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MULTIPLE REPRESENTATIONS FOR SYSTEMS OF LINEAR EQUATIONS VIA THE COMPUTER ALGEBRA SYSTEM MAPLE

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ABSTRACT. A number of different representational methods exist for presenting the theory of linear equations and associated solution spaces. Discussed in this paper are the findings of a case study where first year undergraduate students were exposed to a new (to the department) method of teaching linear systems which used visual, algebraic and data-based representations constructed using the computer algebra system Maple. Positive and negative impacts on the students are discussed as they apply to representational translation and perceived learning.

KEYWORDS. Computer Algebra System, Multiple Representations, Linear Algebra, Maple.

BACKGROUND

As Taylor (in Sankey, 2005) has pointed out, traditional methods of teaching (and learning) are no longer adequate to meet the demands of higher education. Modern student populations are diverse in terms of educational, social and economic backgrounds – different models of teaching are needed in order to cater for the different learning styles of the students. This paper offers a discussion of a recent classroom experiment where a multiple representations approach to teaching linear systems of equations to undergraduate students was employed with an aim to increasing student understanding and more completely catering for the diverse range of learning styles of first year undergraduates.

The use of multiple representations in teaching mathematics, such as in this case study, refers to learning environments where students are offered different representations of the concepts being studied; usually employing various different learning tools each with its individual benefits and disadvantages. Teaching which involves the use of multiple representations aims to cater for a wider range of learners and learning styles than traditional teaching and to allow students to construct bridges between different representations in order to provide a more complete understanding of the concepts under investigation.

Before embarking upon an examination of the classroom experiment carried out in this work, one should first obtain some understanding of the subtleties of the concept of

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'representation'. In the terminology of Ainley et al. (2002), representations may be internal – hypothesised mental constructs, or external – material and physically present such as mathematical notations or diagrams. In mathematics, thinking tools for multiple representations include pencil and paper (or board and chalk), calculators, symbolic notations, algorithms, computer algebra systems and online spaces (such as online whiteboards and discussion forums). In this work, the use of multiple representations aims to provide, and to allow the construction of, mathematical metaphors which translate structure from one representational domain to another thereby affecting a deeper form of learning.

A functional taxonomy of multiple representations based on their three main functions of complementing, constraining and constructing was proposed Ainsworth (1999). While multiple representations are commonly used to capture the learner's interest and promote effective learning, the use of alternate representations in the classroom in this study was intended to complement the algebraic representation traditionally used for this course. It was expected that by offering students multiple representations, an understanding of the different scenarios possible when solving three linear equations in three unknowns could be more easily and completely constructed.

The alternative representations considered here are linked with the computer algebra system Maple (but may use other systems such as Mathematica) to take advantage of the system's matrix manipulation commands, equation solving, and most importantly its visualization capabilities. Computer-based learning environments providing multiple representations which support a variety of learning activities were discussed by Ainsworth (1999). While it is often pointed out that introducing computers too early in the undergraduate curriculum can cloud the mathematical theory with computational syntax, there are many examples of successful coupling of introductory mathematics and computer based learning tools. Cretchley et al. (2000) noted that students were found to become more mathematically confident when computational tools were used as part of the learning program.

Mackie (2002) discussed computer algebra systems (CAS) and their usefulness in new approaches for presenting concepts in calculus and the related improvements offered to student learning. Mackie noted that by offering graphical, algebraic and numerical approaches to presenting and understanding mathematics, computer algebra systems function as multiple representation systems. It is with this in mind that the computer algebra system Maple is employed in this study to aid students in understanding two- and three-dimensional linear algebra. Computer algebra systems such as Maple offer not only fast algebraic manipulation tools, but also, the ability to quickly construct manipulable graphical representations of functions.

Moyer et al. (2002) discussed the concept of representation via "virtual manipulatives" which are dynamic visual representations of concrete manipulatives. Static representations are essentially pictures and diagrams and therefore not true virtual manipulatives since they lack the capacity to be manipulated and cannot be "used". Dynamic visual representations on the other hand are able to be manipulated in the same manner as concrete manipulatives (e.g. spinning a three-dimensional graph with the computer mouse) and are therefore truly virtual manipulatives. In this study, a virtual manipulative provided by the computer algebra system Maple is used to investigate the solutions of systems of linear equations and this is discussed in further detail later. Using multiple representations when teaching allows learners alternative descriptions and a possible "preferred choice" when attempting to understand a concept. The use of computer-visual representations in this context provided students with a tool to construct an understanding which in the past has been left to their imagination.

In the sections to follow a background discussion of multiple representations in the teaching and learning of mathematics (in particular linear algebra) is presented. Following this the context of this research is described and the representations used are explained. Then the impact on the students is discussed as well as a reflection on translation between representations. Finally discussions and conclusions arising from the study are presented.

MULTIPLE REPRESENTATIONS AND ELEMENTARY LINEAR ALGEBRA

As has been accepted practice in the past (prior to the widespread use of computers) in mathematics teaching and learning, it is possible to expose students to concepts using different representational modes. This multiple representation of mathematical ideas can be exemplified most simply by the common use of a two dimensional graph to represent functions such as $f(x) = x^2$ or $f(x) = \sin(x)$. Using different representations allows for a more complete coverage of the different learning styles of students. Representations mediated by new technologies such as graphing calculators and computer algebra systems strengthen student learning through the facilitation of interpretations of mathematical models (Yerushalmy 2005), especially since such representations are often non-textual and non-algebraic. Representational technologies such as these offer enhanced understanding in mathematics, especially when multiple representations are linked (Romberg, Fennema and Carpenter, 1993). Yerushalmy (2005) also notes that visual language can be used to promote thinking and new ideas. When students make translations between representations and transformations within representations, these demonstrate a deeper level of understanding on the part of that student and the emergence of critical thinking skills.

At the introductory level, linear algebra is a field in which visualizable, low (two or three) dimension concepts are difficult enough to understand, without abstracting notions to multiple dimensions that are not easily visualised. The availability of various technologies makes it possible to offer students several new ways to learn about linear equations and their solutions – the foundation of linear algebra. Various uses of such technologies in the teaching and learning of linear algebra have been investigated previously (see for example Lin, 1993 and Lindner, 2003).

In this paper, a recent experiment regarding the use of multiple representations in the teaching and learning of linear systems of equations is discussed. Teaching using multiple representations caters for the fact that students prefer to learn in an environment which is appropriate to their preferred cognitive style (Kordaki, 2005; Hazari, 2004). Lindner (2003) noted the recommendations of the National Council of Teachers of Mathematics which state that every mathematical concept should be presented numerically, graphically, algebraically and descriptively. Three representations – algebraic, data-based and visual – are considered in this work, with descriptive representation implicitly included in all three.

In a similar study regarding computer algebra systems (CAS) and function approximation techniques, Klincsik (2003) reported that learners more actively participated in the class and were able to analyse problems in new ways not available to them without the use of a CAS. It was expected that similar outcomes would result from the use of multiple representations, facilitated by the use of CAS, in this study also.

CONTEXT

This paper focuses on a portion of the content of Mathematical Sciences 1C. Mathematical Sciences 1C (henceforth MAB112) is a first year, first semester undergraduate unit at Queensland University of Technology studied by Mathematics Major students and other students requiring a firm basis in mathematics for their future degree studies in for example, science or education. Students are required to have passed Queensland Senior Mathematics C high school studies (see Queensland Board of Senior Secondary School Studies, 2000 for syllabus details) or some equivalent, with particular emphasis on previous studies in matrices, vectors, calculus and number systems.

MAB112 is a prerequisite unit for a host of second and third year mathematics units in the university, including those covering linear algebra and computational linear algebra. Students are exposed to various different topics in MAB112, with none of the areas forming a majority component of the unit. The topics are summarised in Table 1.

Table 1. Topics of study in MAB112.

Real number system	Complex numbers
Trigonometry	Linear systems
Polynomial and rational functions	Matrices
Algebraic systems	Differential equations
Vectors	Determinants

The emphasis in this discussion is on linear systems and, to a lesser extent, matrices and vectors in general. Multiple representations are recognised as having potential to facilitate understanding (see for example Ainsworth, 2002 and Yerushalmy, 2005) and are used here to help students understand what "linear equations" are, what "systems of linear equations" represent and finally, what it means to be a "solution" or "solution set" of a system of linear equations.

Refreshed in the Linear Systems topic of MAB112 is students' high school knowledge of the concepts of linear equations and systems of such objects. Matrix representations of the systems are then considered along with the accompanying terminology. Solution techniques such as Gaussian elimination with back substitution and Gauss-Jordan elimination are discussed. The main extensions from high school level study in this area are

- discussion of solution geometry in 2d and 3d,
- existence and uniqueness of solutions and relationship with rank of matrices,
- the concept of solutions forming a vector space with basis and dimension,
- relationships between homogeneous and nonhomogeneous systems.

Students were required to attend a computer lab tutorial during the period in which linear systems were discussed in class. These tutorials occurred after the lectures had been presented. In the lab, the tutor introduced students to the basic use of the computer algebra system Maple, with specific attention paid to the matrix and linear algebra functions of the CAS.

Students then interactively completed a Maple worksheet which stepped through the use of the CAS to carry out individual row operations or automatically use elimination methods on matrix versions of linear systems of equations (see Appendix 1). Students were not required to actually learn the syntax. In itself this was a valuable exercise, allowing students to gain a valuable skill in computerised linear algebra. However the main reason behind this part of the computer lab session was to investigate a system of equations in numerous different ways.

THE MODES OF REPRESENTATION

In MAB112 in the past, standard practice for presenting linear systems of equations and their solutions has been to use algebraic representations. In this investigation of the use of multiple representations, two further modes are considered – namely graphical visualisation via computer algebra systems (Maple) and numerical data analysis (also generated in Maple). Maple is used because of its symbolic nature and the advantages this provides in allowing students an easier transition from algebraic to computational representations and hopefully greater possibilities for inter-representational translations. This overcomes the hurdles related to using numerical software (such as MATLAB) as experienced by Tonkes et al. (2005) in a similar study.

Algebraic Representation

Figure 1 shows an example of the algebraic representation commonly used in MAB112. This mode is that which is commonly presented in textbooks and static online teaching materials. It involves presenting students with a set of linear equations, converting them to a matrix equation and then to the associated augmented matrix form. Various elementary row operations are then employed to the extent of either Gaussian elimination or the more complete reduction of Gauss-Jordan, and finally either back substituting or inspecting to find a solution to the original equation set.

Figure 1. Extract from class notes used to demonstrate the algebraic representation method as it is applied to the solution of a linear system of equations.

Solving the linear system

$$\begin{aligned} x+y+z&=1\\ 2x+y+z&=2\\ 3x+y+z&=3 \end{aligned}$$

for the unknowns x, y, and z, involves first writing the system in augmented matrix form:

$$\begin{pmatrix} 1 & 1 & 1 & | & 1\\ 2 & 1 & 1 & | & 2\\ 3 & 1 & 1 & | & 3 \end{pmatrix}$$

Next, a sequence of row operations is applied to the augmented matrix to arrive at the following, reduced row echelon form matrix:

$$\begin{pmatrix} 1 & 0 & 0 & | & 1\\ 0 & 1 & 1 & | & 0\\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Any attempt at back substitution requires letting $z = t$, $t \in (-\infty, \infty)$ is a free parameter. Then we see $y = -t$ and $x = 1$, giving the following solution vector:

$$\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 1\\ -t\\ t \end{pmatrix}$$

Hence, the system of linear equations has infinitely many solutions parameterised by the value of t.

While formally this is a correct and exact method, the algebraic presentation of solutions to linear systems is vague to many students – especially when the system exhibits infinitely many solutions and the algebraic representation involves parameterised vector solutions as is the case in Figure 1. The student finishes a problem with some object such as a unique vector solution, a vector with a parameter or the conclusion that they can't solve the problem because of the appearance of the reduced augmented matrix.

But what does that mean and why can't they solve that problem which supposedly has "no solution"? The two alternate representations considered in this study are at least two ways to assist students in answering these questions and in understanding the ideas of systems of linear equations, especially in the two and three dimensional cases. Such understanding might then be transferred to the multidimensional problems considered in more advanced studies.

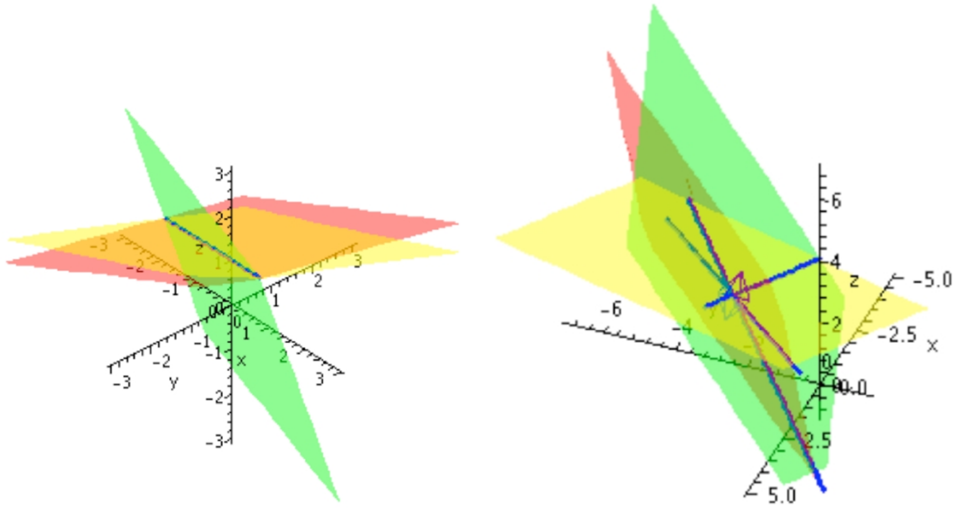
Graphical Representation

As part of the Maple worksheet completed in the computer lab tutorial, the students solved linear systems with a) no solution, b) a single solution and c) infinitely many solutions. They then plotted the three sets of linear equations to see what there algebraic reductions actually meant. Examples of this plotting exercise are shown in Figure 2.

Figure 2. Maple 3d plots of a line of solutions

(a) where three planes (red, yellow and green) intersect on a line, and a single solution point,

(b) where three planes intersect at one point only. On the right, the blue lines show the intersection lines for each possible group of two planes. The lines then intersect at the single solution point



Maple's 3d plots are interactive, allowing the student to change colour and lighting characteristics to suit their particular preferences. The student can also change the plot from a surface, to a wire frame, transparent or contour plot if they find the 3d visualisation difficult to understand at first. The most important advantage of this representation is that Maple's 3d plots are virtual manipulatives than can be rotated in any direction, allowing the student to investigate the spatial situation represented by the equations from any viewpoint. Overlaying the solution point or points (for the infinite solution case) is a further advantage of the CAS representation that permits the student to see what it means for a system of linear equations to have one, many or no solution(s).

Numerical Data Representation

Another representation of the linear equations was considered – that of data points. Maple was used to construct arrays of data values for two linear equations in two variables. This involved making an assumption regarding one variable being an input and the other an output, then calculating output values from each equation corresponding with various different input values. An example is shown below.

The equations considered are

$$4x+2y=4 \quad (1)$$

$$-2x+y=-5 \quad (2)$$

Students chose different x values and then used Maple's "solve" command to print out the corresponding y values for the two equations. Provided an appropriate range of x values was used, the two equations would show the same y values for some x value or show *similar* y values. Then either the exact solution of the system of equations is found (where the equations have corresponding y values) or the discretization can be refined in the area of the similar y values.

Table 2. Chosen x values along with corresponding y values from each of the two equations. The solution is highlighted.

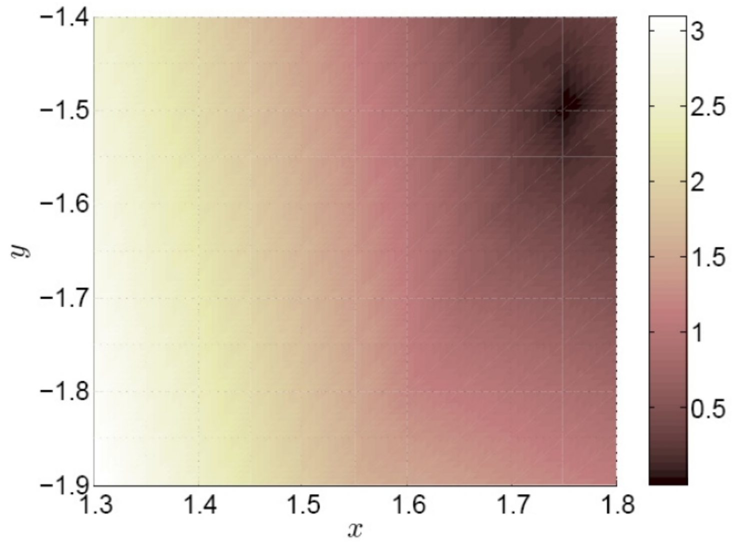
***x* value**	***y* value from equation 1**	***y* value from equation 2**
1	0	-3
1.25	-0.5	-2.5
1.5	-1	-2
1.75	**-1.5**	**-1.5**
2	-2	-1

A slightly more complicated version of this data value representation can be constructed which couples with the graphical representation. A 10**x**10 grid is constructed such that each row in the grid corresponds to an x value and each column represents a y value. Each element in the grid is the sum of the absolute values of the two functions

$$\begin{aligned} F_1(x,y) &= 4x+2y-4 \\ F_2(x,y) &= -2x+y+5 \end{aligned} \quad (3)$$

formed by rearranging equations (1) and (2) and evaluating at the relevant x and y values. That is, the function $|F_1(x,y)|+|F_2(x,y)|$. The idea being that the "solution" to the linear system is the element in the grid where the sum of the absolute values is zero (see Figure 3 and Appendix 2 for a discussion of Figure 3). If no zero is found then either the grid is centering on the wrong area in the plane or the grid is not fine enough. This allows students to develop an understanding of approximating solutions, regions of solution and could even be used to motivate study of the bisection method or finding extrema of multivariable functions. Again, the outcomes of this idea are discussed in the following section.

Figure 3. The grid method representation using an interpolated contour plot which shows the simultaneous solution of linear equations $F_1(x,y)$ and $F_2(x,y)$, via the function $|F_1(x,y)|+|F_2(x,y)|$. The function value at the point *(x,y)* is represented using different depths of colour. The darkest region is where $|F_1(x,y)|+|F_2(x,y)|$ is closest to zero. (See Appendix 2 for further discussion.)



TRANSLATION AND IMPACT

Translation

When considering translation between representations it is important to note that students are most likely to attempt to make connections between mathematical concepts and their preferred representation, and between other representations and their preferred representation. Keeping this in mind when designing lessons can aid in directing students away from simple memorisation and towards conceptual understanding and the development of connections between various representations.

In this course, students were most familiar with pencil and paper calculations from their high school studies of linear equations. At no point was this neglected and in fact students were encouraged to simultaneously employ Maple to solve the equations analytically. A possible future exploration involves students setting up a matrix representation of the linear system and working through the individual row operations (using Maple) in order to solve the system while simultaneously plotting the transformed system of equations and noting the changes caused by each row operation.

Computers, and in particular CAS such as Maple, can be used to allow students to become more active in their learning – they may explore multiple representations easily. An equation may be entered into the CAS which is then used to plot the function or output a table of function values, or even to convert to matrix form and carry out row reductions. Maple proved to be an ideal computer-based tool for presenting multiple representations and for allowing an easier translation between representations for the students.

An important point made by both Mackie (2002) and by Ainsworth (1999) is that instructor-experts can have a very different understanding of representations from that of the students that they teach. The professional may be able to quickly and easily move between representations of a single concept, while the same task may be quite demanding for the student. In the classroom this can lead to confusion. This should be kept in mind whenever employing multiple representations in the classroom and it should not be assumed that students will simply see all the connections immediately (or for that matter, at all).

Impact on the Students

The two modes of representation introduced here, namely Maple's visual manipulative representation and the data value representation, were chosen so as to offer students different ways to understand the concepts underlying systems of linear equations. The visual manipulative was chosen to cater for those learners who prefer the visual representation. On the other hand, many students in the class are interested in statistics and/or scientific studies with statistics playing an important role in their studies. The data representation was aimed primarily at these students. Of course, the algebraic representation was also available to those who prefer the traditional, written mathematics.

A number of students in the class commented that moving from the classroom to the computer lab was appreciated as it provided a different atmosphere and a different way to interact with the subject matter apart from simply writing about it. Most students who attended the sessions and commented on them were in favour of looking at the linear systems in different ways, although quite a large proportion were critical of Maple and its complicated syntax – even though students were not required to learn it or to develop it themselves.

It is difficult to provide a comprehensive quantitative measure of the impact of using alternative representations on the performance of the students. However, the following class average assignment marks form the basis for one attempt at such a measurement.

1. Linear systems homework (pre Maple labs): 85%
2. Maple assignment (post Maple labs): 91%
3. Advanced linear systems homework (post Maple labs) 86%

The advanced nature of the third assignment, coupled with the increase in the average mark indicates some impact, possibly due to the alternative representations offered in the Maple lab, and cemented through the Maple assignment.

It was pleasing to note that students were able to translate some of the understanding gained through the visual and data-based representations to their algebraic work for the advanced assignment. This indicated an increase in the students' "cognitive flexibility" discussed by Lindner (2003). Without prompt or requirement, numerous students interpreted solutions to systems of equations using phrases such as "an infinite number of points on a line", "two planes may intersect, but never all three at once" and "a line where the planes intersect".

On the other hand, the data value representation did not appear to have been transferred to the algebraic work. This may be a result of this representation not being as obviously linked to the algebraic representation or possibly the lecturer's personal preference for (and therefore possible bias towards) the visual representation. Furthermore, the data value representation involves a multi-step jump from the existing algebraic understanding of the students. Most students in the group would be familiar with graphing straight line equations and claiming that the simultaneous solution is the point of intersection, and could therefore be expected to make an easier transition to the graphical representation presented in Section 4.2. However, the data value and interpolated contour plot representation requires first a discrete representation of the equations in terms of data values on a grid of *(x, y)* values, then an interpretation of the process of searching for a "zero" of the function $\left|F_1(x,y)\right|+\left|F_2(x,y)\right|$ and why exactly this represents a solution of the system of equations. It may be appropriate to present students with a stronger link between this representation and the algebraic and graphical presentations. For example, a proof could be presented to demonstrate that $\left|F_1(x,y)\right|+\left|F_2(x,y)\right|=0$ corresponds with a solution of the system of linear equations. Also, the graphical representation of Section 4.2 could be built in a discrete manner from the data values found using the method of Section 4.3. This translational linking is the subject of further research to be undertaken in the coming semester.

CONCLUSION

Overall, it would seem that this initial introduction of multiple representations for linear systems and their solutions via the CAS Maple was a success. More learning styles are catered for, students commented on their enjoyment of using the computer representations, and students provided evidence that translation occurred between at least two of the representations. Furthermore, a preparation of sorts was provided for students future mathematics studies in, for example, computational mathematics, differential equations and linear algebra, where further use of Maple is encountered and a deeper understanding of linear algebra is required.

In the future, it may be appropriate to think more carefully about how to represent the linear systems and solutions using data tables/values. Without being a statistical expert, this representation did not work as it was originally intended. A possible alternative would be to introduce data from an application of some type, leading to the construction of equations (rather than the reverse process).

As a final note, some other interesting representations with slightly different purposes could also be considered. Lin and Hsieh (1993) for example, consider graphical investigations of how changes in equation coefficients lead to alternative solution behaviours. Lindner (2003) considers a representation which is applicable in the two equation-two unknown case (but could be extended to three and three). In this representation, the CAS is used to plot the equations given at each step of the Gaussian elimination process – students observe that while the lines change, the solution does not, thereby demonstrating the concept of matrix/system equivalence.

Through the provision of environments where multiple representations are employed, instructors allow students to experience alternative representations and to choose that which best suits their style of learning. This is vital to improvement and modernisation of undergraduate mathematics education – lecturers must move away from simply teaching what they were taught and how they were taught and towards more inclusive styles where different learning styles are embraced.

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APPENDIX 1

Figure 4. The self-paced worksheet completed by students in the Maple lab class. Syntax was provided and students were required to read and work through the worksheet in preparation for other activities.

MAB112 Maple Tutorial

The restart command clears everything (such as names of variables) from Maple's memory.

```
> restart;
```

Maple has many packages of useful commands that are used in different mathematical areas. When dealing with matrices and linear systems it is often useful to use the "linalg" package...have a look at some of the commands you can use when this package is loaded:

```
> with(linalg):
```

If you want to look up help on any maple command, you can use the help menu (like any other Windows program). Or you could try the following:

```
> ?gausselim
```

Let's try to work an example from the lecture notes using Maple. Topic 9 Example 4.

We can tell maple about the equations first of all:

```
> eqn1:=x1+3*x2+x3=-1; eqn2:=2*x1+x2-x3=0; eqn3:=x2+4*x3=3;
```

Now use the "genmatrix" command to generate the coefficient matrix and right hand side vector:

```
> A:=genmatrix({eqn1,eqn2,eqn3}, [x1,x2,x3], 'b'); print(b);
```

Now to solve the system we have a number of options open to us. We can Gaussian eliminate and then use backsubstitution, we could Gauss-Jordan elimination and read off the solution, or we can go row by row doing the individual row operations. These methods are all shown below. Whatever we do, we will need the augmented matrix first:

```
> S:=augment(A,b);
```

Gaussian elimination

Maple has its own built in Gaussian elimination command. Gaussian eliminating the augmented matrix gives:

```
> T:=gausselim(S);
```

Note that the system has a unique solution. We can find the unique solution by backsubstitution - which Maple also has built in!

```
> x:=backsub(T);
```

Gauss Jordan

Similarly, Maple has a Gauss Jordan elimination command:

```
> U:=gaussjord(S);
```

And we can now read off the solution.

Individual row operations

Maple has all of the elementary row operations built in also. You might use these to check your answers. Recall the augmented matrix S:

```
> print(S);
```

We need to take 2 of row 1 from row 2 to get a zero in element (2,1) of the matrix:

```
> G1:=addrow(S,1,2,-2);
```

Now we can multiply row 2 by -1/5:

```
> G2:=mulrow(G1,2,-1/5);
```

Now subtract row 2 from row 3:

```
> G3:=addrow(G2,2,3,-1);
```

Finally, we can backsubstitute to find the solution as before:

```
> x:=backsub(G3);
```

APPENDIX 2

Figure 3 presents graphically the value of $\left|F_1(x,y)\right|+\left|F_2(x,y)\right|$, over a two dimensional domain using a shaded and interpolated contour plot. That is, the single-valued, two-variable function $\left|F_1(x,y)\right|+\left|F_2(x,y)\right|$ is computationally evaluated throughout the domain and a depth of colour associated with differing numerical values. The higher the value of the function, the whiter the shade on the interpolated contour plot. As the function value moves closer to zero, and hence as the location in the plot nears a solution of the system of linear equations, the colour becomes darker. The darkest region is where $\left|F_1(x,y)\right|+\left|F_2(x,y)\right|$ is closest to zero.

An alternative, especially useful for students who are already familiar with multivariable functions and three-dimensional visualisation of such functions, is to plot the function $\left|F_1(x,y)\right|+\left|F_2(x,y)\right|$ in three-dimensions (as opposed to the interpolated contour plot above), along with the plane $z = 0$. The solution can then be clearly identified as the location where the surface corresponding with $\left|F_1(x,y)\right|+\left|F_2(x,y)\right|$ meets the $z = 0$ plane.

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