure of the parabola. The analysis focused on whether participants recognized $-\frac{b}{2a}$ as a structural reference point and u as a symmetric displacement. This focus enabled and evaluation of what participants were expected to notice—specifically, the transition from seeing roots as isolated numbers to understanding them as functional properties centered around the vertex (Rittle-Johnson et al., 2001, 2015). Accordingly, in the introductory task, u was deliberately presented both as an algebraic result (±4) and a visual distance on the number line. This design was intended to create a conceptual tension to probe whether learners recognize how the symmetry of the parabola makes the choice of sign for u algebraically redundant in the $(\frac{x_1+x_2}{2}) \mp u$ form.

Each worksheet was analyzed individually for deeper insights. For the second question, the focus was on the procedural steps, result accuracy, and sources of mistakes.

Table 1. Example of codes and categories

Code	Category	Example from data
Formula reliance	Procedural understanding vs. conceptual understanding (Correct/incorrect formula use)	*PT16: I always find myself trying to solve it using a formula, and I tend to rely on formula. Right now, I'm also thinking about the solution in terms of a formula. (interview)*
Terminology differences	Terminology variations (Aritmetic mean (average) vs. midpoint, point)	*PT4: Then, the midpoint is marked, and since both roots are equidistant from this point…(worksheet)* *PT19: When students find the average of the sum of the roots…(interview)*
Reference to parabola	Conceptual understanding of quadratics	*PT19:…the vertex is equidistant from the roots of the parabola." (worksheet)*
Positive u value	Interpretation of u value	*PT20: The distance must be positive, it can't be negative, since u is the unit of distance between them. That's why we need to take the absolute value. (interview)*

To organize the data, especially for the first question, the answers were categorized into four sections: i) formulation of the roots' sum and product without formulas, ii) number line representation of the average of the roots, iii) explanations of "u²" and "u", and iv) solution set and interpretations. Each part was coded separately and categorized into themes such as procedural explanations vs. conceptual understanding, differences in terminology, positive "u" values. Once initial coding was completed, categories were refined into broader themes.

As the fourth step, the interview responses were integrated with the worksheet data to check consistency between written responses and verbal explanations. This helped capture deeper insights into the PTs' understanding and highlight points not mentioned in their worksheets. Through the interviews, it was revealed that confusion about the arithmetic mean and the sum and product of the roots was more pronounced than expected. Additionally, PTs did not fully address the first part of the methodology and relied on the sum of the roots (-b/a) instead of providing conceptual explanations.

The answers were first coded and categorized individually, and then analyzed together for inconsistencies. Responses to the third question were also re-examined for discrepancies with earlier answers. Finally, the categories and codes were applied to both worksheet responses and interview data. Examples of codes, categories, and some PTs' explanations are presented in **Table 1**.

As shown in **Table 1**, the codes and categories illustrate the main patterns in PTs' responses. The examples demonstrate that most participants tended to rely on formulas, while only a few provided deeper conceptual connections. In addition, variations in terminology (e.g., using "arithmetic mean," "midpoint," or simply "point") were also observed, reflecting differences in how PTs articulated the same concepts. These distinctions formed the basis for the broader themes discussed in the Results section. All the categories developed have been presented in the results section according to the research questions.

RESULTS

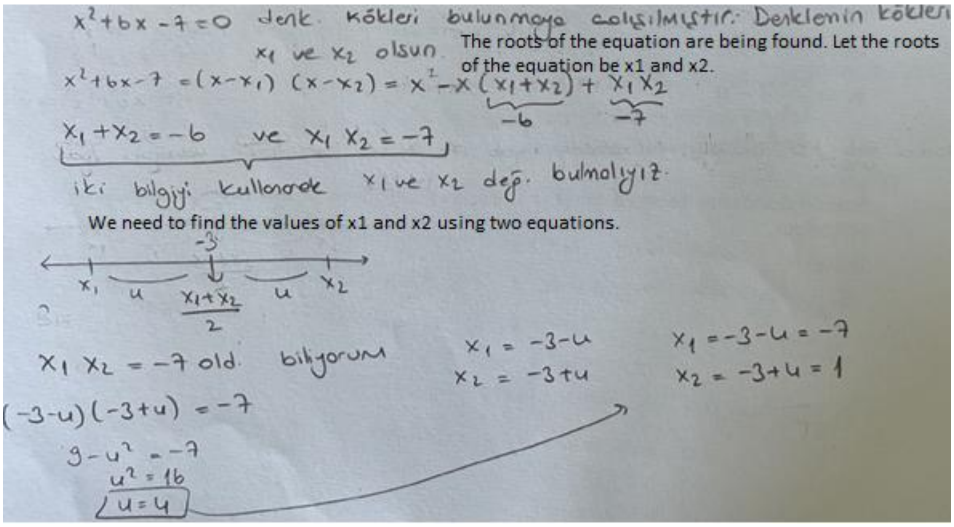
In the present study, PTs were asked to respond the three tasks: 1) explain the purpose of the given alternative method and what it attempts to achieve; 2) solve the provided equation ($x^2 - \frac{4}{3}x - 5 = 0$) using this method; and 3) indicate whether they would use this method in their future teaching.

Findings Related to the First Task

As the first question of the study, PTs were asked to answer: "Explain and evaluate what is being attempted in the alternative method". PTs expected to explain the aim of the method. The answers given by the PTs have been categorized in **Table 2**, drawing on the classification of student explanations suggested by Drageset (2021). As seen in the table, the explanations of the PTs have been evaluated in six categories: Procedural explanations, semi-conceptual explanations, concept explanations, incomplete and no understanding.

Table 2. Explanations of the method

Types of explanations	Description	Participants	Example explanations
Semi-Conceptual explanations	• Describes the stages of the process step by step • Explains that in the equation, $-b$ and c represent the sum and product of the roots, respectively. • Not providing explicit explanations or reasons for the aim of the method	PT5, PT13	*PT5: In the equation $ax^2 + bx + c = 0$, it expresses that b is actually the sum of the roots and c is actually the product of the roots. By distributing the parenthesis (x-x_1).(x-x_2) =0, It actually states that in the expression $ax^2 + bx + c = 0$, b represents the sum of the roots and c represents the product of the roots. In other words, it actually helped us understand where these (b and c) came from... (Interview)*
Procedural explanations	• Describes the stages of the process, providing rule-based explanations for the addition ($-b$) and multiplication (c) of the roots. • Not providing explicit explanations or reasons for the aim of the method	PT1, PT3, PT4, PT6, PT8, PT9, PT11, PT18, PT12, PT14, PT16, PT21, PT22	*PT4: First, the roots (x_1, x_2) are placed on the number line. Then, the midpoint is marked, and since both roots are equidistant from this point, they are represented by the symbol u. The focus has been on how to locate the roots on the number line. The expression for x_1 and x_2 is written on the number line. In the equation ($ax^2 + bx + c = 0$), the product of the roots is c. In the given example, this value is listed as -7. $u^2 = 9 + 7 = 16$, By substituting the value of u into x_1 and x_2 we get x_1=−7 and x_2=1, which gives us the roots of the equation. (Worksheet)*
Concept explanations	• Explains the process stages, linking them to the concept of "parabola" • Uses the formula $(-\frac{b}{a}, \frac{c}{a})$ for the sum and the product of the roots,	PT2, PT17, PT19	*PT 19: When students find the average of the sum of the roots, they are actually reaching the x-coordinate of the vertex of the parabola. At the same time, it has been suggested that the vertex is equidistant from the roots of the parabola.*
Incomplete	• Superficial explanations, unclear meaning	PT7, PT10	*PT10: I can't explain it even though I understand. (worksheet)*
	• Explanations that they could not understand the method	PT15, PT20	*PT15: I couldn't fully understand what the expression proves because I don't grasp the meaning of u. (worksheet)*



$x^2+6x-7=0$ denk. kökleri bulunmaya çalışılmıştır. Denklemin kökleri x_1 ve x_2 olsun

The roots of the equation are being found. Let the roots of the equation be x1 and x2.

$x^2+6x-7=(x-x_1)(x-x_2)=x^2-x\underbrace{(x_1+x_2)}_{-6}+\underbrace{x_1x_2}_{-7}$

$\underbrace{x_1+x_2=-6 \quad \text{ve} \quad x_1x_2=-7,}$

iki bilgiyi kullanarak x_1 ve x_2 değ. bulmalıyız.

We need to find the values of x1 and x2 using two equations.

$x_1x_2=-7$ old. biliyorum

$(-3-u)(-3+u)=-7$

$9-u^2=-7$

$u^2=16$

$\boxed{u=4}$

$x_1=-3-u$

$x_2=-3+u$

$x_1=-3-4=-7$

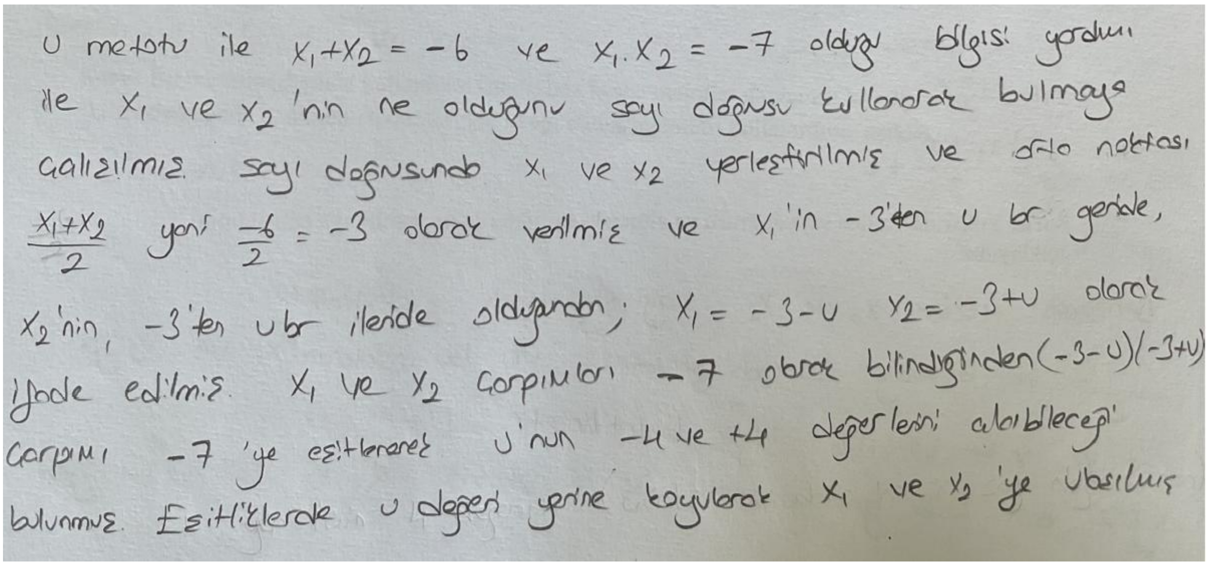
$x_2=-3+4=1$

Figure 1. A snapshot of PT13's worksheet

As seen in **Table 2**, the PTs provided explanations about the steps and stages of the method, rather than addressing its purpose. None of the PTs explained clearly the underlying idea of the method or why it works.

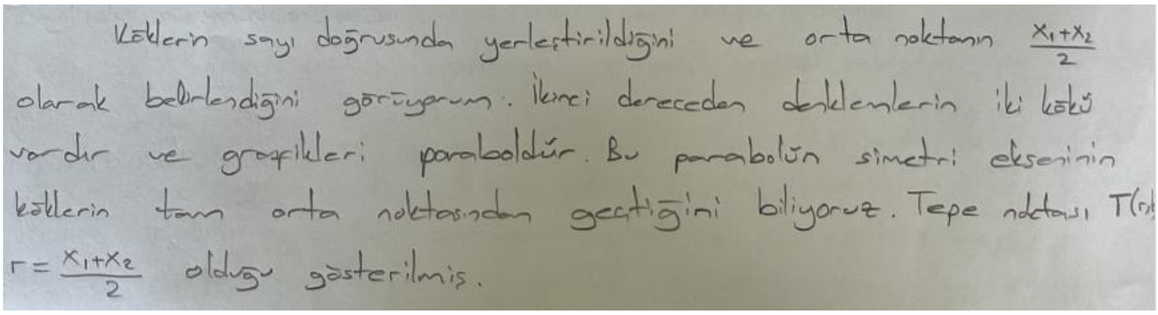
In **Figure 1**, it can be seen that PT13 provided procedural explanations, she found the roots of the equation using the steps of the method and provided a summary of the process. PT13 stated in the worksheet that the goal is to find the roots of the equation $(x^2 + 6x - 7)$ using the method.

PT18's work provides an example of the responses from other PTs who gave semi-procedural explanations to the question. As can be seen from **Figure 2**, PT18 wrote that "*Using the method, an attempt was made to find the values of x_1 and x_2 with the information that $x_1 + x_2 = 6$ and $x_1 . x_2 = -7$. The values of x_1 and x_2 were placed on the number line, and the midpoint, which is $\frac{x_1+x_2}{2}$, was given as $-\frac{6}{2} = -3$. Since x_1 is u units behind -3 and x_2 is u units ahead of -3, they are expressed as $x_1 = -3$ and $x_2 = -3 + u$.*"



u metotu ile $x_1+x_2=-6$ ve $x_1.x_2=-7$ olduğu bilgisi yardımı
ile x_1 ve x_2'nin ne olduğunu sayı doğrusu kullanarak bulmaya
çalışılmış. Sayı doğrusunda x_1 ve x_2 yerleştirilmiş ve orta noktası
$\frac{x_1+x_2}{2}$ yani $\frac{-6}{2}=-3$ olarak verilmiş ve x_1'in -3'ten u br geride,
x_2'nin, -3'ten u br ileride olduğundan; $x_1=-3-u$ $x_2=-3+u$ olarak
ifade edilmiş. x_1 ve x_2 çarpımları -7 olarak bilindiğinden $(-3-u)(-3+u)$
çarpımı -7'ye eşitlenerek u'nun -4 ve $+4$ değerlerini alabileceği
bulunmuş. Eşitliklerde u değeri yerine koyularak x_1 ve x_2'ye ulaşılmış

Figure 2. A snapshot of PT18's worksheet



Köklerin sayı doğrusunda yerleştirildiğini ve orta noktanın $\frac{x_1+x_2}{2}$
olarak belirlendiğini görüyorum. İkinci dereceden denklemlerin iki kökü
vardır ve grafikleri paraboldür. Bu parabolün simetri ekseninin
köklerin tam orta noktasından geçtiğini biliyoruz. Tepe noktası T(r
$r=\frac{x_1+x_2}{2}$ olduğu gösterilmiş.

Figure 3. PT2's explanation related to the first question

During the interview, PT16 stated rule/formula based explanations:

> PT16: I always find myself trying to solve it using a formula, and I tend to rely on formula. Right now, I'm also thinking about the solution in terms of a formula. But now I understand why b/a gives the sum of the roots, and why c/a gives the product of the roots (interview)

Some PTs provided conceptual explanations. For instance, PT2, wrote that "I see that the roots are positioned on the number line, and the midpoint is determined as $\frac{x_1+x_2}{2}$. A quadratic equation has two roots, and their graphs are parabolas. We know that the axis of symmetry of this parabola passes through the exact midpoint of the roots. It has been shown that the vertex $T(r,k)$ $r=\frac{x_1+x_2}{2}$.

Figure 3 presents an example of a concept explanation provided by PT2. In the written response, PT2 noted that a quadratic equation has two roots and referred to the midpoint of these roots, the axis of symmetry, and the vertex of the parabola while explaining the method. Therefore, the response was categorized as a concept explanation rather than a description of procedural steps.

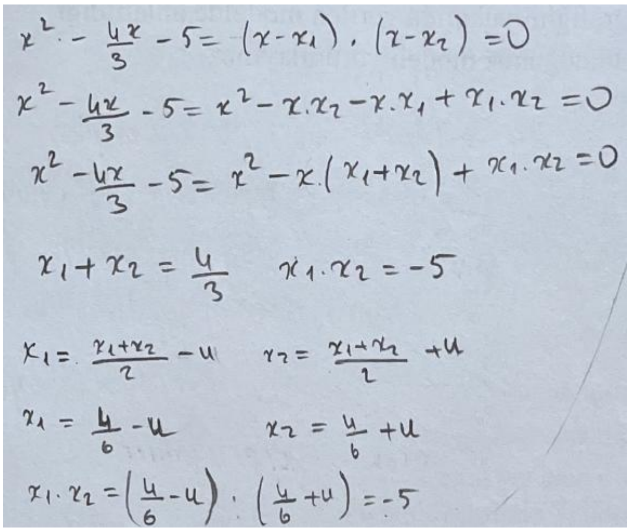
Four of the PTs (18%) provided superficial explanations or indicated that they did not understand the method.

Findings Related to the Second Task

To solve this problem, the PTs followed the procedures outlined in the method. The analysis of the data revealed two distinct categories. **Table 3** shows that out of 22 PTs, 10 (45%) attempted to reach a solution by following all the steps outlined in the method, while 11 (50%) PTs attempted to reach a conclusion by skipping the initial steps and instead using the previously memorized formula for the sum of the roots. Among 22 PTs, 10 PTs (45%) either found incorrect solutions or were unable to reach a solution.

Table 3. PTs solutions to $x^2 - \frac{4}{3}x - 5 = 0$ equation by using this method

Categories		**Results of the Solutions**	**Participants**	**f (%) (n=22)**
Procedural solutions (following steps given in the method)	• Proceeding by following all the steps outlined in the method.	Correct	PT5, PT11, PT13, PT18, PT20	5 (%23)
		Incorrect	PT4, PT8, PT10, PT14, PT21	5 (%23)
	• Skipping the initial steps and proceeding directly with a formula (sum of roots as $-\frac{b}{a}$)	Correct	PT2, PT3, PT6, PT9, PT12, PT17, PT19	7 (%32)
		Incorrect	PT1, PT7, PT16, PT22	4 (%18)
Irrelevant			PT15	1 (%0,5)



$$x^2 - \frac{4x}{3} - 5 = (x - x_1).(x - x_2) = 0$$

$$x^2 - \frac{4x}{3} - 5 = x^2 - x.x_2 - x.x_1 + x_1.x_2 = 0$$

$$x^2 - \frac{4x}{3} - 5 = x^2 - x.(x_1 + x_2) + x_1.x_2 = 0$$

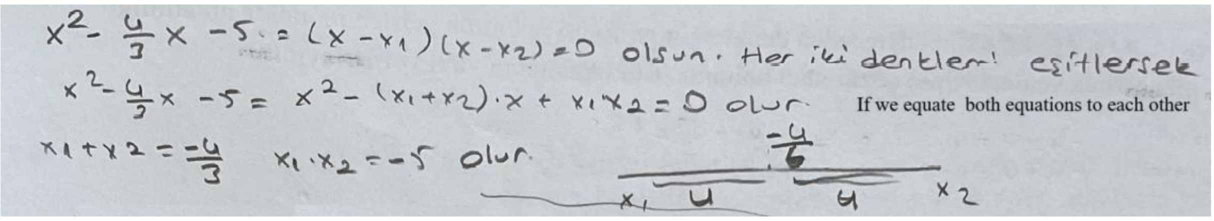
$$x_1 + x_2 = \frac{4}{3} \qquad x_1.x_2 = -5$$

$$x_1 = \frac{x_1+x_2}{2} - u \qquad x_2 = \frac{x_1+x_2}{2} + u$$

$$x_1 = \frac{4}{6} - u \qquad x_2 = \frac{4}{6} + u$$

$$x_1.x_2 = \left(\frac{4}{6} - u\right).\left(\frac{4}{6} + u\right) = -5$$

Figure 4. A snapshot from PT5's work on solving the equation



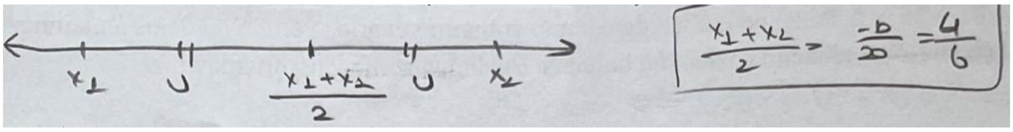
$x^2 - \frac{4}{3}x - 5 = (x - x_1)(x - x_2) = 0$ olsun. Her iki denklemi eşitlersek

$x^2 - \frac{4}{3}x - 5 = x^2 - (x_1 + x_2).x + x_1x_2 = 0$ olur

If we equate both equations to each other

$x_1 + x_2 = \frac{-4}{3}$ $x_1.x_2 = -5$ olur.

Figure 5. A snapshot from the work of PT14 on the solution of the equation



$$\boxed{\frac{x_1+x_2}{2} = \frac{-b}{2a} = \frac{4}{6}}$$

Figure 6. A snapshot from the work of PT17 on the solution to the given equation

Figure 4 shows PT5's work on the solution of the equation.

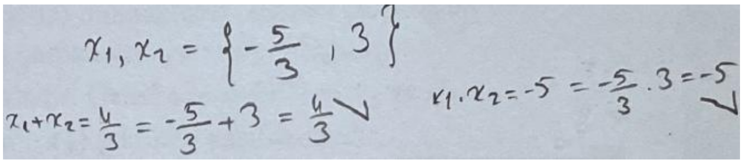
In the first category, all of the PTs who reached an incorrect solution (except PT21) (36%) took the sum of the roots as "negative" ($-\frac{4}{3}$), instead of $\frac{4}{3}$. **Figure 5** is part of PT14's work on solving the equation.

PT21 made a calculational mistake in the multiplication operation.

In the solution process of the equation, some PTs skipped the initial steps and progressed directly by a formula (the sum of the roots is equal to $-\frac{b}{2a}$). **Figure 6** shows the part of PT17's solution to the equation.

Again, as in the first category, the reason for the incorrect solutions by the PTs is that they took the sum of roots as negative ($-\frac{4}{3}$) instead of positive ($\frac{4}{3}$). Thus, in total, nine PTs (41%) took the sum of the roots as negative.

Upon examining the worksheets, it was observed that the verification of whether the result (roots) were correct or not was conducted only by PT5.



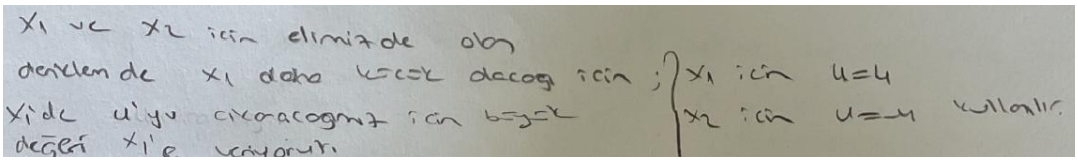
$x_1, x_2 = \{-\frac{5}{3}, 3\}$

$x_1+x_2=\frac{4}{3} = -\frac{5}{3}+3 = \frac{4}{3}$ ✓ $x_1 \cdot x_2=-5 = -\frac{5}{3}\cdot 3=-5$ ✓

Figure 7. A snapshot of PT5's work on the solution

Table 4. Interpretation of the "u value" for the first question

Categories	Explanation	Participants	f (%)
Correct explanations	- $+u$ or $-u$, it wouldn't make a difference	PT5	1 (0,5%)
Incorrect explanations	- For x_1; $+u$ for x_2; $-u$ should be used for respectively	PT1, PT8, PT16	4 (14%)
	- Since one of the roots is positive and the other is negative, instead of adding the roots, the distance between them is calculated.	PT17	1 (0,5 %)
	- Only $+u$ calculations	PT3, PT10, PT11, PT13, PT14, PT18, PT19, PT20, PT22	9 (41 %)
No explanation		PT2, PT4, PT6, PT7, PT9, PT12, PT15 (couldn't understand), PT21	8 (36%)



x_1 ve x_2 için elimizde olan denklemde x_1 daha küçük olacağı için; x_1'de u'yu çıkaracağımız için büyük değeri x_1'e veriyoruz.

x_1 için $u=4$
x_2 için $u=-4$ kullanılır.

Figure 8. Part of PT8's work on the first question

Table 5. Interpretation of the "u value" for the second question

Categories		Participants	f (%)
Sign of "u" is positive	Denotes first $u = +/-$, then taking "*u" value as positive*	PT1, PT2, PT3, PT4, PT 5, PT 7, PT 9, PT 10, PT 12, PT 13, PT 14, PT 16, PT 17, PT 18, PT 21,	14 (%81)
	Denotes $u^2 = +$ and taking "u" value as positive	PT6, PT11, PT 13, PT19, PT20 PT22	5 (23%)
Incomplete		PT8	1 (%1)
No calculation for u		PT15	1 (%0,5)

Finally, it is unclear what PT15 was attempting to do. She also stated that she could not understand the method.

Other points: General Tendencies of Prospective Teachers

This part outlines additional findings about the PTs' general tendencies, with a particular focus on their interpretations of the value of *u* (Part A) and their reasoning about the arithmetic mean and midpoint (Part B).

Interpretation of the u value: The value of "u" should be "positive".

Table 4 summarizes participants' interpretations of the value of (u) in the first question. While only one participant provided a correct explanation, most participants either focused solely on the positive value of (u), provided incorrect interpretations, or did not offer an explanation.

PT5 explained on the worksheet: *"From $u^2 = 16$ we found that u is +4 or -4. Now that we have found u, let's find the roots of the equation. Let's substitute +4 (It wouldn't matter if we used -4 but since it represents a distance, it is more appropriate to express it positively.)*

Figure 8 presents an example of PT8's interpretation of the value of (u) in the first question. The participant associated the positive and negative values of (u) with different roots and explained that (+u) and (-u) should be used accordingly.

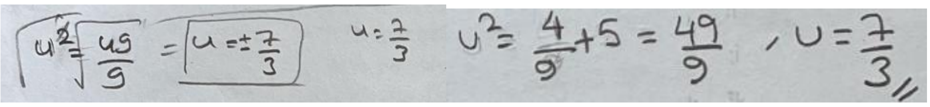
PT8 explained on the worksheet: "*For x_1 and x_2 since x_1 is smaller in the given equation, we assign the larger value of u to x_1, as we will subtract u from it. Therefore, $u = +4$ is taken for x_1 and $u = -4$ taken for x_2.*"

The findings indicate that many PTs struggled with interpreting the sign of u. While a few recognized that both positive and negative values were possible, most insisted on a positive distance, treating u as an absolute value. This shows a strong procedural orientation and a lack of reflection on the dual nature of solutions.

Similar difficulties were evident in the second task, where a majority of PTs again emphasized that *u* must be positive. Although this approach yielded technically valid solutions, it suggests a limited conceptualization of the meaning of "±" in quadratic contexts.

Participants' interpretations of the value of (u) in the second question are summarized in **Table 5**.

The codes in Category 1 (*Sign of "u" is positive*) are illustrated in **Figure 9**.



$\boxed{u^{2}=\sqrt{\frac{49}{9}}}=\boxed{u=\pm\frac{7}{3}}$ $u=\frac{7}{3}$ $u^2=\frac{4}{9}+5=\frac{49}{9}, u=\frac{7}{3}$

Figure 9. A snapshot of PT14’s and PT11’s work, respectively

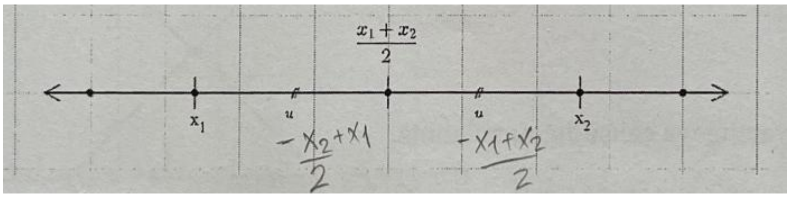
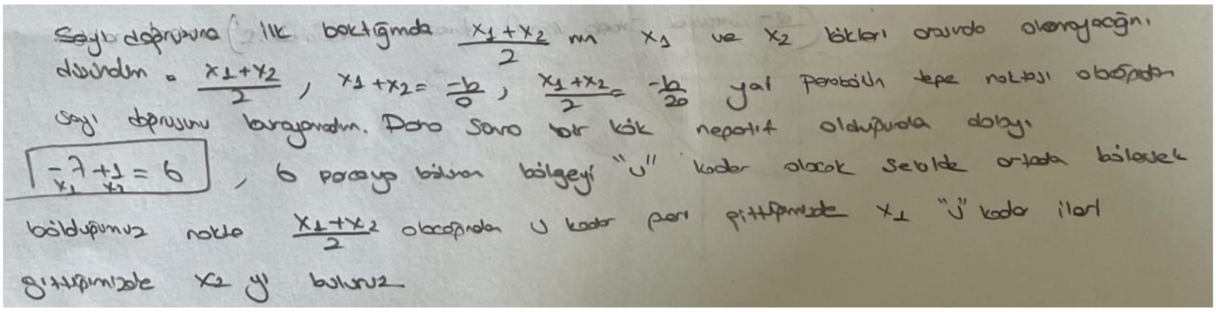


Figure 10. A snapshot of PT19’s work



Sayı doğrusuna ilk baktığımda $\frac{x_1+x_2}{2}$'nin x_1 ve x_2 kökleri arasında olamayacağını düşündüm. $\frac{x_1+x_2}{2}$, $x_1+x_2=\frac{-b}{a}$, $\frac{x_1+x_2}{2}=\frac{-b}{2a}$ yani parabolün tepe noktası olduğundan sayı doğrusunu kavrayamadım. Daha sonra bir kök negatif olduğundan dolayı $\boxed{\underset{x_1}{-7}+\underset{x_2}{1}=6}$, 6 parçaya bölünen bölgeyi "u" kadar olacak şekilde ortadan bölerek böldüğümüz nokta $\frac{x_1+x_2}{2}$ olacağından u kadar geri gittiğimizde x_1 "u" kadar ileri gittiğimizde x_2'yi buluruz

Figure 11. A snapshot of PT17’s work

Average and midpoint: PTs’ Different Uses and the Effect of Negative Root

In the analyses, different explanations regarding the value of $(\frac{x_1+x_2}{2})$ have been observed in the PTs’ explanations. These explanations fall into four categories: Average/arithmetic mean, midpoint, point, and average could not lie between the roots.

The analysis related to the categories is presented in **Table 6**.

Two of the PTs (PT17 and PT19) thought that because one of the roots is positive and the other is negative, the average value of the roots cannot lie between the two.

> PT19: That’s what an average is! The average of any set of numbers is the number in the middle of the set. For example, let’s say you have two numbers, a and b. For simplicity, we’ll assume that a is the smaller of the two. The distance between these numbers is b - a, so half of that distance is (b - a) / 2. The halfway point is then a + (b - a) / 2; you start at a and move halfway toward b. But a + (b - a) / 2 = (2a + b - a) / 2 = (a + b) / 2. In other words, the halfway point is the average. However, I find it difficult to visualize, especially when one number is negative and the other is positive. (interview)

PT17 similarly stated on the worksheet that since one of the roots is negative, (x1 + x2)/2 cannot be between the two roots.

> PT17: “When I first looked at the number line, I thought that (x1 + x2)/2 couldn’t be between x1 and x2. The equation (x1 + x2)/2 becomes (x1 + x2) = -b/a and (x1 + x2)/2 = -b/2a, so it represents the vertex of a parabola, which made me not fully grasp the concept on the number line. Later, since one root was negative, I realized that -7 + 1 = 6, the region is divided into “u” parts. If we divide this region in half, the point we get is (x1 + x2)/2. So, by going “u” units back from this point, we find x1, and by going “u” units forward, we find x2. (worksheet)

Table 6 presents the categories related to how PTs interpreted the arithmetic mean and midpoint. While more than half described the midpoint correctly, some PTs used vague terminology or misunderstood the position of the average when negative values were involved. Variations in terminology (e.g., “arithmetic mean,” “midpoint,” “point”) further illustrate inconsistent use of mathematical language. A small group even believed that the average of a positive and a negative root could not lie between the roots, reflecting conceptual misconceptions. These differences point to challenges in connecting symbolic procedures with graphical or conceptual interpretations.

Table 6. PTs' interpretations of "average" and "midpoint" in relation to the roots

Categories	Participants	Explanations	f (%)
Average/aritmetic mean	PT1, PT5, PT7, PT9, PT16, PT21	*PT1: "On the number line, the roots of the equation are located. The arithmetic mean of the two roots has been found. The average is equidistant from both roots on the number line, and this distance is u.*	6 (27%)
Mid-point	PT2, PT3, PT4, PT6, PT8, PT10, PT11, PT12, PT13, PT14, PT20, PT22	*PT22: The roots x1 and x2 have been placed on the number line, and then the midpoint has been marked.*	*12 (%55)*
Point	PT15, PT18	*PT18: It is stated that the point (x1 + x2)/2 is equidistant from x1 and x2 by a distance of u.*	*2 (10%)*
Average of x_1 and x_2 could not lie between the roots	PT17, PT19	*PT17: At first glance, I thought that the average of x1 and x2 could not lie between the roots.*	*2 (10%)*

Table 7. PT's preferences for using the method

	Participants	Explanations
Prefer using this method	PT2, PT3, PT4, PT5, PT11, PT13, PT14, PT15, PT16, PT17, PT18, PT21, PT22	*PT4: "I'm seeing this method for the first time, but I like the way the roots are placed on the number line. I think it effectively demonstrates to the student how to proceed with only the information they have.*
Not prefer	PT1, PT6, PT7, PT8, PT9, PT12,	*PT12: "I think the method is complicated; there are many calculations, and the plus and minus signs can be confusing."*
No explanation	PT10, PT19, PT20	

Overall, the general tendencies observed across these tasks suggest that PTs primarily relied on procedures, often misinterpreted signs, and displayed inconsistent use of terminology. Such tendencies highlight the need to strengthen conceptual understanding and precise use of mathematical language in teacher education.

Findings Related to the Third Task: Preference for Using the Method

Finally, PTs were asked whether they would use this method and if they would prefer it when they become teachers. In **Table 7**, overall answers for the question outlined. The analysis of the data were given in **Table 7**.

As shown in **Table 7**, 13 out of 22 participants (59%) stated that they preferred using this method, 6 (27%) did not prefer it, and 3 (14%) did not provide an explanation. While these frequencies provide a general overview of participants' tendencies, the emphasis in this qualitative study is placed on the explanations and reflections shared by the participants. Those who preferred the method explained that they liked its visual representation of roots on the number line and believed it could help students solve equations using the knowledge they already have. In contrast, those who did not prefer the method emphasized its complexity, the number of steps involved, and the potential confusion caused by positive and negative signs.

DISCUSSION

This study provides a detailed account of PTs' thinking about how they understand the new method for solving quadratic equations. The results showed that instead of grasping the conceptual aspects, PTs primarily focused on explaining the procedures and writing down the steps, sometimes facing difficulties with the process. In the present study, although most of the PTs did not fully understand the method from the worksheet, they managed to reach solutions by following the given steps, and at times, they proceeded by using previously memorized methods while trying to understand the new approach. The analysis revealed that while the new method was understood procedurally by many students, there were gaps in their conceptual understanding. Most of the students (72%) used the formula for the sum $(-\frac{b}{a})$ and product of the roots $(\frac{c}{a})$ in the first part of the first question. This indicates that the PTs started the solution of the problem using the formula, following the methods they were familiar with. During the interviews, some of the PTs stated that "*they often tend to rely on the formula*" when solving equations. The fact that most of them use the formula for the sum and product of the roots in the initial stages of the solution supports this observation. Furthermore, four students (18%) either did not understand the method or provided very superficial and unclear explanations, while only two students offered semi-conceptual explanations and described the method as presented, independent of the formula. These semi-conceptual explanations allowed them to use method without fully understanding or being able to explain it, thus serving as a bridge to the formation of true explanations.

The observed gap between the PTs' procedural success and their limited conceptual articulation can be understood through the iterative model of conceptual and procedural knowledge (Rittle-Johnson et al., 2001). According to Rittle-Johnson et al. (2015), these two types of knowledge do not develop in isolation but rather in a "hand-over-hand" fashion. The participants' ability to follow the steps of Loh's method, even without a robust grasp of its structural logic, may represent the initial procedural phase that provides the baseline for subsequent conceptual gains (Rittle-Johnson et al., 2001, 2015). Their reliance on formulas for the sum and product of roots indicates that they were attempting to integrate new procedural actions into their existing algebraic schema (O'Connor & Norton, 2024). Therefore, the "procedural reliance" seen in the results is not necessarily a failure of the method, but rather a snapshot of a developmental process where procedural fluency is currently leading conceptual depth (Rittle-Johnson et al., 2001, 2015). Our results align with Newton et al. (2022), who found that pre-service teachers often succeed in procedural tasks related to quadratic equations but struggle to draw on specialized content knowledge for conceptual explanations.

Where procedural or calculational explanations describe a procedure to solve a problem while conceptual explanations describe the reasons for the steps. These two types of explanation (process vs. reason and concept) should not be seen as fundamentally different, but instead as belonging to each other in a bidirectional relationship (Bowers & Doerr; 2001; Rittle-Johnson et al., 2015).

The interviews provided deeper insight into why some of these gaps in their conceptual understanding occurred, particularly in the understanding of the arithmetic mean and the representation of the roots on the number line. Data analysis revealed that some PTs had difficulties with equations involving negative coefficients and negative roots. Instead of calculating $\frac{x_1+x_2}{2}$ to find the average, as the second root is negative, they tended to use $\frac{x_1-x_2}{2}$ to eliminate the negativity. Another issue related to the negativity is that some PTs, as if recalling the formula incorrectly, calculated the sum of the roots as $\frac{b}{a}$ instead of $-\frac{b}{a}$ led to calculation errors. This might be due to the given equation having a negative coefficient ($x^2-\frac{4}{3}x-5=0$). This finding suggests that some PTs may not have engaged in a control phase during their problem-solving process. As memorizing formulas can be misleading as in this case, they were expected to verify whether the roots they found satisfied the equation, and in this way, they might have been able to identified their mistakes. The findings align with O'Connor and Norton's (2024) study, which also revealed that both errors reflect a lack of conceptual understanding of quadratics and their forms. The study highlighted that difficulties with foundational algebraic concepts hindered students' ability to successfully understand and work with quadratics.

On the other hand, the roots of the quadratic equation ($u^2=16$) result in two values of u, but PTs took the positive one. The majority of the PTs indicated that the value of "u" should be positive, as it represents a distance. Some interpreted the plus and minus signs as "addition" and "subtraction". They explained that the value of "u", when positive, would be added to the average to find one root, while the other root would be obtained by subtracting "u" from the average. However, in this case, what they were actually doing involved a single value of u, which was positive.

The analysis showed that PTs encounter difficulties even with some basic concepts. These results are consistent with Nielsen's (2015) thesis, which, although conducted with high school students, also focused exclusively on the positive solution. Similarly, Thorpe (1989) suggested that students may expect equations to have only one solution, and students do not realize that equations of the form $x^2=a$ have two solutions, one positive and one negative, and that the meaning of the plus or minus symbol (±) in the quadratic formula might not be fully understood.

This tendency to prioritize the positive value of u confirms that the intentional ambiguity in the introductory example served its purpose as a diagnostic tool. By presenting u as both a symbolic result of ±4 and a visual distance on a number line, the task revealed a "procedural embodiment" (Tall et al., 2014) where participants felt more comfortable treating u as an absolute distance. This preference highlights a common gap in relational understanding, where the formal algebraic properties of the variable are overshadowed by the more intuitive, yet limited, geometric concept of unsigned distance.

The concepts of arithmetic mean and midpoint are fundamental, yet misconceptions about these concepts are observed in PTs. Some PTs made errors as one of the roots is negative, which caused confusion among them. The relationship between a midpoint and an average (mean) highlights the interplay between procedural and conceptual knowledge in mathematics education for teaching, with a relational understanding requiring the establishment of meaningful connections across various aspects of the mathematical concept (Bu, 2013). Students could struggle to understand that the arithmetic mean of two roots, one positive and one negative, is the midpoint of those roots for several reasons. One reason could be visual understanding, as the visual representation on the number line could be confusing for some students, making it hard for them to understand what the length between the two roots represents. Additionally, students may struggle to connect the mathematical definition of the arithmetic mean with its visual representation on the number line. Bu (2013) explained that the complexity of the midpoint formula could stem from the multiple meanings of the term "point". The term "point", such as B, could be interpreted as a geometric point, a number, the distance from the origin O to point B, or as the length of segment OB. These factors can hinder students' understanding of the concept and indicate the need for teachers to explain these ideas more effectively. Using more visual aids and concrete examples may help overcome these difficulties and support conceptual understanding. Additionally, the data revealed differences in the use of mathematical language among PTs. Differences were observed in the use of mathematical language among pre-service teachers, with terms such as "average of the roots", "midpoint", and "point" being used to describe the same concepts. Tsamir and Tirosh's (2023) research emphasizes the critical role of mathematical language in students' understanding and problem-solving. They highlight that mathematical language not only facilitates the expression of concepts but also shapes conceptual understanding, with teachers playing a key role in helping students effectively use this language to grasp abstract mathematical ideas.

Although the majority of the PTs (around 60%) stated that they would be able to apply the method in their future classrooms, 6 students (27%) indicated that they found the method confusing, as there are many calculations, and 3 teacher candidates (13%) did not provide any comments. This might result from the PTs' difficulties in understanding the method, which may stem from their unfamiliarity with its approach, as they are accustomed to formula-based methods for solving problems. Another reason could be that although Loh's method offers a way for students to solve quadratic equations without memorization, some PTs found it difficult and complex, possibly because they assumed that this new method was just another approach requiring memorization, as they are already able to solve equations with the formula. They may have viewed it as an additional cognitive load, rather than a method that avoids memorization. This finding aligns with the work of Arcavi (2003) and Wilkie (2021), emphasizing the need to address the affective dimension in the learning and teaching of such tasks, especially as some PTs (future teachers) encountered difficulties.

CONCLUSION AND IMPLICATIONS

The findings suggest that the Po-Shen Loh's method should be introduced not as a "shortcut", but as a way to integrate procedural fluency with conceptual depth (Rittle-Johnson et al., 2015) Instruction should emphasize the bi-directional relationship between the algebraic $\frac{x_1+x_2}{2} \mp u$ form and the visual symmetry of quadratic functions (Rittle-Johnson et al., 2015). By focusing on why the method works - grounded in the fact that real roots are always equidistant from the axis of symmetry – it is suggested that teachers can help students avoid common rote-learning pitfalls and build a more robust schema for algebraic structures (Edwards & Chelst, 2019; O'Connor & Norton, 2024).

The findings of this study revealed that while many PTs were able to follow the procedural steps of Loh's method, their conceptual understanding remained limited, which suggests that difficulties in learning quadratic equations are closely linked to the way they are taught. Therefore, being able to grasp and apply new methods in teaching is an important skill for PTs. This ability allows them to engage with different approaches to teaching mathematical concepts and enhances their problem-solving capabilities. As Pimm (1982) argues, a teacher's role is to guide students through the problem-solving process, helping them to make connections between ideas and to develop critical thinking skills that are far more valuable than simply memorizing formulas. In this context, understanding why a procedure works and when to apply it allows a student to handle new problems more effectively than one who lacks this knowledge, underscoring the importance of understanding the reasoning behind methods. Therefore, teachers should recognize that solving quadratic equations involves more than just following procedures and rules, and should explore alternative methods for teaching quadratic equations. (Didis & Erbas, 2015). Po-Shen Loh's approach to solving quadratic equations emphasizes understanding the relationship between the average of the roots and their distance, rather than just memorizing formulas for the sum and product of the roots. This method encourages students to explore the underlying connections between the roots, offering a more intuitive and conceptual understanding of quadratic equations. By focusing on the reasoning behind these relationships, Loh's approach helps students move beyond simple memorization and fosters a deeper understanding of why the solutions work. This initial reliance on procedural steps, even when using a method designed for relational understanding, aligns with Jupri and Sispiyati's (2017) finding that even experts often default to standard procedures when first encountering algebraic structure tasks. This suggests that the transition from procedural to structural thinking is not immediate; experts themselves often pay less attention to the equation's structure initially and only reconsider their strategies upon further reflection or specific prompts. In this study, the PTs' tendency to use the steps as a "recipe" can be interpreted through the iterative process model (Rittle-Johnson et al., 2015), where procedural skill and conceptual understanding develop in a bidirectional, hand-over-hand fashion. As noted by Rittle-Johnson et al. (2015), gain in one type of knowledge often triggers gains in the other. Therefore, the procedural adoption of the Po-Shen Loh's method by PTs represents a developmental stage in building a more robust relational schema, rather than a failure of the method's conceptual affordance. However, it is vital to recognize that the method's potential for promoting relational understanding (Vaiyavutjamai & Clements, 2006) is not automatic. As the results suggest, without intentional instructional scaffolding, students may simply treat the Po-Shen Loh's approach as another "magic" algorithm or procedural embodiment (Tall et al., 2014). To resolve the tension between the method's design and its initial procedural use, instruction must explicitly focus on the bi-directional links between algebraic variables and geometric symmetry (Rittle-Johnson et al. 2015). Teachers should use the procedural steps as prompts for reflection, asking "why" each step works to help students move from instrumental to relational understanding (Leavy & O'Loughlin, 2006; Vaiyavutjamai & Clements, 2006).

The method should be framed as a *framework for exploration* rather than a replacement procedure (Nemirovsky & Rasmussen, 2005), ensuring that students reach a higher level of abstraction through consistent reflection on the underlying connections (Rittle-Johnson et al., 2015; Verhoef & Broekman, 2005).

Mathematics instruction in schools typically emphasizes teaching students the rules and formulas needed to arrive at correct answers, rather than focusing on the underlying mathematical concepts and the reasoning behind the procedures (Sarwadi & Shahrill, 2014). The purpose of school mathematics should not be limited to providing students with basic mathematical knowledge and memorizing formulas. Rather, if the goal is to develop students' mathematical thinking skills, this approach would be highly beneficial. If the aim of mathematics education is to help students acquire the ability to analyze problems and develop solution strategies, as well as engage in discussions about these strategies, Loh's method would enable them to learn concepts through understanding, thereby enhancing their interest in mathematics. The method in question, Loh's approach to solving second-degree equations, allows for solutions without relying on predefined formulas, focusing instead on applying fundamental concepts. This approach emphasizes conceptual understanding rather than rote memorization, aligning with current educational paradigms that promote deep learning and critical thinking (Hiebert & Grouws, 2007). However, the discomfort and confusion expressed by the PTs may reflect a broader challenge within education: the transition from traditional methods, which often emphasize procedural knowledge and formulaic solutions, to more conceptual, understanding-based methods. Research has shown that students and educators alike may struggle with new methods that depart from familiar, structured ways of thinking (Cai, 2004).

In the context of teacher education, these findings highlight the importance of providing adequate support and scaffolding when introducing alternative solution methods. While the Loh's method is a powerful tool that enables students to solve quadratic equations using basic principles, it requires a shift in thinking that some PTs may find challenging. This resistance to non-formulaic methods may be rooted in the reliance on algorithmic approaches that dominate traditional mathematics curricula. Furthermore, the lack of familiarity with such approaches in teacher training programs may contribute to the initial difficulty in grasping the method. Despite these challenges, the Loh's method offers significant advantages over traditional methods, as it fosters a deeper understanding of mathematical concepts and equips students with flexible problem-solving skills.

Some PTs attempted to understand quadratic equations by interpreting them as parabolas and finding the roots through factoring. Future research could focus on exploring the conceptual understanding of quadratic equations using geometric approaches (Allaire & Bradley, 2001) graphical methods (Kristen, 2021), digital tools (Abramovic, 2016; Carballo et al., 2022), and investigating quadratic expressions as the product of two linear factors. Additionally, Edwards and Chelst (2019) suggest that seperating the quadratic formula into two terms enhances its meaning for students by clarifying the connection between the graphical representation and the equation's parameters. Understanding the components of the quadratic formula better equips students to apply it and interpret the resulting solutions, while examining the formula also provides students with opportunities to gain insight into the structure of algebraic expressions (Edwards & Chelst, 2019; Kristen, 2021). The effects of these different approaches on how they influence the conceptual understanding of quadratic equations/functions can be explored for future research. Teacher training programs should focus on increasing familiarity with such methods, as well as providing strategies to ease the transition from formulaic approaches to conceptual, understanding-based ones. By integrating these approaches gradually, teachers will be better prepared to use and teach these powerful methods in their classrooms.

Although the study was developed under the assumptions of the previous curriculum, its findings strongly align with the pedagogical vision of the new curriculum. The findings of this study are particularly relevant given the transition to the Türkiye Century Education Model (MoNE, 2024), which represents a substantive shift from a procedural focus toward a functional and modeling-based approach (MoNE, 2024). Unlike the 2018 curriculum, which often treated quadratic equations as isolated algebraic tasks, the most recent curriculum integrates them into functional themes, emphasizing relational understanding and graphical reasoning (MoNE, 2024). By moving away from formula-heavy instruction toward a method grounded in the geometric properties of functions, such as parabolic symmetry, educators can better align with this new curricular emphasis. Integrating Loh's approach into classrooms supports the transition from seeing equations as mere symbolic tasks to understanding them as functional models (MoNE, 2024).

By encouraging students to engage in algebraic reasoning through symmetry and simple algebraic operations, the study reflects the curriculum's shift toward concept-based, meaningful mathematics education in Türkiye. Future studies should continue to explore methods like Loh's, which harmonize procedural fluency with conceptual understanding within modern curricular structures.

Incorporating Loh's method and abovementioned methods into textbooks would be extremely beneficial in terms of increasing students' interest in mathematics and developing their mathematical thinking skills. Such an educational approach not only strengthens mathematical achievement but also enhances overall thinking and problem-solving abilities and encouraging them to become more creative and independent in their problem-solving processes. This approach should be integrated into the textbooks, ensuring they support teachers in applying it effectively and integrated into the curricula to support conceptual understanding of not only quadratic equations but also fundamental algebraic concepts (O'Connor & Norton, 2024) and should be designed in a way to achieve pedagogic excellence in learning mathematics (Hendel, 2018).

This study was conducted with a limited number of PTs, and a suggestion for future research is to involve a larger group of PTs and teachers. Additionally, for future research, the method could also be explored further as a whole class discussion, focusing on enhancing argumentative discussions in both whole class and small group interactions. This could offer valuable insights into how these interactions contribute to student learning and the development of mathematical ideas (Ofri & Tabach, 2025). It is believed that this study supports the conceptual learning of PTs, which in turn helps enhance their students' algebraic thinking and learning algebraic topics conceptually.

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APPENDIX

An Alternative Method

Below, the process of finding the roots of quadratic equations with one variable is explained using a different method. Please review the explanations provided regarding this method, which is illustrated through an example.

In general, the factors of a second-degree equation with roots x_1 and x_2 can be expressed as

$(x - x_1).(x - x_2)$.

For instance, if the roots are $x_1 = 7$ and $x_2 = 1$, then we obtain below.

$$(x - 7).(x - 1) = x^2 - x - 7x + 7 = x^2 - 8x + 7$$

In this case, for the below equation to hold, one of the factors must be equal to zero.

$$x^2 - 8x + 7 = (x - 7).(x - 1) = 0$$

Then when we add the numbers 7 and 1 we find the coefficient $-b$ in the equation and when we multiply 7 and 1, we obtain the constant c in the quadratic equation.

To clarify this situation let's try another equation.

What is expected of you is to understand what is being attempted in the explanations below and then answer the questions asked.

The solution of the equation $x^2 + 6x - 7 = 0$ is modeled step by step below.

Let, x_1 and x_2 the roots of the equation $x^2 + 6x - 7 = 0$

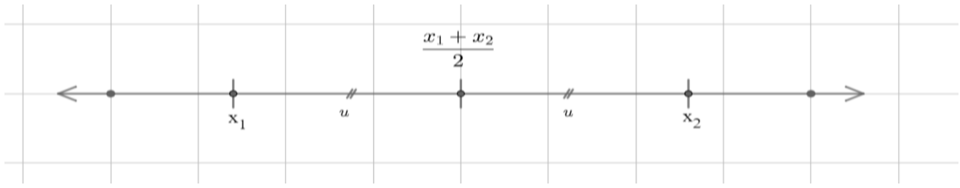
Then, we had seen that this equation could be written in the form below from the example provided above.

$x^2 + 6x - 7 = (x - x_1).(x - x_2) = 0$

If we equate both equations, we get

$x^2 + 6x - 7 = x^2 - (x_1 + x_2).x + x_1.x_2 = 0$

This equation gives us $x_1 + x_2 = -6$ and $x_1.x_2 = -7$



Let $u \in R$,

$x_1 = \frac{x_1+x_2}{2} - u$, $x_1 = \frac{x_1+x_2}{2} + u$

$x_1 = -3 - u$ and $x_2 = -3 + u$

$x_1.x_2 = (-3 - u)(-3 + u) = 9 - u^2$

$9 - u^2 = -7$

$u^2 = 16 \Rightarrow u = \pm 4$

$x_1 = -3 - u = -3 - 4 = -7$, $x_2 = -3 + u = -3 + 4 = 1$

$x_1 = -7$ ve $x_2 = -1$ are found.

Questions:

1) Explain and evaluate what is being attempted in the method described above. Explain what is the aim of the method?
2) Use this method to solve the equation $x^2 - \frac{4}{3}x - 5 = 0$
3) Would you use this method in your lessons?