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Mathematics students' conceptions and reactions to questions concerning the nature of rational and irrational numbers

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ABSTRACT

In this paper, we present parts of an online qualitative research project which lasted a whole semester about the understanding of real numbers by sixty first-year Greek mathematics undergraduate students during the COVID-19 pandemic. We focus on the distinction between rational and irrational numbers and their location on the number line using straightedge and compass. The results, came of questionnaires, focused interviews and tasks, disclosed some gaps regarding real numbers that the students had from school and revealed confusion between irrational numbers and their decimal approximation. The results also led us to group students' conceptions of rational and irrational numbers into five categories related to the number and pattern of decimal digits. Teaching practices and perspectives as well as the, negative, impact of the pandemic on teaching and learning mathematics are discussed.

Keywords: real numbers, number line, geometrical constructions, ICT in mathematics education, categories of conceptions

INTRODUCTION

The notion of number, in general, although seemingly simple, is one of the most fundamental and at the same time most complex mathematical concepts. The history of real numbers' foundation begins in 300 BC with Euclid, who in his *Elements* saves the relevant work of the Pythagoreans, Theaetetus, and Eudoxus, and reaches up to the early 20th century and Dedekind cuts (Klein, 1972). Today, the teaching of real numbers and their subsets constitutes a fundamental part of the mathematical curriculum in elementary and high school as well as a standard section of core subjects (e.g., algebra or real analysis) in many university faculties.

A deep knowledge of the structure of real numbers is a necessary condition for understanding the foundations of mathematics (NCTM, 2006) but also for many basic concepts of calculus such as ε-δ definitions, limits, continuity and derivatives. However, as has long been observed, the majority of both high school and university students face several challenges in understanding the above concepts (Adhikari, 2021; Adiredja, 2021; Artigue, 1997; Bansilal & Mkhwanazi, 2021; Biza et al., 2008; Cornu, 1991; Roh, 2008; Tall, 1992; Tirosh et al., 1998; Vinner, 1991).

To the best of our knowledge there is no any comprehensive study on undergraduate students' thinking about real numbers, which would focus on the identification of rational and irrational numbers in relation to their placement on the line of real numbers, with distance education methods during the COVID-19 pandemic (Pitta-Pantazi et al., 2020; Uegatani et al., 2021).

The main aim of this paper is to explore the conceptions and reactions of first-year Greek mathematics students regarding real numbers and their representation on the number line, using only straightedge and compass, in a calculus course. Considering that the students' understanding of the fundamental concepts of calculus presupposes the knowledge of rational and irrational numbers, this paper aspires to contribute to the existing literature, mainly with the "categories of conceptions", which emerged from the analysis of the research results. In parallel, the effect of the adoption of information and communication technologies (ICT) within the COVID-19 pandemic era is mentioned.

After this short introduction, we review the literature on students' difficulties about rational and irrational numbers and we provide a view of the Greek educational reality. In the next paragraph we clarify our didactical approach and pose our research questions. Then we perform an overview of the research project, we analyze the results of the didactical experiment we conducted by remote methods, closing with the main implication of the study and some comments and suggestions.

THEORETICAL BACKGROUND

Literature Review

Students' difficulties with rational numbers start already in primary school (Siegler & Lortie-Forgues, 2017; Smith et al., 2005; Tian & Siegler, 2018). Some of these difficulties are related to the extension of properties of natural numbers to rational numbers (Vamvakoussi & Vosniadou, 2004) like the existence of the "next" number (Merenluoto & Lehtinen, 2002) or bias connected to the multiplication and division of natural numbers (Christou & Vamvakoussi, 2021).

This fundamental difference between the discreteness of natural numbers and the density of rational numbers creates a lot of misconceptions that follow students in high school (Vamvakoussi & Vosniadou, 2007). For example, they continue to manage the fraction p/q as two independent natural numbers p and q (Siegler & Pyke, 2013) or incline to the additive rather than the multiplicative expression of the relationship between p and q, i.e., they perceive 2/5 as 1/5 + 1/5 rather than $2 \cdot \frac{1}{5}$ (Stafylidou & Vosniadou, 2004). They also consider that the more decimal digits a decimal number has, the larger it is (Moskal & Magone, 2000; Roell et al., 2017) and that a fraction increases when its terms increase (Chinnappan & Forrester, 2014; van Hoof et al., 2013). Roell et al. (2019) claim that the whole number bias is at the root of errors in decimal magnitude comparison tasks.

Particularly for rational and irrational numbers and their properties (representation, ordering, density, and operations), many studies show that both high school and university students encounter various difficulties (Arcavi et al., 1987; Giannakoulias et al., 2007; Kidron, 2016; O' Connor, 2001; Patel & Varma, 2018; Zazkis & Sirotic, 2004, 2010). For example, they find it difficult to define rational and irrational numbers accurately, resorting to incorrect interpretations and models related to their grade (Fischbein et al., 1995) or they base their understanding about irrational numbers on intuition, rather than on formal knowledge (Guven et al., 2011). Besides, the tendency of students to oversimplify problems when they find it difficult to elaborate a subject mathematically by constructing imaginative models and naive interpretations even where they are not applied is well known (Rizos et al., 2017).

But even prospective mathematics teachers who can render the exact definition of irrational numbers and know their basic characteristics, face serious difficulties in handling tasks relative to the different representations of irrational numbers, as well as in placing them on the number line (Peled & Hershkovitz, 1999; Sirotic & Zazkis, 2007; Tall, 2013).

Especially for placing numbers on the number line, the ancient Greek tradition of geometric constructions with straightedge and compass offers students a good opportunity to work with classical methods (Gulikers & Blom, 2001). The value of this historical tradition lies (and) in the plethora of mathematical problems that can be solved with this technique. After all, according to Arcavi et al. (1987), a view that we also share, the historical origin of irrationals in general, and the connections to geometry in particular, can provide an insightful understanding of the concept of irrationality as well as teaching ideas for the introduction of the topic in the classroom.

The Greek Educational Reality

In Greek educational system, the real numbers are taught successively in elementary and secondary education, according to a spiral extension of the basic subsets that compose the set of real numbers (natural, integers, rational, irrational) equipped with operations and relations and their properties. Nevertheless, the teaching and the textbooks have essentially a formalistic character, namely are focused on solving techniques of theoretical exercises (Tzekaki et al., 2003). In this way, however, students–all over the world–acquire fragmentary knowledge (Moseley, 2005) and lose the opportunity to get the feeling of the beauty of mathematics as a fundamental human achievement (Fischbein et al., 1995).

Compared with the teaching of real numbers and, in general, algebra and calculus, the teaching of geometry in the Greek school has been limited and *arithmetized*, meaning it is taught as an axiomatic presentation which presupposes the system of real numbers (Patronis & Thomaidis, 1997). Especially in recent years, geometry has been degraded mainly to comparison of triangles, geometric calculations, and analytic geometry. There is no discussion on historical and cultural contexts that influenced the formation of Euclid's *Elements* and the evolution of geometry up to the analytic geometry of Descartes and Fermat. Of the plane loci, only the circle, the bisector, and the perpendicular bisector appear sporadically in some paragraphs of the textbooks, while the geometric constructions with straightedge and compass are no longer part of the daily school practice.

In contrast to other countries (e.g., Netherlands, Sweden, and Finland) which do not apply control over textbooks and in which the main responsibility for selecting the most appropriate textbook on the market lies with the teachers, in Greece the educational system is highly centralized, there is a national mathematics curriculum and one mathematics textbook for each year (Potari et al., 2019), thus making the official textbook particularly important. At the end of each chapter there is a historical note, which however is not part of the examinable material, so it is usually ignored by both students and teachers. After all, most math teachers have very little historical experience, since there is essentially lack of history of mathematics in pre- and in-service experiences of teachers.

The transition from secondary to tertiary education for Greek students who had taken mathematics as a major subject in high school, in order to study mathematics at university, faces several challenges. The main ones are the increase in the difficulty of the curriculum (e.g., the rigor of the mathematical language), as well as the pursuit for a new identity of studying mathematics (Bampili et al., 2019).

In the first semester of studies, in all mathematics departments of Greek universities, students are taught calculus, usually six hours per week. The basic concepts included in the curriculum are: real numbers and their subsets, sequences, real functions of one variable, limits, continuity, and derivatives. The school knowledge of real numbers is taken for granted and students should be familiar with them in order to face the fundamental calculus concepts (Giannakoulias et al., 2007). Most university calculus books contain short historical notices regarding geometric constructions with straightedge and compass, but experience has shown that few students read them, as is the case with school textbooks.

During the closure of universities and schools in Greece due to the pandemic (March 2020–August 2021), the teaching model that almost all of university teachers followed consisted of live distance lectures using digital communication platforms like MS-Teams and WebEx (synchronous distance learning), in combination with the e-courses management system eClass (asynchronous distance learning). From March 10, 2020, when the first lockdown was announced, universities across Greece have been working to convert their face-to-face classes to online, without, of course, ignoring technical problems, at least in the beginning. Besides, a similar situation prevailed in higher education during lockdown period of pandemic all over the world (Mishra et al., 2020). Thus, distance education using ICT became from a very rare choice to a common practice. In particular ICTs was de facto established as the basic and in many cases the only tool of synchronous and asynchronous teaching and learning (Karalis & Raikou, 2020), saving the interaction, at least in part, between teachers and students.

OUR DIDACTICAL APPROACH

The traditional sequence "definition→theorem→proof→exercise" does not seem to have didactic success neither historical basis. For example, the concept of function, when taught in junior high school (grades 7-9) as a "box" with input and output, gives students the impression that in general mathematics is a "black box" with input and output, which processes numerical data and observations (no matter how it does) and feeds the next similar process. But also in high school (grades 10-12), where only the set-theoretic definition of the function is taught, the students remain uninvolved in the wonderful evolution of the concept of the function from the ancient Babylonians (Neugebauer, 1957) to Euler (Valiron, 1971). Consequently, students cannot appreciate the generality and power of the concept of function. In fact, the set-theoretic definition of this concept does not tell them anything (Sierpinska, 1989).

Our didactical approach combines mathematical with historical components, by taking advantage of an introductory mathematical course (calculus) together with elements from the History of mathematics, namely the ancient Greek tradition of geometric constructions with straightedge and compass, offered to students. In this way historical documents can be integrated in teaching and learning of mathematics (Fried, 2001; Furinghetti, 2020); thus, making the history of mathematics a *tool*, that is, a means of learning the content of Mathematics (Jankvist, 2009; Tzanakis et al., 2000).

We designed the project in order to investigate students' conceptions and strategies on real numbers. At the same time, we wish to involve students in activities that encourage them to expand their notion of mathematics as a human activity, as Tirosh et al. (1998) did, since we consider it crucial for the education. Additionally, we took seriously into account that a historical approach to specific topics from calculus can provide valuable insights to the students and helps to introduce them to the relationship between mathematics and other aspects of our culture (Katz, 1993).

The participating students had mathematics as the main subject in high school and had already been taught the school definitions of *rational* (as a number which can be put in the form p/q, where p, q are integers and q is not zero) and *irrational* (as a real number that is not rational). In addition, they should know from their school experience, already from the 10th grade, that, given the length unit, every number has a position (exactly one point) on the real number line. Nevertheless, one may find it difficult to believe, especially if one has never seen an irrational point located on the number line (Sirotic & Zazkis, 2007). So, locating numbers on the number line might help undergraduate students understand the density of both rational and irrational numbers.

Our research questions are about the way of thinking and the reactions of first-year Greek mathematics students on particular mathematical topics. The questions are interrelated and are focused on real numbers and their geometrical representation. More specifically, we are interested to find out:

1. By what criteria do first-year Greek mathematics students distinguish numbers into rational and irrational, and what thinking strategies do they follow in order to find their exact place on the line of real numbers?
2. To what extent can the contribution of geometric constructions in a calculus course increase the engagement of students?

DESCRIPTION OF THE RESEARCH PROJECT

During the research project we used the communication platform "MS-Teams", the e-course management system "eClass", the online survey creator "MS-Forms", the presentation program "MS-PowerPoint", email and of course a stylus pen and tablet, where we wrote and sketched the necessary figures. In parallel a web camera came in handy when we had to present to students' geometric constructions with straightedge and compass.

In the winter semester of the academic year 2020-2021, about one hundred and ten first-year mathematics students of a university department in Greece, who had taken the calculus course (two-hour lectures three times a week), were invited to participate voluntarily in a project. The invitation was made with an announcement on the eClass platform, as well as verbally during the first online meeting. At this point, it should be mentioned that the specific department is a new institution (founded in 2019), which is located in central Greece and attracts students of all economic and social backgrounds from all over the country. Thus, in a sense, one could say that the sample of our research is representative.

60 students who accepted the invitation were given an e-questionnaire on MS-Forms. This happened in the middle of the first week of the course, that is, after the second meeting, before there was a substantial effect of the university teaching. The students (34 girls and 26 boys 18-19 years old) were of different educational and sociocultural levels and had been taught calculus (arithmetic and geometric sequences, functions of one variable, limits, continuity, derivatives, the rules, and the basic theorems of differentiation) in the 11^{th} and 12^{th} grade for two and five hours per week, respectively. In addition, all students were familiar with technology, had social media profiles, were able to use their mobile phones and tablets easily, and had quickly adapted to the distance learning process (Rizos & Gkrekas, 2022).

The e-questionnaire aimed to reflect students' pre-existing knowledge and perceptions about rational and irrational numbers as they move from secondary to tertiary education and it consisted of six sets of questions (with justification) regarding the representation, ordering, density and operations of real numbers. Similar questions have been asked by Arcavi et al. (1987), Fischbein et al. (1995), Giannakoulias et al. (2007), Mamona (1987), Vamvakousi & Vosniadou (2004), Voskoglou & Kosyvas (2012), Zachariades et al. (2013), and Zazkis & Sirotic (2010). Some typical questions were, as follows:

1. Q1-*What is the ratio of the diagonal (d) of a square to its side (a)?*
2. Q2-*Is the quotient of two rationals always a rational number?*
3. Q3-*Can you find the rational number immediately following the number 2/5?*

A few days later, after studying the answers collected from the e-questionnaires, we called a restricted number of participating students and in addition we conducted focused interviews through the MS-Teams platform, devoting one day to each interviewee.

Taking into consideration all the above, we designed and implemented two two-hour meetings with about a week difference between them, which can be said to have played the role of didactic intervention. This happened halfway through the semester.

In those meetings, we first set a historical and philosophical framework for rational and irrational numbers, starting with the discovery of incommensurability by Hippasus of Metapontum (von Fritz, 1945). We outlined the evolution of the foundation of real numbers from Eudoxus' synthetic definition of proportionality to Dedekind cuts. Integrating primary sources (Jahnke et al., 2000), mainly Euclid's *Elements*, involved the participants into geometric constructions with straightedge and compass and carried out the construction of an equilateral triangle (*Elements*, I.1), the division of a finite straight line into two and three equal parts (*Elements*, I.10 and VI.9, respectively) and the construction of the fourth proportional when three straight lines are given (*Elements*, VI.12). Then, using elementary mathematics we proved (by contradiction) that $\sqrt{2}$ is irrational and located it on the number line using only straightedge and compass. We also referred in brief to the three famous geometric problems of antiquity (circle squaring, cube duplication, and angle trisection), explaining why they are insoluble using only straightedge and compass. In other words, we tried to cultivate a positive attitude in our students towards geometric constructions, thus providing them with an additional tool in supporting the learning of rational and irrational numbers (Helfgott, 2004; Kleiner, 2001). Finally, we encouraged the students to experiment and try to make the same or similar constructions themselves.

Due to the pandemic, our experiment classroom was virtual and consisted of students we had never met up close. It is obvious that in such an environment the teaching model changes radically. Although digital learning in Mathematics education was a positive response to the COVID-19 closure period (Mulenga & Marbán, 2020), the interaction between teacher and students is significantly reduced. This parameter seems to be essential because, no matter if somebody values online teaching or not, one cannot deny that face-to-face interaction is fundamental to any learning that occurs in Mathematics education (Borba, 2021).

The next phase of our experiment was implemented at the end of the semester. With an announcement in eClass and oral information, we invited the 60 students who participated in the project to attend one last hourly meeting. We gave them a task in order to take feedback for our research, namely, to find out what the students acquired during the entire process. The task we set was the following:

For the numbers given as $2/5, 1/3, \pi, \sqrt{5}, \sqrt{2}/2$*, find the exact location on the number line. For any number that is not possible to locate, fully explain the reason. For any number that is possible to locate, indicate the exact location, describing in detail the process*.

The students dealt with the task in their notebooks, which they scanned with proper application and emailed us their answers in .pdf format. In general terms we would say that most students converted the given numbers to decimals in order to place them on the number line. So it seems that unlike square roots and fractions, decimal numbers provide a certainty.

The majority of students do not have a clear criterion as to whether a real number can be placed exactly on the number line and in what exact way; what seemed to be crucial was the number of decimal digits. This issue is detailed in the next paragraph. However, there have been some responses that tend to justify, at least to some extent, our didactical approach. For example, some students combined the geometric construction of $\sqrt{2}$ with the bisection of a linear segment in order to construct the number $\sqrt{2}/2$.

Finally, after the task, we asked the students who participated in the teaching experiment to evaluate the project by writing their thoughts in a paragraph. The texts we have collected undoubtedly show that regardless of the difficulty of the subjects, the students acquired a more positive attitude towards geometric constructions and mathematics generally.

ANALYSIS

During the project, which lasted a full semester, we collected data sets with several methods: e-questionnaire at the beginning of the semester, focused interviews (Merton & Kendall, 1946; Stewart & Shamdasani, 1990) with five students a few days later, a main task and written comments from all the participants towards the end of the semester. We basically followed Denzin's (2009,

Table 1. Q1-What is the ratio of diagonal (d) of a square to its side (a)?

Response	Frequency	Percentage
$\sqrt{2}$	29	48.33
$d = \sqrt{2} \cdot a$	8	13.33
1	7	11.67
2	4	6.67
1/2	3	5.00
Other	7	11.67
No response	2	3.33

Table 2. Q2-Is the quotient of two rationals always a rational number?

Response	Frequency	Percentage
No (counterexample)	18	30
Yes (with explanation)	16	26.67
Yes ("explanation" by an example)	12	20
Yes	6	10
Other	5	8.33
Not always	3	5

p. 310-311) *multiple triangulation strategy*, which encourages several data collection methods at various time periods, and the involvement of investigators with diverse expertise and roles. Particularly for the investigators, they were the first author of the paper, who actively participated in the project by making crucial interventions, and the second author who participated in all phases only by observing the research activities remotely (Burgess, 1984). The validity and reliability of the data are well-documented in the above combined context.

Students' answers to each e-questionnaire and task question were grouped, with minimal verbal changes compared to their original wording, and placed on tables along with their frequency and percentage. There was collaboration and agreement among the researchers in grouping the data, thus making any further statistical control (e.g., inter-rater reliability test) not necessary.

The Questionnaire

Table 1 presents the answers of 60 students about the question Q1 (*What is the ratio of diagonal (d) of a square to its side (a)?*). As expected, students used the Pythagorean Theorem to answer, but only half of them were able to arrive at the correct answer. Eight students while reaching the relation $d = \sqrt{2} \cdot a$, could not display the ratio *d/a*. This, as the interviews showed, is due to two reasons: Some students were unaware of the meaning of the word "ratio", while others were unable to solve a formula for a particular variable. It should also be noted that 21 students came up with results without natural significance. For example, it is not possible for the required ratio to be equal to 1, i.e., the diagonal of a square to be equal to its side!

About the question Q1, students were quite familiar with performing algebraic operations, although they seemed to have difficulty solving a formula for a specific variable. However, this does not mean that they always realized the meaning of these operations and the results they come.

Table 2 presents the answers about the question Q2 (*Is the quotient of two rationals always a rational number?*). This question seemed to be more difficult for the students than the first one, because while more than half of them answered correctly, only 26.67% made a rigorous proof about the quotient of two rationals. The other students either based their claim on an example or relied on what they thought was a "counter-example" to justify that the quotient of two rationals is not rational.

From this question, as it appeared from the progress of the experiment, several misconceptions of the students about the nature of the rational numbers and the operations between them emerged. Here, the well-known issue of the mathematics "proof" arose, which has been extensively studied in other researches (Alibert & Thomas, 2002; Hanna, 2000; Weber, 2001; Williams, 1979).

In question Q2, the attention has to be paid to students who answered that the quotient of two rational numbers may not be rational. One student answered that "*while the numbers 1/3 and 1/6 are rationals, their quotient is equal to 2, which is not*". In other words, it seems that the student considers that *only* fractions are rationals and not numbers that can take the form of a fraction in integer terms. That is, for the student, integers are not rationals. There were a total of 10 responses like this. Eight other students responded that the quotient of two rational numbers may not be rational, invoked repeating decimals as counter-examples. "*1/3=0.333…, and 0.333… is irrational*", a student answered. Therefore, for these students, those decimal numbers that have an infinite number of decimal digits are irrationals.

Table 3 presents the answers about the question Q3 (*Can you find the rational number immediately following the number 2/5?*). This question may have seemed the most difficult for the students, that is, the one related to the density of rational numbers. Only a small percentage, almost 12%, answered that there is no "next" of a rational number. The rest of the students followed a strategy based on the logic of the "next natural number" adding or subtracting unit fractions that can be said to play the role of the "infinitesimal".

The answers to question Q3 are also of great research interest. The response with the highest frequency, i.e., 3/5, resulted from the addition of 2/5+1/5. In other words, the students considered that the unit fraction 1/5 is essentially an elementary fraction, an "infinitesimal", which added to 2/5 leads to the "next" rational number.

Table 3. Q3-Can you find the rational number immediately following the number 2/5?

Response	Frequency	Percentage
3/5	18	30
1/2 (or 2/4, 3/6, 5/10)	16	26.67
There is no "next" number	7	11.67
It is 0.400…1 but it is impossible to find	3	5
7/5	3	5
41/100 or 401/1000	3	5
1	2	3.33
1/6	2	3.33
Other	5	8.33
No response	1	1.67

The second most common response, i.e., 1/2, resulted from the following procedure: Initially the students performed the division and converted 2/5 to a decimal number (0.4). Then, they thought that the next decimal is 0.5 and finally converting 0.5 to a fraction, they were led to 1/2. Thus, to find the "next" fraction, we need the mediation of decimal numbers. However, there were some students who came up with essentially the same result, either by subtracting one unit from the denominator of the original fraction, i.e., $\frac{2}{5-1}=\frac{2}{4}$, or by adding one unit to both the numerator and the denominator of the original fraction, that is $\frac{2+1}{5+1}=\frac{3}{6}$.

Three students added a unit to 2/5, ending up in the fraction 7/5. This reaction, which would be correct if we studied integers, is reminiscent of Polya's (1945) heuristic method according to which when we cannot solve a problem, then we find another problem analogous to the given one.

The Interviews

In this section, we attempt a qualitative analysis of five dialogues we had with an equal number of students (let as call them *Athena*, *Byron*, *Catherine*, *David,* and *Eleanor*) in order to explore their views on the nature of rational and irrational numbers. The questions we asked are based on the answers we received from the e-questionnaire.

Interview I: Researcher (R) and Athena (A)

R: I see here that you used the Pythagorean Theorem…

A: Yes, that's what we did at school. And then I divided by a^2 and I put square roots.

R: Finally, are the magnitudes d and a commensurable or incommensurable?

A: … What do you mean «commensurable or incommensurable»?

R: Is their ratio rational or irrational number?

A: That is, if I divide them?

R: Let's say yes.

A: … Because the diagonal of a square is an irrational number, divisible by the side that is rational, it gives irrational.

R: Does this apply to every square?

A: Yes. I remember [from a textbook] for example that $a=1$ and $d=\sqrt{2}$.

From the above dialogue, it seems that the student ignores the concepts of *commensurability* and *incommensurability*, although this issue was discussed in a previous meeting, while she perceives the meaning of the ratio of two magnitudes as a division of measures of two quantities. In addition, she has formed the impression that the diagonal of the square is an irrational number, while the side is rational, which is obviously due to school exercises, where the lengths of the sides are usually integers.

Interview II: Researcher (R) and Byron (B)

R: The answer you give is "no" and as a justification you write $\frac{\frac{1}{2}}{\frac{1}{4}}=\frac{4}{2}=2$. I would like you to explain it to me a little more.

B: If we take two rational numbers, 1/2 and 1/4 …

R: Why are they rationals?

B: They are fractions.

R: Go on…

B: Yes, and if we divide them, then we get a result of 2, which is not rational.

R: 2 is not rational?

B: No, it is integer. So the result of dividing two rationals is not always a rational number.

R: When would it be rational?

B: If as a result we get say 3/8.

R: If the result, the quotient, was let's say 1.75, would we accept it?

B: It must be a fraction. No, we would not accept that.

Byron still believes that only fractions with integer terms are rational numbers, while integers and decimals (obviously regardless of the number of decimal digits) are something different; they belong to classes disjoint to rationals.

Interview III: Researcher (R) and Catherine (C)

R: You write "No, it is not, e.g., $1/3 = 0.\overline{3}$". Can you explain it to me?

C: $0.\overline{3}$ has infinite decimal digits… It is irrational.

R: What is the fraction 1/3?

C: And this is irrational, because division never ends.

R: The fraction 1/2 what is it?

C: It is rational, because if you do the division, you get a result of 0.5.

R: Is 0.5 rational?

C: Yes, it ends there.

R: While 0.333…?

C: It is irrational. It goes on indefinitely.

Many students believe that the number of decimal places in a decimal number determines whether the number is rational or irrational. Specifically, Catherine converts the given fraction into a decimal number and considers that the number 0.5 is rational because it has a finite number of decimal digits, while the number 0.333… is irrational because it has an infinite number of decimal digits. As we will see below, the number of decimal digits also seems to be related to the ability to locate numbers on the number line.

Interview IV: Researcher (R) and Daniel (D)

R: Is the number 0.111… rational or irrational?

D: It is rational.

R: How do we understand this?

D: Because we know its decimal digits. It's all 1, indefinitely.

R: What is the number 4.127127127… Not all decimal digits are the same.

D: But it is also rational. Its decimal digits are repeated.

R: What about π?

D: It is irrational. It is 3.14… I do not remember how it goes, but we don't know its digits. Never ends. Nor is there a pattern…

R: That is, if there was a pattern… as in the number 1.010010001…, what this number would be?

D: Here… [short pause] there is a pattern. Each time we add a zero. So we know how this number goes… It is rational.

Table 4. About the location of 2/5

Response category	Frequency	Percentage
Exact construction, dividing a unit section into five equal parts using straightedge and compass	2	3.33
Verbal explanation only: "*We divide a unit section into five equal parts using straightedge and compass. Then we place the number 2/5 two sections further away from the beginning*"	4	6.67
Convert the fraction to decimal number (that is 0.4) and approximate placement between 0 and 1	20	33.30
Convert the fraction to decimal number (that is 0.4), calibrate a line using ruler (cm and mm) and placement	13	21.67
Convert all the given numbers to decimals and order them linearly (without other numbers in reference) "$\frac{1}{3}<\frac{2}{5}<\frac{\sqrt{2}}{2}<\sqrt{5}<\pi$"	4	6.67
"*The numerator is less than the denominator, therefore 0<2/5<1*"	2	3.33
"*2/5 is rational, so we can find the exact location on the number line*"	5	8.33
No response	10	16.67

Few students seem to know more than two decimal digits of π, although many books and websites have various mnemonic rules, presumably for this purpose. On the contrary, the students seeing 3.14 spontaneously conclude that it is indeed the number π.

Another issue is that the existence of a pattern in the decimal expansion of a number is a sufficient criterion for some students to be convinced that the number is rational. For example, in the previous dialogue, David describes a Liouville number as rational because he «*knows how this number goes*». Indeed, such a number has a very close rational approximation, but it is irrational; in fact, it is transcendental number (Apostol, 1997), namely number that is not the root of any integer polynomial.

Interview V: Researcher (R) and Eleanor (E)

R: What do you mean by "it is 0.400…1 but we can't find it"?

E: Actually, we can find it. 2/5=0.4 so the next number is 0.400…1, but this cannot practically be stated.

R: Why not?

E: Because we don't know how many zeros there are between 4 and 1. Therefore, we can't express it accurately.

Eleanor, in order to deal with the question, converts the fraction to a decimal number, a strategy that was observed in most students. But what is important here is that she believes that "*we can find the next number but […] we can't express it accurately*". We consider that Eleanor's conception is equivalent to the popular conception among students that the number 0.999… is immediately preceded by the number 1.

The Task

The most important part of the project turned out to be the task, for it gave us the opportunity to go deeper into how our students conceive rational and irrational numbers and in what ways these conceptions are related to the thinking strategies they follow in order to find the exact place of real numbers on the number line. In contrast with Sirotic and Zazkis (2007), we did not give students a line drawn on a Cartesian plane with a visible grid. So, in our project, students had to draw a straight line themselves and carry out the constructions.

We point out that during our didactic intervention we explicitly constructed $\sqrt{2}$, while in dividing a line segment into two and three equal parts we did not directly explain that in this way we can construct the numbers 1/2 and 1/3.

The first activity was about locating the fraction 2/5 on the number line. As it can be seen in **Table 4**, two students succeed in giving a complete answer doing the construction, while four more correctly stated that a unit section should be divided into five equal parts using straightedge and compass, although they did not explain how this would be achieved. Therefore, we can say that six students (10%) approached the correct answer with more or less success.

It was also positive that two more students, even though they gave a rough approximation (0<2/5<1), they managed the fraction correctly, explaining that "*the numerator is less than the denominator*". It should also be noted that five students replied that "*2/5 is rational, so we can find the exact location on the number line*"; thus, stating that rationality implies the exact location.

On the contrary, the majority of students (62%) converted the fraction to a decimal number in order to place it on the number line, either by approximation (between 0 and 1) or by scaling the line using a ruler, while a fairly large percentage of students (about 17%) did not respond at all.

The second activity was about locating the fraction 1/3 on the number line, which although completely similar to the previous one created more difficulties. As it can be seen in **Table 5**, only a small percentage of undergraduate students (almost 8%) approached the activity with some success. The majority of the other students converted 1/3 into a fraction in order to decide which strategy to follow next.

As is clear from the students' responses, what plays a key role in the possibility to locate a number on the number line is the number of decimal digits. Up to two decimal digits mean that the number can be located (using a ruler or between two integers). Infinite decimal digits mean that the number cannot be located. Another conclusion that emerges about the way some students think, a conclusion stronger than that of the previous activity, is that a number can be located in the number line if and only if it is rational.

Table 5. About the location of 1/3

Response category	Frequency	Percentage
Exact construction, dividing a unit section into three equal parts using straightedge and compass	2	3.33
Verbal explanation only: "*We divide a unit section into three equal parts using straightedge and compass. Then we place the number 1/3 one section further away from the beginning*"	3	5
Convert the fraction to repeating decimal (that is 0.333…) and approximate placement between 0 and 1	9	15
Convert the fraction to repeating decimal (that is 0.333…). "*It is not possible to find the exact location due to infinite decimal digits*"	13	21.67
Convert the fraction to terminating decimal (that is 0.3 or 0.33), calibrate a line using ruler (cm and mm) and placement	8	13.33
"*The numerator is less than the denominator, therefore 0<1/3<1*"	2	3.33
Convert all the given numbers to decimals and order them linearly (without other numbers in reference) "$\frac{1}{3}<\frac{2}{5}<\frac{\sqrt{2}}{2}<\sqrt{5}<\pi$"	4	6.67
"*1/3 is irrational, so we can't find the exact location on the number line*"	4	6.67
"*1/3 is repeating decimal, so it hasn't exact location on the number line*"	3	5
"*1/3 is repeating decimal, so it has no place on the number line*"	1	1.67
"*1/3 is rational, so we can find the exact location on the number line*"	1	1.67
No response	10	16.67

Table 6. About the location of π

Response category	Frequency	Percentage
"*π is transcendental so we can't locate it*"	9	15
Decimal approximation using two decimal digits	7	11.67
Approximate placement between 3 and 4	8	13.33
"*It is not possible to find the exact location due to infinite decimal digits*"	11	18.33
"*π is irrational, so we can't find the exact location on the number line*"	9	15
"*π is a mathematical constant, is equal to 3.14*"	2	3.33
"*π is irrational, but it can be written as a decimal number, specifically π=3.14, so it can be placed on the number line*"	1	1.67
"*We can find the exact location on the number line because π=3.14, is rational*"	1	1.67
Convert all the given numbers to decimals and order them linearly (without other numbers in reference) "$\frac{1}{3}<\frac{2}{5}<\frac{\sqrt{2}}{2}<\sqrt{5}<\pi$"	4	6.67
No response	8	13.33

In the next case, see in **Table 6** about the location of π, the percentage of correct responses (π is transcendental so we can not locate it) rose to 15%, perhaps because we mentioned π several times during the teaching intervention and especially regarding the circle squaring. Therefore, these reactions may be partly an automatic recall from memory.

However, the number of decimal digits plays a decisive role in this case as well. Since it is widely known that the decimal approximation of π is the number 3.14, many students (over 25%) assumed that π is equal to or approximately equal to 3.14 so it can be located on the number line, as in the previous cases of 2/5 and 1/3. Another 33% of students answered that it is not possible to find the exact location of π due to infinite decimal digits or due to its irrationality. It is also worth noting that some students perceive certain numbers such as π as "mathematical constants", regardless of the nature and properties of these numbers. We could say that students create a correspondence between the physical constants (e.g. gravitational acceleration or speed of light) and the "mathematical constants" like π, ϕ, or *e*.

Table 7 presents the frequency and percentage of undergraduate students which response about the location of $\sqrt{5}$ on the number line. Five of the sixty participants, maybe inspired by the teaching intervention, "expanded" the construction of $\sqrt{2}$. They draw a straight line, arbitrarily placed 0 and 1, then with the compass placed 2, brought a vertical line to 2, considered a linear segment on the vertical line with length equal to 1 and formed a right triangle with vertical sides 2 and 1 and hypotenuse $\sqrt{5}$. Then with the compass they located the number $\sqrt{5}$ while explaining the application of the Pythagorean Theorem. We consider this geometric approach as an encouraging fact, which gives a perspective for the integration of geometric constructions in the teaching of calculus.

The responses of the other students are about similar to those regarding the location of 2/5, 1/3, and π, and up to a certain point expected, taking into account the results of previous research (Sirotic & Zazkis, 2007). The infinite number of decimal digits, the irrationality of the number, as well as the possibility of rounding the number, play crucial role for the location of $\sqrt{5}$ on the number line. Since several students used a calculator, either conventional or from their mobile phone, to calculate some of the first decimal digits of $\sqrt{5}$, it should come as no surprise that beyond rounding, a student concluded that $\sqrt{5}$ has exactly ten decimal places (those displayed on the calculator screen) and thus the decimal is terminated, "*so it is rational, so we can find the exact location on the number line; however, it is difficult due to the many decimal digits*".

The effect of computational technology is obvious here and confirms, at least to some extent, the existence of the phenomenon of "demathematization" by technology (Gellert & Jablonka, 2009; Keitel, 1989) that is the replacement of thinking and mathematizing processes by technological "black boxes", i.e., programs that simulate mathematical constructions and thus do not require the users to understand their underlying mathematical structures.

Table 7. About the location of $\sqrt{5}$

Response category	Frequency	Percentage
Exact construction, using straightedge and compass and applying the Pythagorean Theorem	5	8.33
"$\mathbf{4 < 5 < 9 \Rightarrow 2 < \sqrt{5} < 3}$" and placement between 2 and 3	5	8.33
"$\sqrt{\mathbf{5}}$ *is irrational, so we can't find the exact location on the number line*"	9	15
"*It is not possible to find the exact location due to infinite decimal digits*"	13	21.67
Convert to decimal (2.2 or 2.24) and approximate placement between 2 and 3	4	6.67
Convert to decimal (2.2 or 2.24), calibrate a line using ruler (cm and mm) and placement	10	16.67
"*The number* $\sqrt{\mathbf{5}} = \mathbf{2.2360679775}$ *has a finite number of decimal digits, so it is rational. We can find the exact location on the number line, however it is difficult due to the many decimal digits*"	1	1.67
Convert all the given numbers to decimals and order them linearly (without other numbers in reference) "$\frac{1}{3} < \frac{2}{5} < \frac{\sqrt{2}}{2} < \sqrt{5} < \boldsymbol{\pi}$"	4	6.67
No response	9	15

Table 8. About the location of $\sqrt{2}/2$

Response category	Frequency	Percentage
Exact, using straightedge and compass and applying the Pythagorean Theorem	10	16.67
"$\mathbf{1 < 2 < 4 \Rightarrow \frac{1}{2} < \frac{\sqrt{2}}{2} < 1}$" and placement between 0 and 1	5	8.33
"$\frac{\sqrt{2}}{2}$ *is irrational, so we can not find the exact location on the number line*"	6	10
"*It is not possible to find the exact location due to infinite decimal digits*"	12	20
Convert to decimal (0.7) and approximate placement between 0 and 1	5	8.33
Convert to decimal (0.7), calibrate a line using ruler (cm and mm) and placement	8	13.33
"*In general, irrational numbers cannot be placed on the number line, but exceptionally* $\frac{\sqrt{2}}{2}$ *can*"	1	1.67
Convert all the given numbers to decimals and order them linearly (without other numbers in reference) "$\frac{1}{3} < \frac{2}{5} < \frac{\sqrt{2}}{2} < \sqrt{5} < \boldsymbol{\pi}$"	4	6.67
No response	9	15

Table 8 presents the frequency and percentage of undergraduate students which response about the location of $\sqrt{2}/2$ on the number line. Two strategies were observed for this construction; in the first one the students constructed $\sqrt{2}$ exactly as we had done in our teaching intervention and then they brought the perpendicular bisector. In the second case a student draws a square of side 1 and brought the diagonals so that four linear segments were appeared. Then she proved that each such segment is equal to $\sqrt{2}/2$.

In this question, however, the numerical approaches continued to be the most often observed, indicating once again that it is quite difficult for students to break free from the numerical/ computational way of thinking.

Interesting is the response of a student who wrote that while "*irrational numbers cannot be placed on the number line,* $\sqrt{2}/2$ *is an exception*". Although the influence of teaching is obvious here, this answer could indicate a spontaneous intuition of constructible real numbers.

From the analysis of the above results, it seems relevant to group the students' conceptions of rational and irrational numbers into five categories:

1. **C1**: Rational numbers are the integers or the quotients of exact divisions, while irrational numbers are the decimals in general and the fractions (quotients of divisions that leave a remainder).
2. **C2**: Rational numbers are only fractions with integer terms. Integers are not rationals; they belong to a class disjoint to rationals, and the same happens with decimals in general. Irrational numbers are the square roots of non-square numbers.
3. **C3**: Rational numbers are the decimals which have a finite number of decimal digits and fractions that are converted to such decimals. Irrational numbers are the decimals which have an infinite number of decimal digits and fractions that are converted to such decimals.
4. **C4**: The decimal numbers that have a finite number of decimal digits (and fractions converted to such decimals) are rational. But if a decimal number does not end, in order to be rational it is necessary and sufficient to have a pattern, otherwise it is irrational.
5. **C5**: The exact formal definitions: Rational is a number which can be put in the form p/q, where p and q are integers and q is not zero and irrational is a real number that is not rational.

The above *categories of conceptions* are not "levels of real number understanding", such as van Hiele's (1957) levels of geometry understanding. The fundamental difference is that at the levels of geometric thought (van Hiele, 1957; van Hiele & van Hiele-Geldof, 1958) a student starts from the lower level and progresses to the higher levels, while in our empirical model there is no element of succession. For us, a student can be led directly to the fifth category without going through the others.

However, as the analysis of the project indicates, even if a student belongs to the fifth category, it does not necessarily mean that they are able to locate a real number (if it is located) on the number line.

Students' Comments About the Project

In order to find out the students' opinions about the project and to take it into account in a possible future study, we asked them to evaluate the whole process.

From the answers we received it emerged that the students felt quite comfortable during the project, developed positive emotions, recognized the value of the material they involved, felt that they learned new things about real numbers, and acquired a positive attitude towards geometric constructions with straightedge and compass. In fact, some students have recognized that there is still much to be done in order to clarify some basic mathematical concepts and procedures.

Of particular value is the comments of a student, who stated that «at school I generally listened to geometry and history and subjects like that and I did not like them at all, but here [in the project] we followed a completely different approach. Although at the beginning I had some reservations, in the end I "kept" a lot of things […] We discussed the evolution of mathematics, we met great mathematicians, we learned about the relationship between geometry and rational and irrational numbers, and we made geometric constructions, which I experienced for the first time. I believe that if the meetings were done face-to-face, they would have to offer us more things». Indirectly this student responds partly to the objections raised against the integration of elements from the History of Mathematics in teaching and specifically to those objections concerning the background and attitude of the learners (Tzanakis et al., 2000, p. 203). Also, according to the student, distance learning is a hindrance to learning Mathematics.

All of the above give a positive answer to the second research question. Therefore, the contribution of geometric constructions in a calculus course can increase students' engagement.

DISCUSSION

Our research aimed to investigate students' conceptions of rational and irrational numbers in the transition of students from secondary to tertiary education. Then with our teaching intervention we wanted to find out what strategies do the students follow in order to locate a real number on the number line. In addition, we were interested whether geometric constructions in a distance learning calculus course could increase the involvement of students.

The Pythagorean Theorem, which all undergraduate students used in the first question (Q1), is taught in the Greek educational system in the 8^{th} grade and is repeated in the 11^{th}, while students know in advance that in this section they will be assessed in tests and final exams. It should also be noted that the Pythagorean Theorem is a special mathematical topic in popular culture and a source of inspiration for many art forms, from statues and poems to films (*The Wizard of Oz*), T-shirts and stamps (Greenwald, 2016; Pantsar, 2016). It is therefore not surprising that this theorem is the first thing that comes to students' minds when they come to geometric problems. Nevertheless, students' difficulties on display the requested ratio and understand its irrationality are remarkable. Some participants were unaware of the meaning of the word "ratio", while others were unable to solve a formula for a particular variable. Also, some students (in Q1) came up with results without natural significance, which indicates that critical competence is considered as a resource to be developed further through students' participation in the educational process (Skovsmose, 1994), in elementary education.

The task at the end of the project gave very interesting information. As it turned out, for most undergraduate students *irrational* number means *indeterminate* due to infinite decimal digits (that's why many students considered that the fraction 1/3 is an irrational number), in the sense that *irrationality* is equivalent to *ignorance*, while in terms of placement of numbers, our results made clear that the dominant strategy of the students was to convert the given numbers into decimals, round them and then place them on the number line, ignoring the geometric methods taught during the project. This finding is completely in keeping with a discovery of an earlier study, according to which irrationality relies upon decimals and not connected to geometry (commensurable and incommensurable segments) as occurred historically (Arcavi et al., 1987, p. 18). This important, for the students, role of decimal digits, became in fact the determining factor that led us to group students' perceptions regarding rational and irrational numbers into five categories. So the answer to our first research question is that the criterion by which students distinguish rational from irrational numbers and place them on the number line is the number of decimal digits and the existence (or not) of some pattern.

The five *categories of conceptions* we considered are not "levels of real number understanding" such as van Hiele's (1957) levels of geometry thought. We should also emphasize that the above categories are not related to Sierpinska's (2005) model of theoretical thinking, nor to Zachariades et al. (2013) thinking categories that can be exploited in the analysis of prospective teachers' reasoning and knowledge of real numbers. Our model is empirical, in the sense that it emerged from the analysis of specific results. It would therefore be interesting to conduct future research aimed at investigating whether similar results could be drawn from teaching experiments with younger students.

From the analysis of the questionnaires and the interviews, some gaps that the participants have from school came to surface. For example, students viewed fractions as a *process*, not as a numerical *result*, while they saw decimal numbers as "the final number". It is possible that the above conceptions are based on abridged interpretations of elementary school teachers or some high school teachers but also on unclear expressions of textbooks. On the other hand, students are accustomed to using and trusting–usually uncritically–measuring instruments such as scales, calculators, rulers, protractors, etc. whose results are decimals. But also, daily economic transactions are made based on decimal numbers and very rarely on fractions. It is therefore difficult to weaken the effect of decimal numbers and to make clear the equivalence between the different forms of a number.

In some cases, they concerned the placement of the given numbers on the number line, students gave "automatic" answers like the spontaneous conversion of all numbers to decimals. This "automation", in combination with the fact that the course of geometry is degraded in the Greek high school, shows the difficulties that students face with problems that involve geometry. Thus, it seems very difficult for students to overcome the algebraic way of thinking and to get rid of the "confidence" provided by the *formalistic character* of the teaching and learning of Mathematics in Greece (that is blind memorization of algebraic rules, arithmetic operations and solving techniques without conceptual content and without any geometric interpretation), with which children have been familiar with since the early secondary education. At the same time, this students' practice demonstrates the "algebraization" of geometry, i.e., the arbitrary algebraic reformulation of a geometric problem and the consequent loss of meaning of the mathematical concepts involved. This conclusion is in line with recent relevant research finding (Rizos et al., 2021).

Distance education using ICT, as demonstrated by the results of the task but also according to the students' evaluation in the project, was a limiting factor to the learning of geometric constructions with straightedge and compass and consequently made it difficult for the students to grasp the meaning of irrationality and become able to distinct rational and irrational numbers, as well as to locate their position on the number line. After all, research converges that it finally seems almost impossible to teach geometry online without proper interaction (Borba, 2021). In this sense the pandemic had negative impact on the teaching and learning of mathematics. We believe sharing the view of the student that if our teaching intervention had been conducted face-to-face in a real classroom, then the results would be better since in these times, it is visible that human contact and interactivity have faded, because of the exclusive use of ICTs. Of course, new research remains to be carried out in order to confirm (or reject) this conjecture.

In the didactical recommendations of a similar research, Fischbein et al. (1995) suggest the students to face the possible incommensurability of two magnitudes, namely, to be involved in the process of searching a common unit. We believe that such a teaching intervention could be the exploration of *geometric* anthyphairesis of $\sqrt{2}$ (Chrystal, 1889, p. 270). It would be interesting to conduct such research, in order to determine whether a historically inspired approach (like geometric anthyphairesis) could help students find out if a number is rational or irrational.

CONCLUDING REMARKS

What emerges from our research is that the fundamental criterion by which Greek undergraduate students distinguish rational from irrational numbers is the number of decimal digits and the existence of some pattern, if any. Furthermore, the most common thinking strategy they followed in order to find the exact place of a given real number on the number line was the conversion into decimal, the rounding and finally the location. At the same time however, the contribution of geometric constructions with straightedge and compass in a calculus course appears to increase students' engagement. Our findings could be universally significant. After all, research converges that it finally seems almost impossible to utilize geometry online without proper interaction. In this sense the pandemic had negative impact on the teaching and learning of mathematics. Although the students recognized the value of the material they involved, they felt that they gained knowledge about real numbers and acquired a positive attitude towards geometric constructions with straightedge and compass, we believe, sharing their opinions that, if our teaching intervention had been conducted face-to-face in a real classroom, then the results would be better, since in these times it is common knowledge that human contact and interactivity have faded, because of the exclusive use of ICTs. Of course, new research remains to be carried out in order to confirm (or reject) this conjecture

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