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Numerical Solution of Linear Volterra Integral Equation with Delay using Bernstein Polynomial

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ABSTRACT

Bernstein polynomial is one of the most valuable and attractive method used to develop numerical solution for several complex models because of its robustness to demonstrate approximation for anonymous equations. In this paper, Bernstein polynomial is proposed to present effective solution for the 2nd kind linear Volterra integral equations with delay. To evaluate proposed method, the experiments are contacted using two examples and it is obtained the validity and applicability of the proposed method.

Keywords: Bernstein Polynomials, Volterra integral equation with delay

INTRODUCTION

Approximate methods for solving numerically various classes of integral equations (Shihab & Mohammed Ali, 2015) are very rare.

Several methods have been proposed for numerical solution of these equations (Mustafa & AL-Zubaidy, 2011), Bhatta and Bhatti (2006) presented numerical solution Kdv equation using linear and non-linear differential equation both partial and ordinary by modified Bernstein polynomials. Bhattacharya and Mandal (2008) used of Bernstein polynomials is numerical solution of Volterra integral equations. AL-Zawi (2011) used Bernstein polynomials for solving Volterra integral equation of the second kind. Alturk (2016) presented application of the Bernstein polynomial for solving Volterra integral equations with convolution kernels as well. Mohamadi et al. (2017) introduced Bernstein multiscaling polynomial and application by solving Volterra integral equations. A solution for Volterra integral equation of the first kind based on Bernstein polynomials. Maleknejad et al. (2012) demostrated Analytical and numerical solution of volterra integral equation of the second kined.

Many researchers have used Volterra integral equation with delay. Mustafa and Latiff Ibrahem (2008) proposed numerical solution of Volterra integral equation with delay using Block methods. Nouri and Maleknejad (2016) used the numerical solution of delay integral by using Block-pulse functions.

In this work, a robust proposed approach is explored for recruiting Bernstein polynomial to solve linear Volterra integral equation of 2nd kind with delay. Competitive results are obtained after benchmarks evaluation.

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BERNSTEIN POLYNOMIALS AND PROPERTIES

It is worth to mention that, the Bernstein polynomials are useful polynomial formula defined on [0,1]. Its degree n form a basis for the power polynomials of degree n. Bernstein polynomials area set of polynomials (Shihab and Mohammed Ali, 2015) is defined by.

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-1}, i = 0,1,\dots,n \tag{1}$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$, $i = 0,1,\dots,n$

Also Bernstein Polynomials Properties are described below (Nouri and Maleknjad 2016):

1- $B_{i,n}(0) = B_{i,n}(1) = 0\ for\ i = 0,1,\dots,n-1$

2- $B_{0,n}(0) = B_{n,n}(1) = 1$

3- $B_{i,n}(x) = 0, if\ i < 0\ or\ i > n$

4- $B_{i,n}(x) \geq 0\ , in\ [0,1]$

5- $B_{i,n}(1-x) = B_{n-1,n}(x)$

6- $\sum_{i=0}^{n} B_{i,n}(x) = 1$

7- $\frac{d}{dt} B_{i,n}(x) = n\left(B_{i-1,n-1}(x) - B_{i,n-1}(x)\right)$

The polynomials form a partition of unity that is $\sum_{i=0}^{n} B_{i,n}(x) = 1$ and can be used for approximating of any function in [a,b]. Moreover, using binomial expansion of $(1-x)^{n-i}$, it can be defined (Maleknejad et al. 2012).

$$\binom{n}{i} x^i (1-x)^{n-1} = \sum_{k=0}^{n} (-1)^k \binom{n}{i} \binom{n-i}{k} x^{i+k}$$

THE SOLUTION OF LINEAR VOLTERRA INTEGRAL EQUATION WITH DELAY FOR SECOND KIND

We consider the integral of the 2nd kind given by

$$u(x) = f(x) + \int_a^n k(x,t) u(t-\tau) dt,\ 0 \leq x \leq X \tag{2}$$

$$x \in [-\tau, 0) \qquad u(x) = \emptyset\ (x)$$

where u(x) is an unknown function to be determined, k(x, t) is a continuous kernel function, f(x) represents a known function. To determine approximated solution in the Bernstein polynomials basis on [a, b] as (Mustafa & AL-Zubaidy, 2011), the following formula is applied

$$u(x) = \sum_{i=0}^{n} a_i B_{i,n}(x) \tag{3}$$

$$\text{i.e. } u(x) = u_n(x) = a_0 B_{0,n}(x) + \cdots + a_n B_{n,n}(x)$$

where $a_i (i = 0,1,\dots,n)$are unknown constants to be determined by substituting equation (3) in equation (2) we obtain:

$$\sum_{i=0}^{n} a_i B_i(x) = f(x) \tag{4}$$

where

$$B_i(x) = B_{i,n}(x) - \int_a^x k(x-t) B_{i,n}(t-\tau) dt \tag{5}$$

Choosing $x_j (j = 0,1, \dots, n)$ As described above we obtain the linear system

$$\sum_{i=0}^{n} a_i B_{ij} = f_j \,, j = 0,1, \dots, n \tag{6}$$

where

$$B_{ij} = B_i(x_j) = B_{i,n}(x_j) - \int_a^{x_j} k(x_j - t) B_{i,n}(t - \tau) dt \tag{7}$$

where $i, j = 0,1, \dots, n$, and $f_j = f(x_j)$

The system described in equation (6) is solved to obtain the unknown constants $a_i (i = 0,1, \dots, n)$ which are used to obtain unknown function u(x) in equation (3).

NUMERICAL EXAMPLES

In this section, numerical examples are demonstrated to show solution steps of such examples of adding delay time to subjective function. The computations associated with the examples were performed using Matlab\R2013a.

Example (1): Consider the following Linear Volterra Integral Equation with delay of the second kind (Muhammad, 2017):

$$u(x) = sin(x - 1) + sin(1) + sin\, x - xcos(1) + \int_0^x (x - t) u(t - \tau) dt \tag{8}$$

where the exact solution is represented by u(x) $= sinx$.

Table 1 represents the exact solution and absolute error using Bernstein polynomial with n=11 and variable $\boldsymbol{\tau}$. Also, **Table 2** contains the absolute error using Bernstein polynomial with n=11 and $\boldsymbol{\tau} =$ 0.011.

Table 1. The Exact Solution and Absolute Error of Test Example (1) by using Bernstein Polynomial with n=11

τ	x	*Exact Solution*	*Absolute Error*
0.921	0.000	9.999999999999980e-08	2.960243005886170e-40
0.912	0.091	9.078402309593120e-02	1.144971629527700e-09
0.903	0.182	1.808181815895020e-01	6.249671131538960e-08
0.894	0.273	2.693590038395830e-01	5.753030977022780e-07
0.885	0.364	3.556752513294940e-01	2.434596270532100e-06
0.876	0.455	4.390540577996270e-01	6.340053859856200e-06
0.867	0.545	5.188068166477800e-01	1.110745460078040e-05
0.858	0.636	5.942748679744930e-01	1.247952289550710e-05
0.849	0.727	6.648349383053790e-01	6.107111051219810e-06
0.840	0.818	7.299042880650130e-01	4.340506588437910e-07
0.831	0.909	7.889455242905760e-01	4.624889329618040e-05

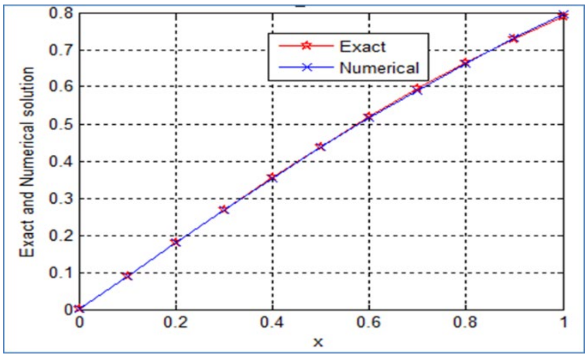

Figure 1. The Exact and the Approximate Solution Using Bernstein Polynomial for Test

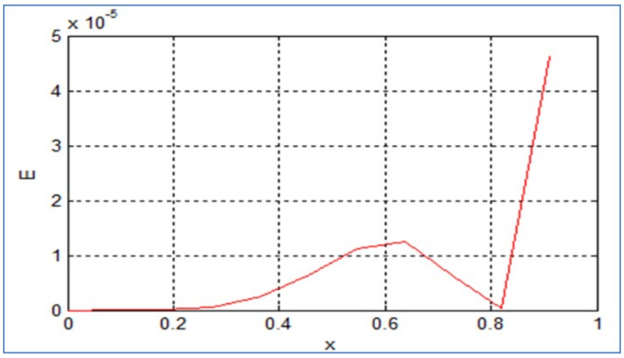

Figure 2. The Absolute Error for Test Example (1) Using Bernstein Polynomial

Table 2. The Absolute Error for Test Example (1) by using Bernstein Polynomial with n=11 and τ =0.011

x	*Absolute Error*
0.000	2.350017576341670e-04
0.100	2.856980012839290e-04
0.200	1.271771117793250e-03
0.300	1.836287273945490e-03
0.400	1.617542535566090e-03
0.500	8.672238476450610e-04
0.600	1.505308398018160e-04
0.700	8.916805012700520e-05
0.800	1.124566298959470e-03
0.900	3.297726396074090e-03
1.000	6.071466686546530e-03

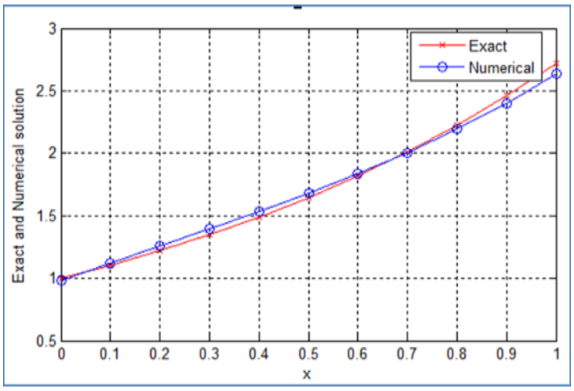

Figure 3. The Exact and the Approximate Solution Using Bernstein Polynomial for Test Example (1)

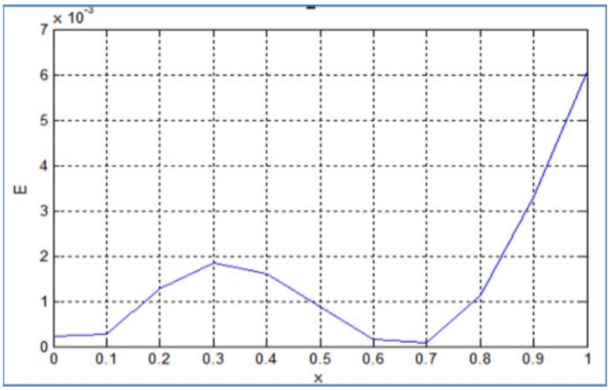

Figure 4. The Absolute Error for Test Example (1) Using Bernstein Polynomial

Example (2): Consider the following LVIE with delay of the second kind (Mustafa and Latiff Ibrahem, 2008):

$$u(x) = e^{x} - \big(x(e^{x} - 1)\big)e^{-1} + \int_{0}^{x} (x\,)u(t-\tau)dt \tag{9}$$

with the exact solution u(x) = e^{x}.

Table 3 represents the exact solution and absolute error by using Bernstein polynomial with n=11. **Table 4** contains the absolute error by using Bernstein polynomial with n=11 and $\boldsymbol{\tau}$ =0.99086.

Table 3. The Exact Solution and Absolute Error of Test Example (2) by using Bernstein Polynomial with n=11

τ	*x*	*Exact Solution*	*Absolute Error*
0.012	0.000	1.000000100000000e+00	8.553020798579380e-05
0.012	0.091	1.095169549391610e+00	1.030153344182570e-04
0.011	0.182	1.199396221975000e+00	4.448944265135480e-04
0.011	0.273	1.313542088608150e+00	6.331143783658950e-04
0.010	0.364	1.438551153432780e+00	5.747772657432590e-04
0.010	0.455	1.575457260936030e+00	3.712314404696440e-04
0.009	0.545	1.725392646005790e+00	1.687830112024190e-04
0.009	0.636	1.889597297690020e+00	5.514755867544300e-05
0.008	0.727	2.069429214099860e+00	2.824486407253070e-05
0.008	0.818	2.266375633266010e+00	1.007494871588140e-04
0.007	0.909	2.482065332829530e+00	6.671654985329340e-04

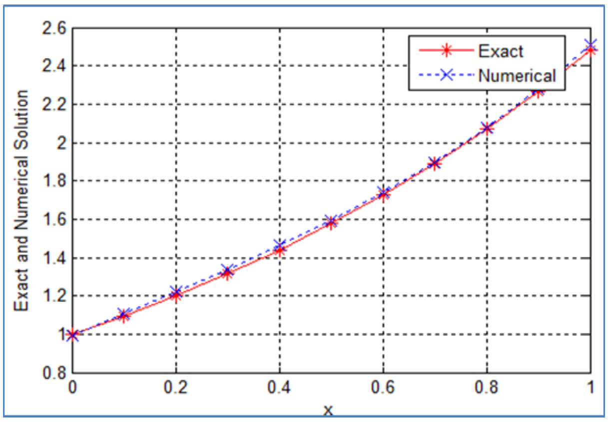

Figure 5. The Exact and the Approximate Solution Using Bernstein Polynomial for Test

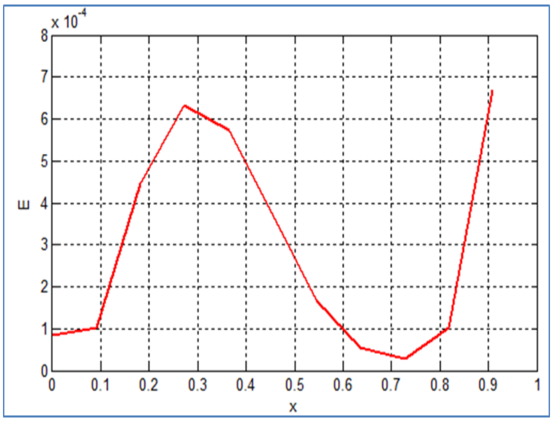

Figure 6. The Absolute Error for Test Example (2) Using Bernstein Polynomial

Table 4. The Absolute Error for Test Example (2) by using Bernstein Polynomial with n=11 and τ =0.99086

x	*Absolute Error*
0.000	0.000000000000000e+00
0.100	2.055633856733260e-11
0.200	1.557687709800940e-09
0.300	2.066152851803160e-08
0.400	1.330457399556670e-07
0.500	5.727724179257330e-07
0.600	1.901697548401350e-06
0.700	5.256496928584300e-06
0.800	1.266511407663450e-05
0.900	2.740980880843180e-05
1.000	5.440701677592340e-05

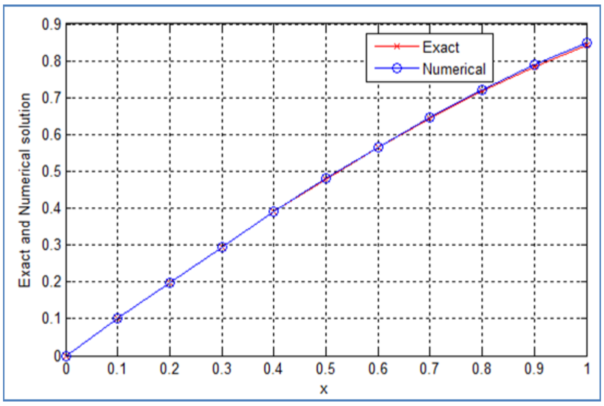

Figure 7. The Exact and the Approximate Solution Using Bernstein Polynomial for Test Example (2)

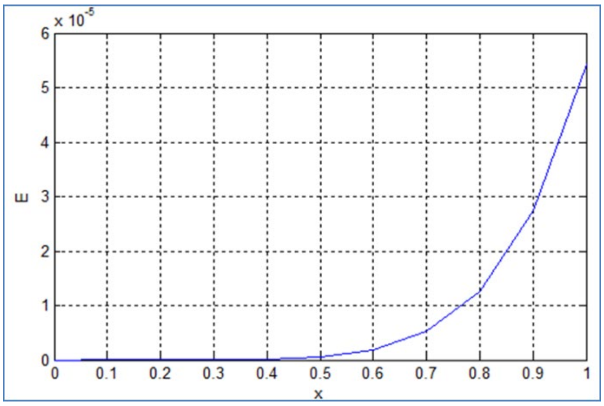

Figure 8. The Absolute Error for Test Example (2) Using Bernstein Polynomial

CONCLUSION AND RECOMMENDATIONS

In this paper, Bernstein polynomial method for solving Volterra integral equations with delay of the second kind is proposed. For this method we used different values of τ because of its effect on Bernstein polynomials for each example above. Thus, we have noticed a difference in the coroner. In general, the results illustrate efficiency and accuracy of the method. The mean absolute error of the numerical examples at the point x in **Tables 1-4** for n=11 are computed. According to the numerical results obtained from the illustrative example, we conclude that:

- The approximate solutions obtained by MATLAB software show the validity and efficiency of the proposed method.
- The method can be extended and applied to nonlinear Volterra integral equation with delay using Bernstein polynomial.
- The method can be extended also for solving nonlinear Volterra integro equation of nth order with delay using Bernstein polynomial.

Disclosure statement

No potential conflict of interest was reported by the authors.

Notes on contributors

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