International Electronic Journal of
Mathematics Education
Volume 2, Number 1, February 2007 www.iejme.com

MODELING WITH THE SOFTWARE 'DERIVE' TO SUPPORT A CONSTRUCTIVIST APPROACH TO TEACHING

Mette Andresen

ABSTRACT. This article reports on a research project, which was part of the research- and development project World Class Math and Science. The objectives of this part of the project were to research the potentials of computer use in upper secondary school mathematics for the teaching of differential equations from a modeling point of view. The project involved small scale teaching experiments with changes at two levels from the traditional viewpoint on school mathematics: 1) Change of view at curriculum level on the subject differential equations and 2) Change of view at the level of didactical reflections on the intentions of modeling and using models. The article discusses how students' use of laptops can serve as a means for both changes by replacing complex, time consuming expressive modeling with more controlled exploration of differential-equations models.

Finally, the perspectives for the teaching and learning of mathematics of such changes are discussed.

KEYWORDS. Constructivists Teaching Approach, Modeling for Concept Formation, Laptops with the Software Derive, Differential Equations Models, Upper Secondary School.

THE TRADITIONAL APPROACH TO DIFFERENTIAL EQUATIONS

The traditional teaching of differential equations in Danish upper secondary school follows an algebraic-analytical approach to the subject. The differential equations are most commonly introduced in connection with calculus and differential equations are considered as algebraic equations in a function and its derivative. Focus is on determination of integrals and only a few types are treated. The students learn to recognize the types and solve the equations analytically, using their compendium of formulas.

For example, the widely used textbooks by MSc J. Carstensen and MSc J. Frandsen take this point of view (Carstensen & Frandsen, 1999 pp77-92). Fig. 1.-2. shows an excerpt (in Danish language) from the textbook. In the excerpt, the concept of differential equations is introduced by a very short example of a model of growth, followed by a general definition. In general, differential equation is defined as an equation, which involves one or more derivatives of a function.

ISSN: 1306-3030

Figure 1. Excerpt from a Danish textbook. In the excerpt, the concept of differential equations is introduced

4 Differential–ligninger

En vækstmodel

Et meget anvendt matematisk redskab er vækstmodeller, hvor vækst skal forstås som alt, der ændrer sig i afhængighed af en variabel, der ofte er tiden. Som eksempler på vækst kan nævnes

- en befolknings størrelse som funktion af tiden
- hastigheden af et legeme i frit fald som funktion af tiden
- betalingsbalancen som funktion af diskontoen.

Man ønsker så ud fra en række funktionsværdier og væksthastigheder til nogle tidspunkter at bestemme funktionsværdier og væksthastigheder til andre tidspunkter – tænk fx på udarbejdelse af prognoser.

Hvis man skal beskrive størrelsen f(t), er det af og til lettere at opstille en hypotese om væksthastigheden f′(t) end om selve f(t). Dette fører ofte til en ligning, der indeholder f′(t), t og/eller f(t), en såkaldt *differentialligning*, det kan fx være $f'(t) = t \cdot f(t)$. Man skal så bestemme funktioner, der opfylder differentialligningen.

Vi indleder med at se på en population, der til tiden 0 har størrelsen 11500, og hvor man har en hypotese om, at dens væksthastighed er proportional med populationens størrelse med proportionalitetsfaktoren 0,02. Hvis vi kalder populationens størrelse til tiden t for y = f(t), har vi altså, at

$$f(0) = 11500 \quad \text{og} \quad \frac{f'(t)}{f(t)} = 0{,}02.$$

Her kaldes den første ligning *begyndelsesbetingelsen*. Vi omskriver den sidste ligning

$$\frac{f'(t)}{f(t)} = 0{,}02 \iff f'(t) = 0{,}02 \cdot f(t). \qquad (1)$$

Det betyder, at fx

$$f'(0) = 0{,}02 \cdot f(0) = 0{,}02 \cdot 11500 = 230,$$

dvs. at $f(1) \approx 11500 + 230 = 11730$.

Figure 2. The excerpt continued. In the red square, the concept of a differential equation is defined as an equation, which involves one ore more derivatives of a function.

Vi har endnu ikke metoder til at løse ligningen (1), men det er nærliggende at gætte på, at f har noget med en eksponentialfunktion at gøre, og efter lidt regneri finder man, at funktioner af typen

$$f(t) = k\,e^{0{,}02t} ,$$

hvor k er en konstant, er løsninger; det kontrolleres ved at differentiere:

$$f'(t) = k \cdot 0{,}02\,e^{0{,}02t} \;\Leftrightarrow\; f'(t) = 0{,}02 \cdot k\,e^{0{,}02t} \;\Leftrightarrow\; f'(t) = 0{,}02 \cdot f(t) .$$

Vha. begyndelsesbetingelsen kan k bestemmes:

$$f(0) = 11500 \;\Leftrightarrow\; k\,e^{0{,}02 \cdot 0} = 11500 \;\Leftrightarrow\; k = 11500.$$

Vi har så, at

$$f(t) = 11500\,e^{0{,}02t} ,$$

er løsning, men ved endnu ikke, om der er flere løsninger; vi skal senere se, at der ikke er andre. Fx er

$$f(1) = 11500\,e^{0{,}02} = 11732 \quad , \quad f(10) = 14046 \quad \text{og} \quad f(30) = 20954.$$

Efter dette indledende eksempel går vi over til den generelle teori for løsning af differentialligninger.

Differentialligninger

Definition. Ved en *differentialligning* forstås en ligning, hvori en eller flere afledede af en funktion $y = f(x)$ indgår. Hvis den højest forekommende afledede funktion er $f^{(n)}(x)$, kaldes ligningen en *n−te ordens differentialligning*.

Enhver funktion, der passer i ligningen, kaldes en *løsning* til differentialligningen, og dens graf kaldes en *løsningskurve* eller *integralkurve*.

Mængden af samtlige løsninger kaldes *den fuldstændige løsning*.

Eksempel 1. Følgende differentialligninger er af første orden:

$y' = 2x$, der også kan skrives $f'(x) = 2x$,

$\frac{dy}{dx} = xy$, der også kan skrives $f'(x) = x\,f(x)$,

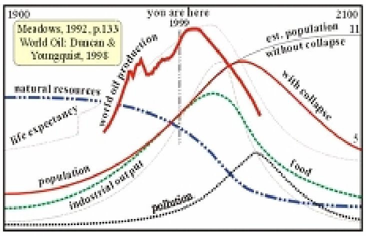
$\frac{dy}{dx} = \frac{x^2}{y}$, der også kan skrives $f'(x) = \frac{x^2}{f(x)}$.

Open-ended, guiding, modeling tasks

The task in Fig.3 (my translation) illustrates a mathematical model's approach, which often supplies the structural one in the traditional teaching:

Figure 3.Task designed to initiate and guide the students' modeling process (Blomhoej, 2001)

ENCL V: Project 3 – Explosive growth of population



This project origins from the **BASE Note 2**[1] and consists of tasks to guide you through the project. You are supposed to set up the model of explosive growth yourself.

How does the world's population grow?
The overpopulation is regarded, as one of the mankind's the most serious problem. Since the book "Limits to growths" (D. H. Meadows & Behrens; 1976) was released, models of the growth of the world's population have been considered in different social contexts, as well as its environmental consequences and global resource-problems. This problem aims at setting up a simple model to describe the growth of the world's population during the last 350 years.

t (year)	1650	1700	1750	1800	1850	1900	1920	1940	1960
N(t)	545	623	728	906	1171	1608	1834	2295	3003

Tabel 3.1 The world's population N(t) estimated for selected years t. N(t) is in million people

a. Examine the data to see if the world's population can be modelled by exponential growth in this period of time.

Instead of exponential growth, one could imagine that the rate of growth was proportional to the population squared. That type of growth is called explosive growth.

b. Examine the data in table 4.1 to see if the world's population can be reasonably modelled by explosive growth in the period of time, covered by data.

(Continues)

[1] Morten Blomhøj, Tine Hoff Kjeldsen and Johnny Ottesen: **BASE Note 2**, Nat Bas RUC 2000

The excerpt illustrates how the design of this task intends to initiate and guide the students' modeling process. The task in the example sets the scene for a technical modeling in the sense that it aims to compare two models and select the one which gives the best fitting of the data. Such technical modeling has been the issue of training tasks as well as tasks for the written examination for more than thirty years in Danish upper secondary school. Now, traditional tools for solving these tasks are graphic calculators with or without CAS, laptops with various software and sheets with pre-printed coordinate systems, having one or two logarithmic scales on the axes respectively. In mathematics, the tasks usually consider models of growth without to involve differential equations explicitly.

The approach in the example, though, is open-ended. For example, the students are supposed to discuss and set up criteria to assess whether "the population can be reasonably modeled" by explosive growth. So, apparently, the task's design intends to encourage the students to focus critically on the dynamics of both models. In this aspect, the example illustrates one step towards a dynamic, modeling approach to the subject.

The project, reported in the rest of this paper, aimed to take the next step: the project researched the potentials of using laptops with the software 'Derive' from Texas Instruments[1] to realize a modeling approach to differential equations in upper secondary school. Derive is a Computer Algebra System, which is useful to solve a number of symbolic and numeric problems. The results can be plotted as 2-D graphs or 3-D color surfaces. For this project, the numerical solution of differential equations and graphing possibilities of slope fields, solution curves etc. were crucial. The students in the project had been using Derive for one or two years, respectively. Hence, they were familiar with the software and the laptops: in terms of instrumental genesis like it is referred in (Trouche, 2005), the students had already accomplished the generation of a number of the Derive commands and other computer facilities as instruments for their work with mathematics.

Changes to a dynamic, modeling approach to differential equations

What could be the research objectives of a change from the classic to a dynamic, modeling approach?

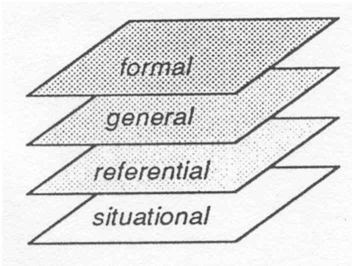
The terms "mathematical models" and "modeling" are used in different contexts with different meanings. In the actual case, modeling is considered at two levels:

At the level of concept formation: Use of the term modeling follows the ideas of Realistic Mathematics Education (RME). The horizontal and vertical mathematizing in RME is illustrated in fig.3, showing K. Gravemeijer's four-level-model. In the model, "horizontal mathematizing" happens by changes from situational to referential level, by the creation of emergent models. Symbolizing is a main issue for these changes. The "vertical mathematizing" happens by

[1] http://education.ti.com/educationportal/sites/US/productDetail/us_derive6.html

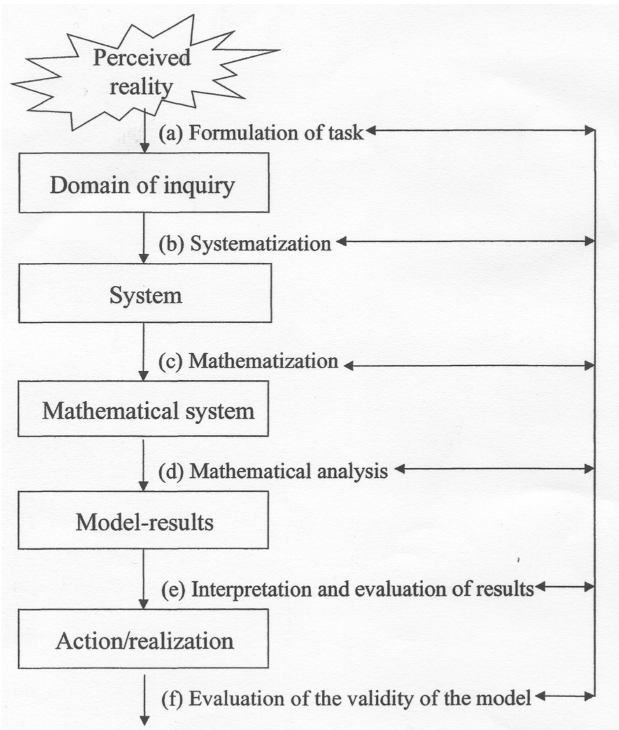
changes from referential to general level, referred to in this paper as changes from "model of" to "model for"- perspective.

Figure 4. Gravemeijer (1997) p340.



At the functional level, "mathematical modeling" is related to applied mathematics and requests a certain degree of modeling competence. The mathematical modeling encompasses technical modeling in the above-mentioned sense. The term "modeling competence", here, is used in the meaning described by Mogens Niss (Niss, 2002). Students' modeling processes take place when the student or, most commonly, a group of students, start with a more or less authentic word problem and build a mathematical model on their own, solve the problem mathematically and transfer the solution back to the real world situation. The process is illustrated in Fig.4., which shows how the process can be divided into a number of sub-processes. The loops are repeatedly carried out during the modeling process.

Figure 5. Blomhoej and Jensen (2003) p 124-125



So, we see two main reasons for the desired change:

First, within the framework of Realistic Mathematics Education (RME) and the constructivists' view on learning, modeling for concept formation is a main heuristic. This means that consciously teaching a modeling approach to the subject may improve the students' concept formation, compared to the teaching of a traditional, non-constructivists' approach.

Second, the classic approach to differential equations represents a structural view on the subject. One main aim of changing to a dynamical systems point of view is to support development of the students' modeling competence at a functional level. This aim follows the claim that a dynamic rather than a structural approach to differential equations is fruitful for modeling.

Accordingly, from a math education point of view, the students' modeling activities should be double-aimed. It follows, that the design of teaching sequences should take the double aim into account.

Expressive and explorative work with models and modeling

In Denmark, the new descriptions of curriculum in terms of mathematical competences are gradually being accepted and adopted by teachers. Since the descriptions encompass modeling competence, there is a growing interest for modeling activities. Some teachers argue against letting the students' do the full modeling process in Fig.4. Important arguments are brought forth: the main arguments claim that the process is too complex and time-consuming. Further, it is hard for the teachers to control the open-ended process to ensure the desired result or learning outcome for all the students.

In particular, the subject of differential equations is regarded as hardly accessible for upper secondary school students. At this stage, the students are not supposed to reach a level of expressive modeling in the sense of being able to build and handle differential equations models for problem solving on their own.

To meet these difficulties, we found it desirable in the project to see how the students could use the laptops to explore models as a forerunner of expressive modeling. The students trained isolated parts of a full modeling process separately by exploring and revising mathematical models, which were already constructed. The idea was to let the students:

- Study the "mechanics" of the single terms in the sense of *symbolizing* and *creating relations* between the single terms
- Focus on the potential roles and the meaning of particular, mathematical conceptions like for example derivative, slope etc.
- Train their ability to *recognize* different types of mathematical models and be critical to their use in the actual context

In this way, the students' modeling activities were less time consuming and complex, and to a high degree the activities were under the control of the teacher. Like an expressive approach, the experiments capitalized on the students' creativity but still allowed the exploration of powerful, conventional symbolizations. This is in accordance with Gravemeijer's description of RME's conceptualization of modeling, which shares some commonalities with both the expressive and explorative approaches to design (Gravemeijer, Cobb, Bowers, & Whitenack, 2000 p 240ff). So, besides the expressive work, the students' explorative work was also designed to facilitate the students' own construction of mathematical conceptions. In the cases of explorative work classroom- and group discussions of shared models and negotiations of symbolizing facilitated and supported the concept formation.

The project's teaching experiments

Our research in changes towards a modeling approach took place in the upper secondary mathematics part of the development project "World Class Math and Science"[2]. Each of the twenty participating teachers in this part of the project was requested to design teaching sequences and prepare materials, which took advantage of the use of laptops. No math education researchers were involved with these preparations. The teachers were expected to do this on their own hand, supported by monthly network-meetings with seminars, discussions and informal exchange of experiences.

A group of three teachers prepared materials for the teaching of differential equations at introductory level using laptops with the software Derive. In accordance with the structure of the entire project, I was only involved with this preparation as a consultant: I made an interview with the three teachers at an early stage and provided them with some materials on differential equations, but we did not discuss the design of the teaching experiment. One of the teacher-authors was graduated from Roskilde University, where a modeling approach to mathematics is the prevailing norm. Four teachers in their own classes tried out the booklet, which took a modeling point of view on the subject. The authors' point of view was presented in the preface (Hjersing, Hammershøj, & Jørgensen, 2004 p3). It can be summarized like this: Focus of attention was moved from finding the solutions to understanding the dynamics of the differential equations. Analytical, numerical and quantitative methods were introduced and attention was paid to geometrical interpretations like slope fields etc. Authentic word problems intended to support modeling and model recognition.

I observed the four teaching experiments without participating in the teaching. Subsequently, I prepared data in the form of cases and episodes, based on these observations and field notes, group interviews with students and with teachers, students' written reports and teaching materials. Each episode and case was chosen to illustrate, inquire or enlighten particular

[2] www.matnatverdensklasse.dk

aspects of the students' individual or collective activities. This material formed the basis for the major part of my Ph.D. project (Andresen, 2006). Data was analyzed qualitatively in accordance with the interpretive framework for analyzing individual and collective activity at the classroom level, described in (Cobb, 2000). Cobb's framework includes social perspectives, such as classroom social norms, socio-mathematical norms and classroom mathematical practices, as well as psychological perspectives such as beliefs about roles, mathematical beliefs and values and mathematical conceptions and activity. The analyses in the Ph.D. project concentrated on identification and interpretation of the flexibility of the students' mathematical conceptions: in the project, the term flexibility was introduced as a technical term to capture the individual student's ability to change between a number of perspectives and media of expression.

The case and analyses in the following paragraph are based on data, picked out from the Ph.D. project's materials (Andresen, 2006 p236ff).

Case: The Rhino task

The theme of this episode is a group of students' explorative work with a differential equation model. Data is based on excerpts from the textbook and from the transcription of an uncut fifteen minutes film recording, besides field notes from the groups work with the task. Words or sentences, omitted from the data, are marked (…).

The episode took place early in the teaching sequence of differential equations, where a group of three students worked with a task, which aimed to explore a differential equations model of a population of rhinos. The students were introduced to simple models of logistic growth. The task asked for a revised version of the model under the assumptions, that there were plenty of space and food for the animals, but if the population went too small, the animals would not be able to find each other for reproducing. In the booklet (Hjersing, Hammershøj, & Jørgensen, 2004 p16), two models of logistic growth were introduced. The first one was (my translation):

Figure 6. Excerpt I from the booklet (Hjersing et al., 2004 p 16)

"Logistic growth of a population

(…) If the population is small, the rate of growth is proportional with the size of the population. If the population is so big, that it may not be fed or kept in the area, then the population will decrease - the rate of growth turns negative. (…) Based on this, several different equations may be set up. We choose a relatively simple one:

$$\frac{dP}{dt} = k \cdot P \cdot \left(1 - \frac{P}{N}\right)$$

(…)"

A logistic model, modified with the factor (p/M-1) to take a lower limit into account, was then introduced (Hjersing et al., 2004 p 18) in a similar way, giving the corresponding equation:

Figure 7. Excerpt II from the booklet (Hjersing et al., 2004 p 18)

$$\frac{dP}{dt} = k \cdot P \cdot \left(1 - \frac{P}{N}\right) \cdot \left(\frac{P}{M} - 1\right)$$

Apparently, the task's hypothetical trajectory of reasoning supposed the students to compare the two models and their corresponding underlying assumptions. Based on the comparison, the students were supposed to build a model where the last brackets in the first equation was substituted with the last brackets from the last equation: $\frac{dP}{dt} = k \cdot P \cdot (\frac{P}{M} - 1)$ (**Figure 8**. the expected answer to the question)

The students' discussions revealed that they expected one of the textbook's two models to give the – unique – right answer:

P2: So, it says… There will be no overpopulation. And that is the N.

P1: Where is the one with the overpopulation (..)

P2: But there must be something to minus, because if it has to be over a certain level…

P3: None of them includes that...

P1: I think that it is maybe included here

P2: It has nothing to do with this

P1: Isn't it just..

The student P1 might suggest something like $\frac{dp}{dt} = k \cdot p \cdot (1 - \frac{p}{N})$ (**Figure 9**. Students' temporary answer).

(Taking their subsequent discussions into account, this is most likely to be their result at that stage).

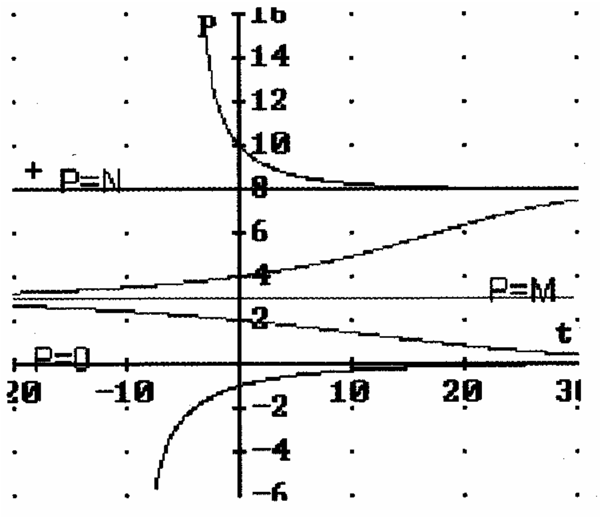
P2: I think so, let us try that! (...)

P1: Should we type it in and then find some values, or how?

P2. What it has to do is to pass a little further than the middle one and then be unlimited... (Looks in the materials): Let us have a look at the logistic ones…look, it must be shaped like this one, but unlimited and it has to pass here...

Here, the students may look at the figure from the materials in the booklet's chapter on modified logistic growth: at page twenty in the booklet, the equation from the model of modified growth was solved for N=8 and M=3, using the numerical method of a 4th order Runge-Kutta. In the booklet, the solutions were graphed under a series of different initial conditions:

Figure 10. Excerpt from the booklet (Hjersing et al., 2004 p 20)



P1: And then we need some values for… (Looks in the paper)...

P2: But we do not have any, they do not give us any

P1: Couldn't we just take some from the plot (looks in the paper)

P2: p equals twelve...twelve hundred (…)

P3: You should substitute the number

P1: (with a little laughter) that is exactly the problem!

P3: Shouldn't we just stop using the N – so it is p over one

Their result may now be the not correct expression dp/dt=kp(1-p).

The students struggled with plotting the equation, apparently without solving it. The teacher arrived and tried to guide the group by giving hints, meant to trigger their shared experiences of classroom practice:

T: earlier, you have seen some lying here, haven't you? Where it is not one minus p but one minus p over some number

P2: It was said, that...

T: It may very well be that... What did the factor one minus p mean?

P2: It was the …"lim" ... what is it called...the straight line (draws with the finger a straight line horizontally in the air)...

P1: Some identity equilibrium something

T: Yes, that is right, but we were out in a case with two equilibriums…

(...)

P1: The only thing that can make it negative is that they have so much space that they never meet each other (...)

T: If it is less than three, then it will be negative,

P2: Yes

T: Then the growth decreases. It should... (Draws a decreasing line in the air)

P2: Then when the rate of growth is higher...

T: Yes, take a look at the factors in the expression

P2: Okay...

T: and then the one minus P over N, maybe we should look down here, for instance, (points in the materials): This factor tells how many animals there can live and M tells how spread out the population is

P2: Should we have p and...

T: instead of... because, in the case of the rhinos, we may assume that there is no upper limit

P1: Okay

Interpretation of the students work

From the beginning, the students sought to recognize and exclude the model, which took the overpopulation into account. P2 referred to the symbol N for the upper limit of the population. P2, further, suggested that the lower level should be symbolized by a negative term, “something to minus”.

At this state, the students changed between the word-problem’s reality perspective and a model perspective represented by the textbook’s models. Unfortunately, the students did not continue the discussion of symbolizing the population’s upper and lower limit, respectively. Apparently, it was not part of their classroom mathematical practice to discuss generally, how the problem should be modeled: The three students did not discuss an overall strategy or plan for their work. From a teacher’s point of view, this lack of discussion could be considered as a waste of learning potentials, as far as even a short discussion and negotiation might have helped to bring the links between model and reality perspective of the single terms in the mathematical model in focus of attention and supported the emergence of the student’s model. Without to reflect on the issue of modeling or negotiate a strategy for the symbolization of the upper and lower limitations, P1 immediately suggested choosing the simpler of the two differential equations models and P2 agreed. The students’ next step was to study the textbook’s graphs of solution curves. Each of the students tried to imagine how the growth would proceed under the different circumstances and based on these imaginations, they compared the expected solutions with the shapes of the curves in the textbook. So, at this stage the modeling took place at referential level, referring to Gravemeijer’s four-level-model in Fig.4.

Moreover, the referential differential equations model was studied by means of its solution curves. In Derive, graphs of the solution curves are at hand without necessarily to solve the differential equation explicitly. Hence, the aspect of equation-solving is not predominant when working with differential equations models in Derive. None of the three students neither reflected consciously on the process of building a differential equations model nor reached the general level, that is, did they not change from “model of” perspective to “model for” perspective.

P1 and P2, however, had different strategies: P1 wanted to try the differential equations model with some values. Maybe because the technical fitting method was the usual classroom practice, like it was illustrated in the task in Fig.3. Simultaneously, P2 took the graphs from the textbook as the starting point and tried to imagine the shape of the solution curves for the equation they were looking for. So, P2 saw an emergent model of solution curves in graphic representation and tried to change in both directions between reality and model perspective on the solutions. But P2 did not manage to model the solution with symbols and P1 took over. P1 did not explain how the try out with actual values could help – maybe the idea was to support symbolizing and changes between reality and model perspective on the differential equations model and/or its solution curves. So the group dropped P1’s strategy since the students had no values to substitute. P3, who supported P1’s idea of solving the problem by substitution of actual values, then suggested that they simply omitted the symbol N from the model. The two other students agreed in this simplification. None of the three did reflect on the meaning of N in the model.

The teacher was aware of the role as a guide. For example, the teacher asked for the meaning of the factor one minus p, rather than simply corrected the students or gave them the final results. The teacher’s question, apparently, intended to support the students’ collective process of symbolizing and provoke a change from model to reality perspective by each of them. Though, P2 changed to the graphic representation of the solution curve, rather than to the real world content of the factor. P1 changed to natural language and explained the general, real world content of the factor. Finally, the teacher explicitly demonstrated how to change between reality and model perspective.

CONCLUSION

Apparently, the rhino task was not at all easy for the three students to solve. One could argue that the right answer could easily be found during a superficial, instrumental routine of labeling the models’ terms to the word problem’s different circumstances and establish an analogy. None of the students did so. Symbolizing the restricting factors, i.e. the upper and the lower limit for the population respectively, seems to be a prerequisite for solving the task. The excerpt shows that P2 identified “overpopulation” with “the N”. A little later the same student revealed an attempt to symbolize the lower limit, namely when P2 looked for “something to minus” in the two models. The design of the task, though, did not offer an explicit, close guidance to support the symbolizing, for example by encouraging the students to interpret the meaning of the terms in the first two models. In the end of the excerpt, the teacher offered such guidance to the students. Before that, instead of interpreting the single terms in the two models, the group turned to concentrate on the shape of solution curves.

It is a key point in my interpretation of the students' activities and dialogue that this change of focus is linked to their conception of differential equations: the three students preferably considered differential equations in the Derive environment as a means to capture the progress over time of the population of rhinos in graphic representation. The process of solving the differential equations was completely out of focus, as were the interpretations of the single terms in the models until the general expression was identified.

To sum up: the case illustrates, how the students in the teaching experiments developed the desired dynamic modeling approach to the subject differential equations during their explorations and reversions of ready made models. The teacher and the textbook's tasks guided the explorations. The software Derive facilitated a change of focus from equations-solving processes to qualitative, graphical interpretations of the system's behavior over time. The interpretations were in the core of the models, which emerged by the students and laid the foundation of the students' symbolizing. The interpretations were supported by the discussion amongst the students.

REFERENCES

Andresen, M. (May 2006). Taking advantage of computer use for increased flexibility of mathematical conceptions. *1*(ISBN 877684069-7), 327.

Blomhoej, M. (2001). *BASE Note 1+2* [Basic notes one and two]. Roskilde, Denmark: Roskilde University.

Blomhoej, M., & Jensen, T. H. (2003). Developing mathematical modelling competence: Conceptual clarification and educational planning. *Teaching mathematics and its applications: An international Journal of the IMA*, 22(3), 123-139.

Carstensen, J., & Frandsen, J. (1999). *Mat 3H* [Math 3H]. Denmark: Systime.

Cobb, P. (2000). Conducting teaching experiments. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307-333). Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers.

Gravemeijer, K. (1997). Mediating between concrete and abstract. Nunes, T. and Bryant, P. *Learning and teaching mathematics. An international perspective* (pp. 315-346). Psychology Press.

Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolising, modeling and instructional design. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolising and communicating in the mathematics classroom. Perpsectives on discourse, tools, and instructional design*. (pp. 225-274). Mahwah, New Jersey: Lawrence Erlbaum Associates Publishers.

Hjersing, N., Hammershøj, P., & Jørgensen, B. (2004). Modeller i Derive. *Differentialligninger og modelbygning. Højt niveau i matematik* [Models in Derive. Differential equations and modeling. High level mathematics]. Copenhagen, Denmark: Matematiklærerforeningen.

Niss, M. (Ed.). (2002). *Mathematical competencies and the learning of mathematics: The Danish KOM project.* Roskilde, Denmark: Roskilde University.

Trouche, L. (2005). An instrumental approch to mathematics learning in symbolic calculator environments. In D. Guin, K. Ruthven & L. Trouche (Eds.), *The didactical challenge of symbolic calculators* (pp. 137-162). USA: Springer.

Authors : **Mette Andresen**

E-mail : mea@dpu.dk

Address : Danish University of Education

Tuborgvej 164, DK-2400 Copenhagen NV

Denmark

Phone Number : +45 8888 9688