Patterns of Metacognitive Behavior During Mathematics Problem-Solving in a Dynamic Geometry Environment

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This paper describes the problem solving behavior of two preservice teachers as they worked individually on three nonroutine geometry problems. A dynamic tool software, namely the Geometer's Sketchpad, was used as a tool to facilitate inquiry in order to uncover and investigate the patterns of metacognitive processes. Schoenfeld's (1981) model of episodes and executive decisions in mathematics problem solving was used to identify patterns of metacognitive processes in a dynamic geometry environment. During the reading, understanding, and analysis episodes, the participants engaged in monitoring behaviors such as sense making, drawing a diagram, and allocating potential resources and approaches that helped make productive decisions. During the exploring, planning, implementation, and verification episodes, the participants made decisions to access and consider knowledge and strategies, make and test conjectures, monitor the progress, and assess the productivity of activities and strategies and the correctness of an answer. Cognitive problem-solving actions not accompanied by appropriate metacognitive monitoring actions appeared to lead to unproductive efforts. Redirection and reorganizing of thinking in productive directions occurred when metacognitive actions guided the thinking and when affective behaviors were controlled.

Keywords: problem solving, metacognition, nonroutine geometry problems, preservice teachers, dynamic geometry software

At the beginning of the 21st century "the rapid mathematization of work in almost all areas of business, industry, personal decision making, and the social and life sciences dictates that most students learn more and different mathematics than school mathematics programs provide" (Fey, Hollenbeck, & Wray, 2010, p. 41) creating unprecedented challenges in schooling practices. Nowadays, topics taught in mathematical classes require more than mere arithmetic or calculation skills, but rather extension and adaptability of previous knowledge, and flexibility in thinking. On the other hand, since the 1980s mathematics educators have agreed upon the idea of developing problem solving ability and problem solving has become a focus of mathematics education as a means of teaching curricular material and seeking the goals of education (Stanic & Kilpatrick, 1989). Nevertheless, mathematical problems have been central in the mathematics school curriculum since antiquity. Lesh and Zawojewski (2007) defined mathematical problem solving as

> the process of interpreting a situation mathematically, which usually involves several iterative cycles of expressing, testing, and revising mathematical interpretation—and of sorting out, integrating, modifying, revising or refining clusters of mathematical concepts from various topics within and beyond mathematics. (p. 782)

Nowadays, problem solving plays a prominent role in the curriculum for several reasons: (1) to build new mathematical knowledge, (2) to solve problems that arise in mathematics and in other contexts, (3) to apply and adapt a variety of problem-solving strategies, and (4) to monitor and reflect on the mathematical problem-solving processes (NCTM, 2000). Despite the emphasis given to mathematical problem solving, however, research (Garofalo & Lester, 1985; Schoenfeld, 1985, 1987; Silver, 1994) shows that students' low problem-solving performance is not due to the inadequacy of mathematical content knowledge and facts, but rather is associated with students' inability to analyze the problem, to fully understand it, to evaluate the adequacy of given information, to organize knowledge and facts they possess with the goal of devising a plan, to evaluate the feasibility of the devised plan before its implementation, and to evaluate the reasonableness of the results. Hence, individual's awareness, consideration, and control of his or her own cognitive processes—metacognitive behaviors—are held to be essential in mathematics problem solving (Flavell, 1976).

Metacognition in problem solving helps the problem solver to recognize the presence of a problem that needs to be solved, to discern what exactly the problem is, and to understand how to reach the goal (solution). For the successful solution of any complex problem-solving task, a variety of metacognitive processes is necessary; regulatory activities of planning, monitoring, testing, revising, and evaluating throughout problem solving, especially in making the mental representation and selecting and assessing the effectiveness of the strategies employed (Brown, 1978, 1987; Flavell, 1992; Schraw, 1998). Therefore, metacognition is a critical component in cognitive function and cognitive development.

Although psychological and educational researchers share a common agreement about the important role of metacognition in problem solving, however, before we as educators focus on promoting metacognitive processes with a goal of improving problem-solving outcomes and performance, we need to better understand the concept of metacognition; that is, how students acquire metacognitive processes, how metacognitive processes emerge in problem-solving situations, and the extent to which students act metacognitively. On the other hand, new and emerging technologies (e.g., Geometer's Sketchpad [GSP], Cinderella, Cabri, GeoGebra) continually transform the mathematics classroom and redefine ways mathematics can be taught (Fey et al., 2010). Leading researchers on the teaching and learning of geometry have emphasized the benefits of using dynamic environments (Fey et al., 2010; Hollebrands, 2007). However, we yet need to obtain convincing evidence concerning students' mathematical achievement with dynamic technology tools. With these considerations in mind, the primary purpose of this study was to investigate the patterns of metacognitive processes preservice teachers exhibit when solving nonroutine geometry problems in a dynamic geometry environment (DGE); that is, to investigate how preservice teachers experience working in a DGE and how these experiences affect their own mathematical activity when integrating content (nonroutine problems) and context (technology environment). The following questions guided the study:

- What are some of the metacognitive processes exhibited by preservice teachers when engaged in solving nonroutine geometry problems using GSP?
- How are these metacognitive processes associated by their use of GSP?

Theoretical Background

Metacognitive Aspects of Problem Solving

In the literature, terms such as self-regulation, monitoring, control, and executive decisions are frequently used interchangeably to describe the concept of metacognition. It is thought to be an elusive concept because of the difficulty distinguishing between cognitive and metacognitive processes. In this study, I use the definition by Flavell (1976):

> Metacognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them, e.g., the learning relevant properties of information or data … Metacognition refers, among other things, to active monitoring and consequent regulation and orchestration of these [cognitive] processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete goal or objective. (p. 232)

In the last 40 years, mathematics educators have begun to focus on the role of metacognition in problem solving. Research on the role of metacognition in problem solving, considered metacognitive processes as "driving forces" that influenced cognitive behavior at all stages of problem solving (Lester, 1994). Schoenfeld (1985) identified that together with heuristics, metacognitive control, and belief systems, resources (factual and procedural knowledge) a problem solver possesses are fundamental for successful mathematics problem solving. However, further research (Carlson & Bloom, 2005; Lawson & Chinnappan, 2000; Schoenfeld, 1992) demonstrated that success in problem solving performance depends greatly on problem solver's ability to retrieve more knowledge, activate links among knowledge schemas and related information, and to coordinate them at the same time. Similarly, Carlson and Bloom (2005) pointed in the direction of the importance of management of different mathematical resources. Thus, "effective metacognitive activity during problem solving requires knowing not only what and when to monitor, but also how to monitor" (Lester, 1994, p. 666).

With respect to problem solving in DGEs, the software provides students with a genuine problem solving activity (Fey et al., 2010; Olive et al., 2010; Wilson et al., 1993). Zbiek, Heid, Blume, and Dick (2007) contend that student engagement in conceptual activities using technology tools allows students their personal problem solving experience through habits of mind (e.g., pattern recognition, conjecturing, generalizing, abstracting) they engage in. Consequently, student' engagement in these habits help develop and increase students' ability to determining on their own how to think mathematically; deciding what information is needed, choosing a particular strategy, testing their conjectures, and examining what is learned and how it can be applied to a different problem solving situation (Goldenberg et al., 1988). Hence, DGEs provide the user a well-tuned system within which different mathematical concepts and mathematical problems may be explored (Hoyles & Noss, 2003).

In summary, different components are crucial for a productive and successful cognitive endeavor; the knowledge base and retrieval of metacognitive knowledge amplifies efficient problem-solving attempts; metacognitive experiences allow students to capitalize on their experience, where the execution of a cognitive action prompts metacognitive experience.

Theoretical Framework

After reviewing various problem-solving models (Garofalo & Lester, 1985; Mayer, 2003; Pólya, 1945/1973; Schoenfeld, 1981, 1985), for the purpose of uncovering and investigating patterns of metacognitive processes two preservice teachers exhibited when problem solving, a model adapted from Pólya (1945/1973), and Schoenfeld (1981) offered a framework describing problem solving behaviors, both cognitive and metacognitive actions, during which a problem solver engaged in a particular activity. The compilation of these models allowed not only capturing together both distinct phases of activity and the complex interplay between metacognitive and cognitive processes in mathematical problem solving, but also the dynamic, cyclic, and iterative nature of these processes. The resulting model was characterized by the following episodes: reading the problem, understanding the problem, analyzing what needs to be done, exploring different possibilities, planning the best solution, implementing the plan, and verifying the answer is a solution, together with junctions between episodes (transition).

In a reading episode, student reads the problem. In the understanding episode he or she may note conditions of the problem, state the goals of the problem, and assess his or her current knowledge relative to the task. In the analysis episode, the student decomposes the problem in its basic elements, examine the relationships between the given information, and choose appropriate perspectives to solve the problem. Whereas an analysis episode is well-structured, an exploration episode is less structured and removed from the given problem. In it the student searches for relevant information that can be used in the following episodes. In a planning/implementation episode, the student creates a plan and implements it. In a verification episode, the student reviews and tests whether his or her solution passes specific or general tests in relation to requirements of the problem. A transition episode is a junction between the other episodes and occurs only when a student assesses the current solution state and makes decisions about pursuing a new problem-solving direction. However, this model does not address local indications of metacognitive behaviors (Artzt & Armour-Thomas, 1992). In order to better understand the nature and interplay of the cognitive and metacognitive processes within each of the episodes, the nature of participants' answers with respect to their metacognitive awareness (individual's awareness of their own thinking), metacognitive evaluation (individual's evaluation of those thought processes), and (3) metacognitive regulation (individual's directing of those thought processes) (J. Wilson & Clarke, 2004) was taken into account.

Methodology

Participants

For this study, a case study qualitative research design was chosen. Such design allows to answer questions such as "How?" and "Why?" the specific phenomenon, such as problem solving occurred (Merriam, 1998) pushing the study beyond description alone and explaining the phenomenon in depth, in real context and holistically (Patton, 2002). Using purposeful sampling, two participants, Wes and Aurora, each serving as a unique case, from the mathematics education program at a large southeastern university in the United States were chosen. Based on both research and personal experience, two participants were determined that would be ideal; not only they had been used to working in a DGE, but worked well individually, were reflective thinkers who articulate their thinking well, had substantial mathematical background and we have established a rapport where they felt comfortable interacting with me on a variety of levels.

Data Collection

Research on problem solving has used different methods, such as think-aloud protocol, clinical interviews, concurrent probing, retrospective probing, retrospective general report and retrospective clinical interview (Artzt & Armour-Thomas, 1992; Ericsson & Simon, 1980), that can help elicit problem solving processes. Having in mind weaknesses and strengths of each of these techniques, and the complex and multi-faced nature of metacognition, to provide the most accurate, thick, and rich description of both cognitive and metacognitive processes, instead of utilizing only think-aloud protocol, concurrent verbalization methods with retrospective methods (probing and clinical interview) were used in this study.

The participants individually solved three mathematical problems taken from different sources, such as web-sites, books (see Appendix for mathematical problem solving tasks) using the think-aloud protocol and concurrent probing. The mathematical problem solving tasks included three nonroutine geometry problems chosen such that they demanded strategy flexibility, thinking flexibility, provided participants with opportunities to engage in metacognitive activity, and covered mathematical content area in geometry. Three types of problems were used for this study: construction, applied, and exploration problem. Having three types of problems allowed the participants to use and apply their knowledge, translate verbal statements into an interactive representation, investigate a mathematical idea, deal with a situation that may not have a single solution, and make, test and verify their conjectures in plethora approaches. Thus, the nature of the problems allowed exhibiting different metacognitive processes, multiple solution paths, and other that consequently enhanced understanding the multi-facet nature of metacognition.

Research Design

Data collection occurred in a one-to-one setting between the participant and the author, and concentrated on the participants' involvement in investigations of three mathematical problems in DGE. They continuously thought aloud and engaged in a conversation with the author while working on the problems describing their thinking and behaviors. However, during extended periods of time I used the following prompts to encourage the participants to speak his or her thoughts: "keep explaining aloud what you are thinking," "keep talking," or "tell me how are you using technology in this situation." They used as much time as they needed in solving each problem. The individual interviews took place shortly after the participants finished solving each problem where we talked comfortably about the participant's problem solving session. The interview protocol consisted of two parts; it intended to elicit the participant's views about the problem solving task as well as to try to understand what situations, and interactions in a DGE promote metacognitive behaviors and to elicit the participant's experience about using technology solving the particular task. The same procedures were used for the following two mathematical tasks. An additional data resource was my own field notes (descriptions of questions, reactions, and behaviors) that were then used during retrospective interview.

Data Analysis and Validity

For the purpose of this study, two stages of analysis, the within-case analysis and the cross-case analysis were conducted using inductive analysis. When using inductive analysis, I focused on creating codes and categories from the data, developing or enhancing theory during the act of analysis and the use of constant comparative method during analysis of the data. After categories were generated from the codes, the data was reanalyzed and the codes were refined by noting key behaviors and characteristics that related each code to its category, and through identification of the level for each problem solving behavior. After within-case analysis was completed, cross-analysis of the cases begun. The cross-case analysis was used to create a sound theory offering general explanations of metacognitive processes and perspectives on the experience of using technology that comply each problem for both participants. To ensure the validity and reliability for the study of preservice teachers, I used several procedures that involve triangulation (triangulation of sources, analyst triangulation), thick rich description, and the audit trail (Patton, 2002). Employing the procedures mentioned above ensured trustworthiness and rigor.

Discussion of Findings

The following is the discussion of the findings related to the research questions addressed in this study. Problem solving behaviors were described within each of seven episodes and associated with the technology use. This model, however, demonstrates exhibited problem solving behaviors within each episode and should not be taken as a rigid model a problem solver goes through; episodes are not linear, but cyclic, dynamic, and iterative. Table 1 offers a short overview of a coding scheme used during the data analysis.

Table 1

Wes's Transcript Excertps and Coding of Problem 1

Excerpt	Metacognitive behaviors	Ep.
(Reads aloud problem statements with pauses between each one of them and from the hard copy.)	Monitoring strategy Control strategy	R
Longest [is important as] it came at the end of the sentence and it's what the entire problem is about, [and] intersecting is key too, given two intersecting circles. That's important.	Organizing information	U
I began by constructing the two circles, and then I read the [statement]. I constructed two intersecting circles…Draw a line through one of the intersection points, say A. So, let me make an intersection Point. A line also intersects circles in exactly two points, say B and C. So I have a line, and it's going through this Point A and, a line goes through A also intersects the circles in exactly two points, say, B and C. B and C is like right there. So, label points, A, B, and C… I broke down the directions and the instruction step by step so I can see, make sure that I am going on the right track.	Awareness-executive strategy Sense making-organizing and labeling information Engaged throughout the process-monitoring and directing his knowledge and thinking	
(He does not read the problem again; because of his step-by-step process he was sure his interpretation of the problem was correct.)	Evaluation- judging the effectiveness of thinking processes and strategy	
I thought of triangles. Triangles seem to be so important in geometry because everything else depends on triangles I have noticed.	Awareness and regulation-mathematical knowledge	
(Places a point that he labeled as Point D on Line BC for dragging, and drags it.) *There is the shortest*. (Moves Point D and stopped when it seemed that BC was at its maximum length.) *Right now I am trying to figure out where BC is the longest. So, I am measuring the length of that segment and I am trying to see where it stops going up, and where it stops going down. So, it seems like it's right here, 19.03 cm (he measures).*	Awareness-executing strategy Regulation Monitoring-reflecting on the problem goal Conjecture Tests conjecture	E
(Assessing the relevance of the new measurement. He explained that at this point he remembered the problem statement in the back of his mind, and the need to characterize Point B in relation to some geometric object.) [Moves point D.] *It seems like it's perpendicular to this segment [segment between the two circle intersections]*. (Draws segment AH.) *Now why would that be the case?*	Evaluation-reflecting on the process and solution and organizing information Visualization and imagining Sense making	A
You have to choose Point B so that the line that joints the two intersection points is perpendicular to BC.	Conjecture	

Reading the Problem

Both participants started each problem solving session by reading the problem statements, which was consistent with Schoenfeld's (1981) model. Although this episode is often labeled as a cognitive episode (Artzt & Armour-Thomas, 1992), both participants exhibited a variety of metacognitive behaviors during this episode. Wes always read the problem aloud and from the hard copy, aiding him not to miss any information. He typically read the entire problem before reading the main problem statement again. These monitoring strategies allowed Wes to maintain focus and identify the problem components. On the other hand, Aurora read it in silence from the screen highlighting the words with the cursor as she was reading through the problem. She typically read just the main parts of the problem if the problem had multiple goals. Similar to Wes, these monitoring strategies helped her not to miss any information, and allowed her to maintain focus and identify the problem components. During reading episode for Aurora, the use of GSP was oriented towards the management of the tool: turning on Text Menu to highlight and bold problem statement or parts of it, *"The bold basically highlights specific things that are very important to the problem to me, so I used that in order to keep it in the back of my mind ... that helps me keep on track."* In addition, during the problem-solving session both participants often reread the problem to review the problem conditions or to see if they had forgotten important parts of the problem, which appeared to be a strategy to control potential missteps. Engagement in these monitoring and control strategies, and management of the tool was a metacognitive behavior. Acting on these metacognitive processes prompted metacognitive behaviors aligned with the understanding episode that contributed to moving through the problem-solving space, *"Now I have an idea what's going on. Now's the time to use this [GSP]."*

Understanding the Problem

Metacognitive behaviors that fit the understanding episode were exhibited immediately after the reading episode for both participants for all of the problems. Behaviors related to the episode usually stood alone or occurred simultaneously with behaviors related to analysis episode. Consistent with previous research, typically both participants first explicitly noted problem conditions, problem goals or key parts of the problem by either stating them aloud (Schoenfeld, 1981), *"So this is the first farmer's land; this is the second farmer's land. Okay. They wanna keep the area the same"* or bolding them. During the understanding episode, the participants needed to consider content specific knowledge and strategies relevant to the problem, which was consistent with previous research (Carlson & Bloom, 2005; Lawson & Chinnappan, 2000; Schoenfeld, 1992). They engaged in a variety of strategies for monitoring their understanding of the problem as reported by previous research (Artzt & Armour-Thomas, 1992; Schoenfeld, 1981, 1985, 1992); they were looking for the given information in the problem and what was being asked of them, restating the problem, reengaging with the problem text, asking for clarification of parts of the problem or the meaning of the problem, making sense of the problem information, representing the goals and givens of the problem by writing them down, mentally or making a representation of the problem, introducing suitable notation, and reminding him or herself of the requirements of the problem. These monitoring strategies were metacognitive behaviors that were an important attribute during problem solving that helped develop an understanding of the problem and access their knowledge, facts, and strategies.

Drawing a diagram representing the problem was a cognitive problem-solving behavior used by both participants when a diagram was not provided as a part of the problem. Although this cognitive behavior is available with paper-and-pen, both participants used the capabilities of the built-in functions of the GSP to represent the problem and quickly add secondary elements, lines, segments, rays, and points to a figure, *"it just helps me visualize it more as a plausible land… If it's presented as land, it helps me to mark it off as plots of land and so I connected those and it just helps with the area calculations".* Typically, both participants monitored their work throughout during problem representation and at the same time they spontaneously accessed and directed their knowledge and thinking. For instance, Wes carefully read statements, restating them or interpreting them before representing them on a paper or on GSP in increments (see Table 1). Aurora also read problem statements, restating them or interpreting them before representing them on GSP, but never on paper-and-pen. When false moves occurred, they were discovered, however, through reengagement with the problem statements again or verbalizing what was done. This aided reevaluating what was done and correcting false moves.

Drawing a diagram using GSP helped them visualize the problem, and attain accurate visual input. It aided accessing mathematical knowledge and facts relevant to the problem when attempting to make sense of the problem, directing their thinking processes towards working through a problem-solving space. Consequently, all these metacognitive behaviors and activities helped participants develop an understanding of what the problem meant concretely, which was consistent with the results by Goldenberg et al. (1988). An accumulation of resources, however, was not sufficient for productive paths but rather led the participants in unproductive directions in the absence of metacognitive monitoring. The ability to manage their resources and actually access useful information at the right moment was an essential metacognitive behavior in making productive and useful decisions.

During this episode they also engaged in metacognitive behaviors, such as pausing to make sense of the problem and of the current effort, and to assess productivity of their thinking (e.g., whether considered knowledge was relevant to the problem) and internal dialogue that aided to productive or desirable thinking and directions. For instance, interpreting the problem statement and sense making was most specific for Problem 3 where both participants related the problem goal of keeping the amount of the land the same with the concept of area, *"So this is the first farmer's land; this is the second farmer's land. Okay. They wanna keep the area the same. I need to somehow make two congruent triangles so I can say their areas are equal."* Neither of the participants wrote down main ideas of the problem. Nevertheless, the main ideas were verbalized by considering and organizing content specific knowledge and strategies relevant to the problem as a result of current problem solving states or previous experience. Internal dialogue consisted of posing metacognitive questions that promoted metacognitive behaviors, *"What can I do to make that [straighten the border]? Can I just look one of the crooks by itself, do the same strategy that we did before in the first problem, and get it down so it's one? [Silence] Yeah, I can!,"* which is consistent with other mathematics education literature (e.g., Carlson & Bloom, 2005; NCTM, 2000; Pólya, 1945/1973). These metacognitive behaviors were important and contributed to move their thinking in productive directions. Metacognitive behaviors consistent with the understanding episode were crucial in solving the problem, highlighting the importance of these preparatory behaviors also recognized by Pólya (1945/1973) and Zimmerman (2002).

Analyzing the Problem

Analysis of the problem occurred as an individual episode after the understanding episode or the exploration episode, or it occurred simultaneously with the understanding episode. Participants devised different perspectives, considered various mathematical concepts, facts and strategies before selecting a perspective,

> I need to somehow make two congruent triangles so I can say their areas are equal and I need those congruent triangles to be in such a way that I have straight line so I can get rid of the hump. Well I know what you can't join these two points [Points C and D] because that would give him some of his land and that just wouldn't be fair. So if you were to continue this way [line through Points C and E]. Would that be?

These behaviors were consistent with the previous research (e.g., Artzt & Armour-Thomas, 1992; Carlson & Bloom, 2005; Schoenfeld, 1981, 1985). Nevertheless, solving problems in GSP prompted or required its use. Having a diagram representing the problem, constructed using the GSP in the understanding episode, triggered accessing, considering, combining and organizing their knowledge when seeking relationships between the conditions and the goals of the problem. It also often served as a tool to recall specific geometry content knowledge to aid in problem-solving behaviors,

> Wait a second, that's the same thing as the other one but that makes sense because it's an equilateral triangle. Ok, this [visual] convinces me that the orthocenter/circumcenter for an equilateral triangle is the only location that the sum is the smallest you can get it.

Thus, having a visual input directed their actions and thinking into understanding the information obtained through the use of GSP. The decision to engage in these activities was a metacognitive act that prompted other metacognitive behaviors.

When choosing a perspective they considered knowledge of what needed to be done, and what might be done in a particular problem-solving context. In addition, both participants reengaged with the problem text and restated the problem in their own words before considering and making a choice of a perspective, which was noted in earlier research (e.g., Carlson & Bloom, 2005) as well. Participants did not always evaluate a choice of perspective with respect to effectiveness of their problem solving strategy or thinking, however, but made random associations with respect to content knowledge and problem perspective. For instance, when solving the extension of the Problem 3 both Wes and Aurora decided to use the same strategy used for the original problem without assessing how to use it, and evaluating if using it at once would be efficient.

> I was trying to deal with both of them [segments] at the same time, and that wasn't working, it couldn't have worked. I did it too fast! And then I thought of the idea; well instead of trying to tackle it at the same time why don't I just try doing just one at a time and see where that gets me and that's how it developed.

Thus, not only they did not evaluate it but also did not direct their thinking if their choice would move them towards a solution or not. Schoenfeld (1992) reported that students often do not know how, when, and whether to use their metacognitive resources to solve a particular problem and identified this as a lack of the control mechanisms. Consequently, absence of such behaviors or the ability to manage these resources sparked lengthy unproductive and unguided efforts in subsequent episodes. Sometimes they considered multiple choices of perspectives such as Aurora when solving the extension of Problem 3 (using technology or not) before deciding on a choice of perspective based on evaluation of its effectiveness,

> Should I jump into trying to do it in mathematically stringent way using the parallel lines or estimate it first and see if I can get to that line segment just by looking at it. Can I use the same strategy? I will probably just start like I did last time with just finding the area… I didn't know exactly where I wanted to go with it.

The analysis episode, interestingly, was the least coded episode for Aurora, while Wes for each problem engaged in metacognitive behaviors consistent with this episode. It appeared that analysis of the problem allowed further understanding of the problem, exploration, and more analysis allowing Wes to combine, and select steps and strategies that might potentially lead to the problem solution, whereas absence of such behavior sparked lengthy pursuits for Aurora's problem solution paths.

Exploring the Problem

Behaviors consistent with the exploring the problem episode were one of the most often coded episodes, which is often the case with novice problem solvers (Schoenfeld, 1992). The problem-solving context led to exploring behaviors since the nature of the software invited participants to explore, experiment, and conjecture in the search for a solution plan, relying on their previous knowledge and experience. Interestingly, although behaviors consistent for this episode were most coded for Aurora, it was least coded for Wes.

For this episode, the participants engaged in a variety of both cognitive and metacognitive behaviors. During this episode, when it was labeled as a cognitive exploring episode, it was characterized by weak structure, and lack of metacognitive strategies and behaviors. For instance, in the search for relevant information Aurora often relied on the use of GSP that was characterized by quick jumps into exploration lacking apparent structure to the work, did not assess of current state of her knowledge, did not assess of relevancy of actions, and lacked perspective on future steps.

> The first two that I'd done, I was really expecting that to work so when it didn't come out to work, I was like: Oooooh I don't really know where to go from it. So I will just pick a random point that could be moved. It can't hurt. I might as well.

She engaged mainly in trial-and-error strategy consciously where she made a guess, tested it, and repeated until she assessed the feasibility of her actions. In addition to the trial-and-error strategy, the GSP allowed the bottom-up strategy where she took the problem as solved and worked backwards to obtain a solution. Aurora then focused more on the result, relying on quick guesses rather than on engaging in productive efforts to select a problem- solving path or to allocate problem-specific resources to obtain such a solution. Nevertheless, the choice of the strategy (trial-and-error and bottom-up) and awareness of its helpfulness was a metacognitive act where their knowledge of the software capabilities guided the way software was used, *"I am good with finding one answer and then working backwards to finding other* *answers that could be possible rather than the other way around.”* Often, taking a step back and regulation of negative affective behavior, namely frustration, were helpful in redirecting participants’ thinking. As reported by Schoenfeld (1981, 1985), lack of monitoring of progress often made the endeavor unsuccessful, which resulted in lengthy pursuits characterized by weak structure, absence of local and global assessment, and impetuous jumps from one particular direction to another through one exploration to another before sense making occurred. Both participants engaged in internal dialogue, such as verbalization of self-questions, conjectures, strategies that appeared to aid efficient movement towards a solution path.

Solving problems in GSP allowed them to engage in conjecturing based on the visual representation of the problem or previous knowledge and to test their conjectures. Also, GSP allowed for a trial-and-error strategy that involved also making purposeful hypotheses that allowed metacognitive behavior associated with the exploration episode to be controlled and focused, *“The picture was static; however, the GSP file was dynamic. I could manipulate the boundaries and explore the problem further. Sometimes, changing the sketch and watching the screen can give me an idea.”* In these situations the feedback provided by the GSP helped again evaluate and directed thinking processes towards devising a solution plan and consequently successfully solving the problem which is consistent with other mathematics education literature (e.g., Hollebrands, 2007; Olive & Makar, 2010). Beside conjecturing and testing of conjectures, both participants tried to imagine their actions in order to assess their efficiency or effectiveness or feasibility. It appeared that such visualization, however, was not an easy task and might have been out of their imagining capabilities.

In summary, both participants used the software’s capabilities of precision, measuring, and dragging to engage in problem-solving activities that proved to be a cyclic process of generation, justification, and refinement of plausible solution paths. Hence, it appeared that metacognitive behavior of both participants considered affordances of the GSP to guide their problem-solving behaviors (Hollebrands, 2007) making the problem-solving process more fluid and allowing flexibility in the problem-solving approaches. The ability to reflect on the feasibility of their thinking processes using different resources (e.g., mathematical knowledge, facts, and technology), and manage those resources at the same time was an essential metacognitive behavior.

Planning

Planning occurred as an individual episode after understanding, analysis, or exploration episodes, or occurred simultaneously with the implementation episode. Behaviors consistent with planning, both cognitive and metacognitive were highly coded for both participants, such as accessing, considering, and manipulating mathematical knowledge, concepts and facts relevant to the problem, assessing the plan through imagining, conjecturing and testing, and monitored and refined, revised, or abandoned the plan according to problem goals until they arrived upon the final plan.

During this episode, when it was labeled as a cognitive episode, both participants described their intended plan or its parts but lacked any visible sequence of strategies and were without apparent structure of the plan; that is, identification of goals and subgoals, global planning, and local planning was absent or possibly not verbalized, *"I did not have any idea how to solve it. There was nothing in the problem, so I just measured and tired different things."* Both participants when planning for solving the second part of Problem 3 did not assess the relevancy and quality of chosen activities or strategies (trial-and-error) with respect to moving forward in the solution process, although they accessed and considered mathematical concepts and facts. However, engagement in trial-and-error strategy, and the ability to piece together different information obtained from devising several problem solving paths at the right time allowed participants, especially Aurora, to attain their goal, that is, to solve the problem, *"And so from there I got a little bit frustrated and I was trying to think of all of the things I knew about it and so then I was just trying to do a few different things."*

The GSP was often used to examine variety of strategies, to examine the details of the plan, check each step carefully, monitor, assess, refine, revise, or abandon the plan according to problem goals until they arrived at a final plan, when needed. Even though the decision to engage in these behaviors was an important metacognitive behavior, however, it was not sufficient for devising an efficient and effective plan. Hence, limited metacognitive behavior, such as lack of the ability to access and coordinate useful information and strategies at the right moment, led the participants in unproductive directions. Moreover, as a result of lack of evaluating and monitoring their work, negative affective behaviors were exhibited, mainly with Aurora, influencing cognitive behaviors to take domination over metacognitive processes.

When selecting steps and strategies for a solution plan accessing resources, such as knowledge and GSP, and experiences relevant to the problem were paramount. For that to happen participants needed to be aware of three components: knowledge of what was done, knowledge of what needed to be done, and knowledge of what might be done in a particular problem solving situation. Manipulating their knowledge and new information obtained from previous episodes led participants to identify the following plans to find the solution to the problem. They also represented the information by adding new elements onto the sketch before using them in a solution plan. They assessed the plan with respect to the process and solution, and problem and solution. Monitoring of their plans and strategies, which was most evident with Wes, was exhibited when they verbalized thoughts and questions about their steps and strategies and by staying mentally engaged through construction of logically connected mathematical statements. There was evidence the students tested that their plans made sense, that they looked for efficient plans, and that they changed their plans during this stage. Often the participants engaged in a metacognitive act of self-questioning *("Can I use the same strategy?," "Will it work?")*, such as verbalization of conjecture, questions and comments related to the plan and strategy (e.g., evaluation of the current problem-solving state, and trying to make sense of it, judging the effectiveness of previous actions). Acting on these metacognitive acts prompted various metacognitive behaviors; internal dialogue contributed to move their thinking in productive directions, movement forward in solution plan, and aided assessing effectiveness and feasibility of their chosen strategies and approaches based on the key features of the problem.

Implementing the Plan

Implementing the plan occurred as an individual episode most often immediately after exploration or planning episode or occurred simultaneously with planning episode. The nature of participants' behaviors differed—cognitive and metacognitive—and were consistent with implementation (e.g., execution of a plan or strategy, proving a conjecture) as exhibited in earlier research (e.g., Artzt & Armour-Thomas, 1992; Schoenfeld, 1981, 1985). Behaviors consistent with the implementation episode were most coded for both participants.

Cognitive behaviors exhibited during the implementation episode included participants having executed their planned activities on paper-and-pen or using GSP in a well-structured way without assessing their activities or quality of their activities or monitoring their work. Lack of metacognitive behaviors such as control (e.g., assessing the plan with the conditions and requirements of the problem, assessing the appropriateness of actions, assessing the sensibility of the solution progress and results) made the problem-solving endeavor fruitless and led to quick jumps from one planning or implementation episode to another. These led to lengthy pursuits characterized by weak structure, and absence of assessment. Similar behaviors were exhibited in research studies by Schoenfeld (1981, 1985).

Metacognitive behaviors exhibited during this episode were consistent with previous research (Artzt & Armour-Thomas, 1992; Carlson & Bloom, 2005; Schoenfeld, 1981, 1985) and included: considering, accessing, and organizing their knowledge relevant to the problem when constructing logically connected mathematical statements, evaluating appropriateness, effectiveness and efficiency of their actions, monitoring of their actions and directing their thinking and actions towards a solution, and assessed the sensibility of the solution process and results, *"O…OH! So now we are back at the original problem! Our boundary is LM and LJ so now I can connect the base of MJ. There we go. MN is our new boundary. Yes!"* When implementation drew on actions that were already automatic, monitoring and evaluations of actions were automatic as well. Presence of ongoing monitoring, evaluation and regulatory processes, and the ability to manage their resources and negative affective behaviors were essential metacognitive behaviors that allowed productive directions through the problem-solving space.

The problem-solving context not only allowed for easy implementation of their plans, but helped with more complex questions that extended participants' competence, such as noting where the problem-solving activity might be leading, *"What I am visualizing is slide that down, slide that up. Yeah, I think it will work. Hmmm [uses GSP]. Hold on, it's not [working]."* As a result of reflecting on the activities and results through feedback provided by the GSP, they were then able to redirect their thinking processes towards a solution to the problem, choosing the strategy or the plan and assessing merits of the new strategy or plan. Hence, as noted in other problem-solving episodes, through stages of personalization and transformation of the tool they transformed it to a valuable instrument as a result of their knowledge of the software capabilities. Through engagement in these activities, they optimized the use of available resources, which was undoubtedly a metacognitive act.

Verifying the Answer

This episode, if it occurred, occurred most often individually. Interestingly, this episode was not coded very often, which will be explained in the following discussion. The cognitive processes typical for this episode included evaluating the result by checking the computations steps, which was noted in earlier research (e.g., Artzt & Armour-Thomas, 1992; Schoenfeld, 1981) or using the Measure functions of GSP. The measuring capability of GSP made quick validations of different solution paths possible, whereas measurement capabilities to double-check the result were used only to satisfy the participant's—namely, Aurora's—validating standard,

> So it's at exactly 90° right now. So if I go up, it should be going down and if I go down, it should also be going down. So yeah 24.7876 [cm] looks like it's the highest it can possibly get it. This convinces me.

Hence, mathematical integrity was held by the GSP, and not herself as a problem solver. Interestingly, Wes also on one occasion contemplated whether to accept the result he obtained using GSP followed by using Algebra to verify his answer differently.

Besides cognitive strategies, the participants engaged in several metacognitive strategies for verifying their results. These included decisions to review their work to make sure they did not forget anything or determine if they had made a mistake, rereading the problem to make sure the solution reflected the problem conditions and answered the question, and checking the results for reasonableness of the solution of the problem. These behaviors were consistent with earlier research (Artzt & Armour-Thomas, 1992; Schoenfeld, 1981, 1985, 1992). They also thought about a way of checking to see if their solution was correct with or without the use of GSP. The decision as to what approach to use seemed to depend on the availability and capability of resources as well as on an affective factor of value; that is, on the participant's standard for verifying that an answer is correct. When examining the solution without the use of technology, they engaged in making logically connected mathematical statements, and assessing the reasonableness of the answer, which was consistent with previous research (Schoenfeld, 1981, 1985). Sometimes they had in addition the feeling that the problem was correctly solved, *"I just had this feeling now when I know I completed a problem and it's hard to describe but you just know it…all my logics seems consistent with my prior knowledge. Things are working how they should."* In these situations, metacognitive awareness dealt with accessing mathematical resources needed to engage in a productive effort. Metacognitive evaluation dealt with situations where they assessed the correctness and the efficiency of the solution, and how and why were particular actions and strategies used. In these situations they relied on their content resources; that is, their conceptual knowledge informed them as to the correctness or reasonableness of the obtained solution. The participants sometimes checked for the quality of the process and rarely assessed the aesthetic quality of the solution. Rarely, however, did they examine if the solution could have been obtained differently, *"When I am done, I am done. I don't tend to solve the same problem using a different path".* As a result of engagement in behaviors aligned with the verifying episode with or without the use of GSP the participants spurred additional metacognitive processes. Through the process of evaluation of their actions, they made a decision whether to accept or reject the solution. The decisions to accept or reject a solution were always exhibited before moving to a new problem-solving cycle. If a discrepancy was discovered, which was often result of verbalization of their actions and thinking, the participant cycled back and engaged in correcting (e.g., refining, revising, abandoning) the incorrect cognitive or metacognitive actions by either abandoning their plan or modifying it, *"O…OH! I need it to be parallel to this Line [CD] and not to this one [right boundary] because then it [altitude] wouldn't be [changing] because that would ensure that the altitude is not changing."* These behaviors were consistent with research on graduate mathematics students (Carlson & Bloom, 2005).

Metacognitive behaviors aligned with the verifying episode were important for a successful problem-solving endeavor, however, not every problem-solving session ended by the participants engaging in metacognitive behaviors of evaluation. It was done implicitly or not at all as it made sense to them but could not explain their metacognitive knowledge. Veenman, Van Hout-Wolters, and Afflerbach (2006) postulated that some metacognitive processes, such as evaluation processes, appear on a less conscious level or run in the background of one's cognitive processes, as they became a regulatory habit. Future research should more closely consider how one decides to evaluate the correctness of one's solutions and how this is influenced by the problem content.

Transition

Between the episodes the participants assessed the current stage in problem solving where either decision were made to salvage or not salvage strategies that might be valuable or assessed the value of a new direction or jump into the new approach as addressed by Schoenfeld (1981, 1985). In these situations, metacognitive acts dealt with reflecting on the current stage in problem solving which most often occurred as a result of feedback provided by the GSP (e.g., Is this choice of perspective getting me anywhere?) that then guided their thinking to current or new directions. However, lack of assessment of the current problem-solving stage between the episodes rendered the subsequent efforts fruitless and unproductive before such occurred as observed earlier by Schoenfeld (1981, 1985, 1992). Nevertheless, I was able to observe a new metacognitive behavior—*"taking a step back."* "Taking a step back" was a reflective behavior that entailed reassessing what was done, putting effort to organize relevant knowledge and redirecting those processes that contributed to efficient movements towards a solution. Wes nicely explained,

> When I got stuck, I tried to step back, take a step back, think over what I've been thinking of because sometimes I get so entangled in the problem that I can become lost or focused on something that doesn't really matter, so taking a step back allows me to clear my head for a second and then I go back in.

The decision of "taking a step back" was a metacognitive act that was essential for productive problem solving, and a type of reflective behavior that promoted participants' metacognitive awareness and monitoring skills.

Conclusions and Implications

The findings of this study, similarly to previous research (Artzt & Armour-Thomas, 1992; Carlson & Bloom, 2005; J. Wilson & Clarke, 2004), showed that a continuous interplay between cognitive, and metacognitive behaviors and strategies was paramount for successful problem solving. Problem solvers develop cognitive actions and strategies to make cognitive progress, while at the same time these are important to monitor cognitive processes (Flavell, 1981). More closely, behaviors exhibited by the two participants provided a detailed characterization of the interplay between metacognitive processes and conceptual knowledge that influenced most of the episodes of the problem-solving process. In addition, the findings of the study showed that affective behaviors, such as perseverance, persistence, confidence, interest, and frustration occurred frequently during the problem solving activity, and acted both productively and counterproductively with metacognitive processes. These affective behaviors changed during the process of solving a problem, and were related to participants' success when problem solving. Management of different affective behaviors allowed both participants to persevere in their problem solving activity. The observations made in this study support the arguments from other researchers (Veenman et al., 2006) that research on metacognition should not be studied in isolation, but take into consideration complex construct of affect and extend it to characterizing these affective states and their use during problem solving.

GSP proved to be an important resource when working on nonroutine problems; it allowed participants to engage in processes, such as pattern recognition, conjecturing, abstracting, and other, and supported flexibility in thinking, transfer of mathematical knowledge to unfamiliar situations and extension of previous knowledge and concepts as reported by Zbiek et al. (2007) when working on conceptual mathematical activity. However, it was apparent that both the dynamics of problem solving processes as well as the dynamics between the participant and technology were different for the two participants. Wes perceived GSP as an incredible *"tool,"* an additional resource for working through novel problems. His knowledge of GSP was more generative, he owned more connections, and he had well-connected knowledge that contributed to his effective use of GSP. In addition, he was often able to manage different resources, which was essential for effective problem-solving paths. On the other hand, Aurora perceived it as a *"crutch"* helping her in working through novel problem-solving situation. Moreover, she lacked the ability to effectively manage her own resources (knowledge, technology) and relied heavily on its use to solve the problem for her. As a consequence, such use was detrimental to quality of her reasoning and outlined plans, as she did not take into consideration why she was doing so. Hence, further research on the effect of tool-use on participants' thinking processes or schemes needs to be investigated. Knowing this would allow better insight to what situations, and circumstances promoted or induced metacognitive behavior.

Last but not least, teachers themselves lack an understanding of the complex and multi-faced phenomenon of metacognition (Veenman et al., 2006). Hence, preservice teachers before becoming inservice teachers should have experience in genuine problem solving as well as opportunities to discuss curricular, pedagogical and learning issues with respect to that mission, and metacognitive aspects of problem solving in variety of contexts.

Characterization of preservice teachers' metacognitive processes may help educators effectively plan, develop and adjust preservice teacher programs to support their development.

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Appendix
Mathematical Problem Solving Tasks

Problem 1. The Longest Segment Problem

Given two intersecting circles. Draw a line through one of the intersection points, say, A. That line also intersects circles in exactly two points, say, B and C. What choice of the point B results in the segment BC such that the segment BC is the longest?

a. Formulate and prove your conjecture.
b. Find the construction for a point B such that the length of BC in the longest.

Justify your answers as best as you can.

Problem 2. The Airport Problem

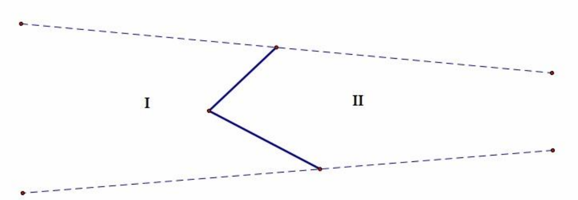
Three towns, Athens, Bogart and Columbus, are equally distant from each other and connected by straight roads. An airport will be constructed such that the sum of its distances to the roads is as small as possible.

a. What are possible locations for the airport?
b. What is the best location for the airport?
c. Give a geometric interpretation for the sum of the distances of the optimal point to the sides of the triangle.

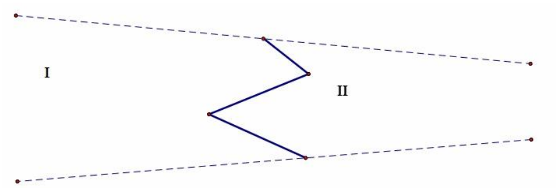
Justify your answers as best as you can.

Problem 3. The Land Boundary Problem

Part I: The boundary between two farmers' land is bent, and they would both like to straighten it out, but each wants to keep the same amount of land. Solve their problem for them. Justify your answers as best as you can.



Part II: What if the common border has three segments?



Justify your answers as best as you can.