

on the set G . Thus, students have the opportunity to draw upon their naive, non-formal knowledge of ERs and partitions and only then, in an already more familiar environment, worry about the correct definition of operations and functions.

Our study, then, apart from verifying if the approach proposed is suitable, should also regard: if the students freely use ERs, partitions and the theorem ER/P that relates them (due to preliminary evidence, we are confident about this); whether their approach to the GIT can draw on that; and additionally, if, in case they approach the (re)construction of the theorem in purely algebraic terms, whether they can succeed in it.

THEORETICAL FRAME

We used the *APOS theory* that was devised by Dubinsky in the 80s (Dubinsky 1986; Arnon et al., 2014). He bases this theory on Piaget's *reflective abstraction* to describe the construction of mental objects, and he distinguishes different types of it, or *mechanisms*: *internalization*, *coordination*, *encapsulation*, *generalization*, and *reversal*. These are the origin of different (mental) *constructions*: *Actions*, *Processes*, *Objects*, and *Schemas*—whence the APOS acronym.

Let us consider a fragment F of mathematical knowledge in Dubinsky's perspective (as in Arnon et al., 2014): Individuals have an Action conception of F if the changes that they make on it are done step by step, obeying stimuli that are and are perceived as external. Individuals *interiorize* an Action in a Process concept of F if they can perform an internal operation that does (or that they imagine does) essentially the same transformations entirely in their mind, not necessarily covering all the specific steps. They can coordinate two or more Processes, or reverse one to obtain a new Process. If individuals think about the Process as a whole and build and perform transformations on the whole Process, they have encapsulated it in an Object conception of F . If they need to return from the Object conception to the Process from which it comes from, they do it by de-encapsulating the Object. A Schema of F is a collection of Actions, Processes, Objects, and other Schemas that are consciously or unconsciously related in the mind of the individual in a *coherent* cognitive structure. Coherence relates to recognizing relationships within the Schema, recognizing whether the Schema can solve a particular mathematical situation, and using it in such a case. In dealing with a mathematical problem, the individual recalls a Schema and unfolds it to gain access to its constituents, uses relationships between them, and works with the whole. A Schema is ever evolving and can be considered as a new Object to which an individual can apply Actions and Processes; in such a case it is said that the Schema has been *thematized*.

APOS methodology has been consolidated since the 80s and it is considered a successful method of research in the area of abstract algebra learning today (Dubinsky & Lewin, 1986; Dubinsky, 1991; Asiala et al., 1996; Brown et al., 1997; Trigueros & Oktaç, 2005, Arnon et al, 2014). It uses a *Research cycle* that aims to allow for empirical evidence of the mental mechanisms and constructions to be put into play in the construction of a fragment of mathematics: theoretical analysis, design, and implementation of instruments and data analysis and verification (Asiala et al., 1996, Arnon et al. 2014).

A *genetic decomposition* is a hypothetical model whose aim is to describe both the mental structures and mechanisms that a student might need—a genetic decomposition is not necessarily unique—in order to learn a specific fragment of mathematics (Arnon et al. 2014), and how those mechanisms and structures are organized in a Schema. It includes prerequisite structures that the students need to have constructed previously—that is, not just a list of concepts and facts that should be 'known' by them, but whether they are required as Actions, Processes, Objects or

Schemas. Specific aspects, such as coordination of Processes, shed light on the constructions made by the students. Thus, a genetic decomposition not only deals with the cognitive aspects of the apprehension of knowledge by the students, but also clarifies the role of its prerequisites—in fact, it is common knowledge that, when students approach specific subjects, their mathematical prerequisites are not necessarily well known by them. (Arnon et al. 2014). It typically starts as a preliminary genetic decomposition, which has to be validated, often after adjustment based on empirical data (Asiala et al., 1996). A preliminary genetic decomposition is based on "the researchers' experiences in the learning and teaching of the concept, their knowledge of APOS theory, their mathematical knowledge, previously published research on the concept, and the historical development of the concept" (Arnon et al., 2014, p. 28).

In what follows, we report some relevant findings on mental constructions related to groups, normal subgroups, and quotient subgroups obtained by using APOS theory.

We assume that the student has encapsulated the general concepts of group (Brown et al., 1997) and of function, epijectivity, and injectivity (Dubinsky, 1991; Baker, Trigueros, & Hemenway, 2001). Hamdan (2006) presents an initial genetic decomposition for ERs (and functions). An explicit genetic decomposition for (an additive group) homomorphism can be found in Roa-Fuentes and Oktaç (2010). Isomorphism is studied in Leron, Hazzan, and Zazkis (1994, 1995); Leron and Dubinsky (1995); Dubinsky and Zazkis (1996); and Nardi, (1996, 2000). Construction and multiplication of cosets is seen in Dubinsky et al. (1994). Asiala et al. (1997) report at length about the mental constructions on cosets and normality: roughly two thirds of the students considered were successful in constructing cosets and normality, while for quotient groups the rate dropped to around one third and, generally speaking, a high percentage of the students seems lost and can end up disconnecting from the subject (see also Ioannou & Nardi 2009). In fact, some researchers doubt the permanence in time of the quotient concept (Dubinsky et al., 1994). Note that if the encapsulation of the quotient-partition is not achieved it is certainly doubtful that the concept has some value for the student. Dubinsky (1986) showed that students' difficulties with mathematical symbolism (e.g., multiplying cosets aH) come from trying to apply labels before achieving encapsulation.

We found no genetic decomposition for the GIT.

CONSTRUCTING THE GIT

We turn now to requirements for achieving an Object conception of the GIT. According to the research cited above, one needs Object conceptions of set (for group), multiple quantification (for isomorphism), function (for homomorphism), group (as a homomorphism is an Action on a group), subgroup (examining normality requires its encapsulation), and homomorphism (examining bijectivity requires its encapsulation). We also need Object conceptions of $\text{Ker}(f)$ (to get $G/\text{Ker}(f)$) and $\text{Im}(f)$ (to compare it with $G/\text{Ker}(f)$). A Schema conception of group consisting of Object conceptions of set, group, subgroup and function as well as a Process conception of binary operation (the coordination of set and binary operation plays a fundamental role in the Schema) are also necessary. A genetic decomposition for the GIT does not need the general concept of quotient group, but of $G/\text{Ker}(f)$, and working with this particular case might help the construction of the general case. Also, the encapsulation of cosets just as sets does not need to be carried out by an "algebraic" process: the encapsulation of a quotient set is first achieved as the partition $G/\text{Ker}(f)$ defined by $aR_f b: f(a) = f(b)$ and then

defined as an operation on $G/\text{Ker}(f)$. As shown in the research cited above, this is not an easy task, but in this approach students are able keep in mind that they are dealing with (sub)sets of elements to start with. Moreover, we did not require an Object conception of $G/\text{Ker}(f)$ but a Schema composed of Object conceptions of group, subgroup (and set, function, and binary operation), and (the naive version of) ER/P.

Over those constructions, the student is supposed to make a number of coordinations—several of them already established in the literature reviewed—such as (in this case, $N = \text{Ker}(f), G/N = G/\text{Ker}(f)$): subgroup with cosets (as classes in a partition) through the normality property in order to obtain a Process conception of a normal subgroup; group with E/R, and subgroup and quotient group in order to obtain a Process conception of G/N ; function and binary operation, so as to construct the homomorphism as a Process which encapsulates by means of the identification of the quotient group and its image; function with ER/P in order to define a function from the quotient, then a bijection and obtaining the SIT; etc.

METHOD

We designed and implemented some instruments to explicitly reflect the constructions through which students can grasp the concepts that are needed. For this, we framed our preliminary genetic decomposition using a case study methodology (Goetz & Le Compte, 1988; see also Arnon et al., 2014) so as to provide, through a purposeful selection of participants, knowledge about the learning of the GIT through the study of mental mechanisms and constructions shown by comparatively advantaged individuals in dealing with our subjects.

We started with four groups of students, 17 in total, each group from a different university in Chile. The universities were chosen so as to ensure: equivalence in admissions processes in terms of basic skills for addressing the mathematical field, existence of both an undergraduate program in mathematics teaching that included group theory, existence of a graduate program in mathematics, and affordability for the study. The students were required to have approved a course in abstract algebra.

We designed a questionnaire (see Appendix) to be administered to the aforementioned students. The answers to the questionnaire were not aimed at determining which students might proceed with the construction of the GIT, but rather which might not. It was applied in a classroom at the location where the students' programs are offered and lasted about one hour. At least one of the researchers was in the room, and provided explanations about symbols or related questions (e.g., the meaning of “inside” and “outside” mathematics in Question 4 of the questionnaire) but not prompting.

Then we applied two semi-structured interviews to students chosen based on their answers to the questionnaire. This type of interview is customarily used in order to be able to interact with the interviewee and to be able to clarify answers if needed (Dubinsky et al., 1994). For the seven students selected in Group 1 (Juan, José, Pedro, Luis, Jorge, Ana, and Manuel), who we could not assume had paid attention to equivalence relations and partitions, we used Script 1 (see Appendix). For the three students in Group 2 (Carlos, Marta, and Leslie) we used Script 2 (see Appendix), since we knew that they had treated equivalence relations and partitions in their curricula. The aims of these interviews were to gather information about the viability of our preliminary genetic decomposition and to refine it if necessary, as well as determining the difficulties encountered by students in dealing with our theorem.

The interviews were individual, were held in isolated rooms at the location of the academic units to which the students belonged, and were videotaped. Each

interview lasted an hour and a half and in all but one of them there were at least two researchers present. The whole team reviewed the videotapes.

Questionnaire

Given the limited aim of the questionnaire, it was not tested beforehand: students' inability to proceed with the construction of the GIT would be shown in the corresponding interviews. Thus, we do not claim that students who did not do well in the questionnaire would not be able to end up constructing the GIT, but that we doubt it, and that it was preferable to interview others.

There were two kinds of questions (enumeration refers to the questionnaire):

A. The students should:

- distinguish between elements of a group and sets of them (1a, b, c): if they do not, they would not be able to consider cosets as elements not of G but of a quotient of G , which is crucial for defining the isomorphism of the GIT;
- realize that the operation of a subgroup is the restriction of the operation on G (1d): otherwise, since it is needed that the operation on cosets of G has to be induced by the operation on G , the students would lose track of the structure they have to deal with;
- have an idea, though vague, of normality (2a): if not, it is unlikely that they would be able to approach the need for the operation in the classes to be well defined;
- be acquainted with the importance of ERs and partitions inside and outside mathematics, or at least be able to come up with some examples (4): this is needed to take advantage of the set isomorphism theorem for the construction of the GIT;
- be able to at least generally express the notions of ER and partition or produce an approach to them, even in an informal way (5a, b): also to make use of the set isomorphism theorem;
- realize or consider that there is a connection between ERs and partitions defined over the same set (5c): the GIT needs for a translation of the ER defined by $\text{Ker}(f)$ on G into cosets defining $G/\text{Ker}(f)$;
- be able to define homomorphisms in “simple” cases (6): even though this does not guarantee to be able to define an isomorphism starting from a quotient, its absence suggests that the students will not succeed in this last one;
- realize that the commutative property is “preserved” by isomorphism (7): thinking that a non commutative group may be isomorphic to a commutative one suggests that the acting idea of isomorphism is only related to bijection and not to the group structure.

B. We would welcome (but not require) that the student:

- have an idea about why operations on cosets work (2b);
- have a possibility of analyzing normality for subgroups of a small group and (even) of the construction of a quotient group (2b);
- show an idea about how to identify isomorphic groups that are not stated in the same way (8).

At this stage, we valued any arguments, even if not formally stated. Based on experience and research, our expectations for the answers were rather low, and we did not ask for the GIT.

We collected the questionnaires and tabulated them in a colored matrix in order to be able to easily glance at the correctness of the answers of each student to each question. We then decided which individuals to interview by means of identifying correct answers in most questions other than 2b, 3, and 8; “good” answers on these

three questions would prevail over some wrong answers in the others. We did not proceed using a checklist, nor did we hypothesize that the selected students had clarity on the requisites stated above as that would be determined in the interviews.

Interviews

We did not wait for interviewees to fluidly answer our questions, which would have been contrary to the existing research as well as our own experience. As stated earlier, the scripts focused on examining the eventual presence of mental constructions and mechanisms that would come into play in connection with the GIT as we propose it: on the one hand, some aspects of group theory, which we would require even in a rather vague form, and, on the other, questions related to ER/P.

The scripts were used as a broad guide for the researchers' APOS semi-structured interviews. Notations were explained as needed. Interviewees were asked to explain their thoughts, to draw, and to give examples. The tone was conversational.

Script 1 began by asking about the GIT and we did not expect a clear answer on it—in fact, our experience suggested that most students would have forgotten about it, a result that was shown in the interviews. We then asked interviewees to elaborate on ERs and partitions. The key issues in our approach were that we expected subjects to realize were that, given $f: A \rightarrow C$ a function, f defines an ER R_f on A , and that there is a one-to-one correspondence between the quotient A/R_f and $f(A)$. We asked this, but with low expectations. Thus, if students in Group 1 were not able to do this, we gave them the definition of R_f in order to see whether they could elaborate from there, that is, relate it to the quotient defined by $\text{Ker}(f)$ on G and thus proceed to the GIT.

Script 2 was prepared for interviews that we expected to proceed in a more direct course.

ANALYSIS AND DISPLAY OF SOME RESULTS

As expected, every interviewee showed Process conceptions of group, subgroup, and normal subgroup. Luis, Ana, Manuel, and Marta manifested Object conceptions of group, subgroup, and normal subgroup and were able to construct a quotient group. Pedro, Juan, and José could not progress much in their interviews and they could not elaborate on examples nor on drawings: Juan and José tried instead to recollect definitions (such as Kernel, image, partition) and theorems (such as " $(G/H)/(K/H) = G/K$ "). On the other hand, everyone used ER/P as a theorem in action: when arguing, they changed from ERs to partitions and vice versa. Group 1 students could not elaborate much on the GIT, as expected; Ana was an exception. Students in Group 2 tended to be more assertive in stating the GIT and reasoning about it. We will turn to these later.

Coordination

We were able to verify the coordination that we expected. We give some examples and brief comments:

Subgroup and quotient group. José drew the usual picture of a set, labeled it G/\sim , and wrote " $x, y \in G: x \sim y \Leftrightarrow xy^{-1} \in H$ ". On the contrary, Jorge did not: he stated "A normal subgroup N of a group G is a group with certain properties." As we expected, Jorge could not elaborate on quotients.

Normal subgroup and quotient group. Ana correctly wrote $N \trianglelefteq G: gng^{-1} \in H, g \in G, h \in G$; and added: "They serve to construct the

quotient groups. It is known that if G is an abelian group and N is a subgroup of G , then N is a normal subgroup." (We will turn to her later).

Function and homomorphism. The absence of this coordination seems to impede the construction of the GIT. For instance, Pedro wrote " $\varphi: (G_1, \star) \rightarrow (G_2, \square); g \rightarrow \varphi(g)$ ". We then had the following exchange:

Pedro: But ... the interesting [point] is that I need the function to be bijective in order to obtain an isomorphism.

Interviewer: You told me that you cannot prescind from the function concept.

Pedro: Oh, but I can prescind from the group concept.

Interviewer: Why?

Pedro: Why can I prescind from the group concept? Because the isomorphism concept is more general, uh...

On the contrary, Luis wrote, not quite correctly: $f: \mathbf{Z} \rightarrow \mathbf{Z}; z \rightarrow 2z; f(a + b) = 2(a + b) = 2a + 2b = f(a) + f(b); \text{Ker}(f) = \{0\}; g\text{Ker}f g^{-1} = \text{Ker}f \Rightarrow \text{Ker}f \text{ Normal}; \Rightarrow \mathbf{Z}/\text{Ker}(f) \simeq \mathbf{Z}$.

Nevertheless, his explanation showed just a Process GIT. Thus, the coordinating function and the homomorphism property would not suffice for an Object GIT—as one might expect.

Normal subgroup, quotient group, and homomorphism. We think that coordinating these Processes allows getting close to Object GIT. Manuel correctly stated the GIT, wrote it, and explained: $f: \mathbf{Z} \rightarrow \{-1, 1\}, f(x) = 1, x$ even, $f(x) = -1, x$ odd, and then wrote $\text{Ker}(f) = \{x \in \mathbf{Z}; f(x) = 1\} = 2\mathbf{Z}; \{-1, 1\} \simeq \mathbf{Z}/2\mathbf{Z} \simeq \mathbf{Z}_2$. Nevertheless, he could not elaborate on ERs and partitions and he could state neither the SIT nor the GIT.

Suitability of the Genetic Decomposition

Ana's work. Ana, in Group 1, showed clarity in several aspects of the GIT. She was interviewed again with Script 2 in order to determine whether she could complete her approach but she could not. She correctly stated the SIT, but when asked about an eventual relationship with the GIT, she added, "The guarantee that one has in sets is that in defining the equivalence relation, partitions can be known and one can easily see the quotient, while for groups we do not have the equivalence relation." Thus, she could not coordinate the SIT with the structure of the group.

Carlos's work. Carlos, when asked for examples of ERs, both in mathematics and in real life, easily provided several ERs and partitions. Questioned on it, he stated ER/P and explained how to define an ER from any partition. He was then given a diagram where two sets, $A, B, |A| = 8; |B| = 6$, are shown and was asked to draw a general function on it. He filled in the picture as shown in Figure 1.

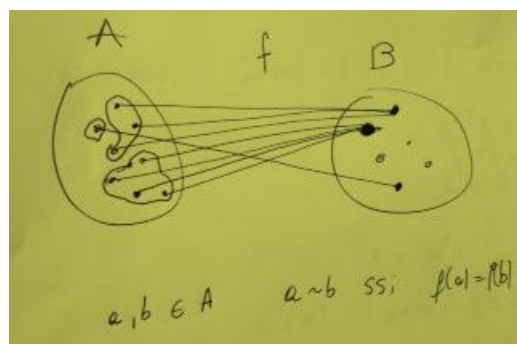


Figure 1. Carlos's work ("ssi" stands for "iff").

Asked whether he saw an ER in the picture, Carlos wrote below it as shown in Figure 1, which he correctly explained. When we inquired whether he saw another function in the picture, he wrote " $g: P \rightarrow B, g(P_i) = b_i$ ", P being the partition induced by g . He explained this correctly, although he called P_i "a partition," as often happens. In answering what kind of function is g , he said that it is injective and that it is epjective over its image.

Turning to groups, we had to remind him what $H \trianglelefteq G$ means and we gave him the definition of R_H . He checked the ERs properties. When asked, he wrote the definition of $\text{Ker}(f)$. His response to our subsequent inquiry about the corresponding ER is shown in Figure 2.

$a \sim b$ ssi $a^{-1}b \in \text{Ker}(f)$ $a, b \in G$
 $f(a^{-1}b) = e' \Rightarrow f(a)^{-1}f(b) = e'$
 $\Rightarrow f(a) = f(b) \checkmark \Rightarrow f(a^{-1}) = f(b)^{-1}$
 $f(a^{-1}a) = f(e) = e'$
 $f(a^{-1}) = f(a)^{-1}$

Figure 2. Carlos's work. (In the third column, he checked what he claimed in the second).

We then asked for $G/\text{Ker}(f)$ and he gave this response, "Two elements are related when they have the same image, which was what we did at the beginning." He explained that the SIT and the GIT are quite similar, the difference being the need for group homomorphisms.

Thus, although Carlos could not work out the GIT by himself, it is clear that if he was given some definitions, he had a clear view of what had to be done. Moreover, he was able to argue using one figure where both the group G and the quotient G/H coincided, which is one of the problems that students have to overcome when reasoning about the GIT.

Carmen's approach. Carmen completely showed that our preliminary genetic decomposition works: her coordinations were readily apparent. Asked for an RE in either mathematics or in common life she immediately said, "What catches my attention more is the one of the pre-images." She wrote " $f: A \rightarrow B, f^{-1}(b)$ ", and stated that an ER is defined by inverse images of elements of B . Asked for more examples, she gave one about buses with a common route and added, "If we think as a partition it comes out much more easily." She stated and explained ER/P. We then gave her a diagram, $A, B, |A| = 11; |B| = 8$ and asked for a general function, and her response is as shown in Figure 3.

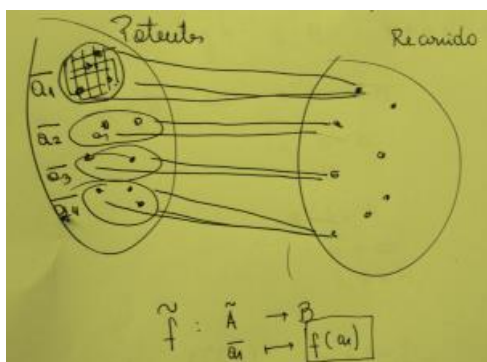


Figure 3. Carmen's work. (*Patentes*: license plates; *Recorrido*: route)

When asked for another function that she may perceive in that setting, she explained that, for example, bus fares are not fixed for individuals but for sets: buses on the same route have the same fare. We then asked her to generalize what is occurring, that is, a sort of theorem that can be stated. Carmen then correctly stated our SIT. When we talked about a homomorphism between two groups, she wrote " $f: G_1 \rightarrow G_2$ ". She remembered that the definition of $H \trianglelefteq G_1$ was " $\text{Ker}(f) = f^{-1}(1_2)$ ", and stated that the equivalent classes in $G/\text{Ker}(f)$ are " $b\text{Ker}(f), b \in G_1$ ". She also wrote " $\bar{f}: G/\text{Ker}(f) \rightarrow \text{Im}(f); a\text{Ker}(f) \mapsto f(a)$ ", and stated that it is an isomorphism. We then had the following exchange:

Interviewer: This theorem is called the GIT. What would you call to the one that you made at the beginning?

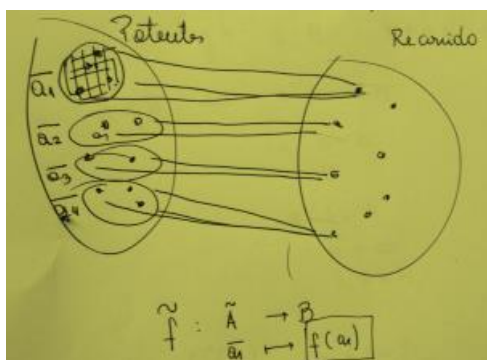


Figure 3. Carmen's work. (*Patentes*: license plates; *Recorrido*: route)

Carmen: Phew!... of the isomorphism because it has to have a form... Here there is no form since there is no structure... but what could it be; in lieu of isomorphism let's talk about the theorem of bijectivity.

We asked her about our approach, the SIT, and then the GIT, she responded, "It seems to be a good idea to me. This theorem could be better understood if one looked at it using the example of the buses, surely." When asked whether something must be taken care of in the group case, she pondered it, and said, "Ah! That it were well defined," and then explained this, both expeditiously and clearly.

DISCUSSION

As stated in our Approaching the GIT, we expected that students would freely use ERs, partitions, and the ER/P theorem even if they could not formally explain them, and all of them did.

The students who showed an approach to the GIT purely in terms of abstract algebra were not able to develop it and could not explain it. Those who at first

seemed able to (re)construct the GIT but in the end could not, had trouble when arguing by means of ERs, partitions, and/or ER/P. Conversely, as shown by Carlos's and Carmen's cases, those who could elaborate on ER/P were able to construct the SIT and the GIT and also worked faster, had a clearer view, and seemed more confident. In addition, they were able to construct the GIT from the SIT as we had expected.

Carlos's and Carmen's cases also show that for the GIT it is not necessary to construct the general quotient G/H for $H \trianglelefteq G$, but rather to realize that one must have $\text{Ker}(f) \trianglelefteq G$ in order for $G/\text{Ker}(f)$ to be a group. When students then have to identify the cosets $a\text{Ker}(f), a \in G$, they already have a glimpse of what they are: just the elements of G whose image by f is the same than that of a . Thus, we think that a student might benefit from proceeding from $G/\text{Ker}(f)$ to the general quotient group G/H : both normality and the good definition $\bar{f}: G/H \rightarrow G'$ are easier to deal with if $H = \text{Ker}(f)$, and this case is one step further than the abelian case.

Having shown that our approach to the GIT is suitable, we feel it is appropriate to try teaching the (first) group isomorphism theorem the way we proposed here in order to gather further data. Furthermore, Mena-Lorca (2010) has shown that of the dozen or so group homomorphism theorems mentioned at the beginning, all but one have corresponding set theory theorems at their bases, which suggests that a general (categorical) perspective might be used for treating them. In fact, when proving homomorphism theorems for rings, for example, it is natural to look to the corresponding group homomorphism theorem, observe that the group homomorphism is (well) defined, and then just prove its compatibility with the product. *Mutatis mutandis*, the same is true for group and sets, respectively. This should be further analyzed from a cognitive perspective: going from the general (e.g., sets) to the particular (e.g., groups) might not always be the best strategy. For instance, the GIT is an immediate corollary of the fundamental homomorphism theorem for groups (that is, if $N \trianglelefteq G$, then every group homomorphism $f: G \rightarrow G'$ such that $N \subseteq \text{Ker}(f)$, factors uniquely as $f = \bar{f} \circ \pi$, where $\pi: G \rightarrow G/H, \bar{f}: G/H \rightarrow G'$ are defined by $\pi(a) = aN$ and $\bar{f}(aN) = f(a)$, respectively). Nevertheless, in some cases, such as the GIT, we may just be turning to more familiar settings.

NOTES

1. Research in Undergraduate Mathematics Education Community (1995-2003), founded by Ed Dubinsky.

2. Although they stressed that their teaching preference was to go from the particular to the general, none of them would try first the “fundamental homomorphism for groups”: the universal property for $(G/\text{Ker}(f); p: G \rightarrow G/\text{Ker}(f), p(a) = a\text{Ker}(f))$.

3. We did not know beforehand that Neubrand (1981) offers a mathematically similar approach, although limited to the GIT.

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APPENDIX

Questionnaire

("≤" stands for subgroup, \bar{z} is the equivalence class of z in \mathbf{Z} .)

- Which of the following are true, if any?
 - $\mathbf{Z}_5 \leq \mathbf{Z}$; b. $\mathbf{Z}_5 \leq \mathbf{Z}_{10}$; c. $6\mathbf{Z} \leq 3\mathbf{Z}$; d. $(\{1, -1\}, \cdot) \leq (\mathbf{R}, +)$
- Let $G = \mathbf{Z}$
 - Assume that $6\mathbf{Z} \leq G$, show that $6\mathbf{Z} \trianglelefteq G$.
 - On $G/6\mathbf{Z}$ define $\bar{z} + \bar{z}' = \overline{z + z'}$ and explain why it works correctly.
- Let $G = S_3, H = \{(1), (12)\}, K = \{(1), (123), (132)\}$.
 - Show that it is not true that $K \trianglelefteq G$; b. Examine whether $H \trianglelefteq G$; c. Show that G/K is not a group.
- Give five examples of ER for each:
 - Inside mathematics; b. Outside mathematics
- Explain:
 - What is an ER; b. What is a partition;
 - A result that links ERs and partitions defined over the same set.
- Define, if possible, a homomorphism.
 - From \mathbf{Z}_2 to \mathbf{Z}_6 ; b. From \mathbf{Z}_4 to \mathbf{Z}_2 ; c. From \mathbf{Z}_2 to \mathbf{Z}_3
- Determine whether $S_3 \simeq \mathbf{Z}_6$
- Let D_3 be the group of symmetries of a triangle. Determine whether the subgroup of rotations is isomorphic to some $\mathbf{Z}_n, n \in \mathbf{N}$.

Script 1

- Let G be a group, $H \leq G$.
 - State the GIT.
 - Explain what normal subgroups are needed for.
 - Describe the quotient group G/H .
- State a theorem that links REs and partitions on a given set. Explain.
- Let $f: A \rightarrow C$ a function.
 - Show that f defines an ER R_f on A .
 - State a theorem that relates the quotient-partition A/R_f with another set.
- A. (If i. is achieved)
 - State a relationship between the theorem in 3.ii and the GIT.
 - Explain what is lacking in 3.ii to be the GIT.
 - Try to fill in what is lacking.

- B. (If 3.i is not achieved)
- i. Let R_f be defined on A by $aR_f b: f(a) = f(b)$.
 - a. Show that R_f is an ER.
 - b. What is A/R_f ?
5. Assume now that A is a group.
- A. Explain how does R_f relates to $\text{Ker}(f)$.
 - B. i. Explain the following: $aR_{\text{Ker}(f)} b; a^{-1}b \in \text{Ker}(f); f(a^{-1}b) = e'$;
 $f(a^{-1})f(b) = e'; f(a)^{-1}f(b) = e'; f(b) = f(a)$
 - ii. What conclusion may you extract from i., related to the GIT?
6. Can you distinguish two parts in the GIT?

Script 2

1. State the GIT. Explain and give an example.
2. What does it mean that a subgroup is normal in a group? What is the purpose of defining normal subgroups?
3. State a theorem that links ERs and partitions defined on a set.
4. Let $f: A \rightarrow B$ be a function.
 - a. Show that f defines an ER R_f on A
 - b. State a theorem that relates the partition A/R_f with another set.
 - c. State a relationship between b. and the GIT. Try to fill in what is lacking.
 - d. Show that $xR_f y: f(x) = f(y)$ defines an ER R_f on A . What is R_f ?
 - e. Assume now that $f: A \rightarrow B$ is a group homomorphism. How does R_f relates to $\text{Ker}(f)$?
- f. Recall the ER defined on G by $H \trianglelefteq G$. Explain G/H .

