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# Mathematical proof and epistemological obstacles: Assumptions of the methodological teaching proposal of the Fedathi sequence

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**ABSTRACT**

The Multimedia Research Laboratory, inserted in the Federal University of Ceará has the Multimedia Mathematics Education Group, in which they study the mathematics teaching with the methodological contribution of the Fedathi sequence (FS). In one of the study meetings, the members raised the issue of how a class in the light of the FS could be, which would address mathematical proof. To this end, the mathematical object used to illustrate the answer to this question was the ordering in the set of rational numbers, aiming at the didactic session in the mathematics undergraduate course. Thus, this research aimed to develop a model of mathematical proof approach based on the FS methodological proposal, attempting to solve the issue raised. To make this work possible, the FS research methodology was used, with a qualitative bias. Regarding the results, it was possible to develop the model with the purpose of performing mathematical proof using the FS as a teaching methodology, considering that this way of mediating knowledge is not a transmission of content, but rather, a way of providing learning.

**Keywords:** rational numbers, ordered field, teaching methodology

## INTRODUCTION

This study aims to provide an explanation about the fundamentals and steps of the Fedathi sequence (FS) in relation to proof in mathematics teaching. The FS seeks to give meaning to the teaching of mathematics, providing the student a learning by means of investigation and with ideas arising from mathematical thinking (Menezes, 2018). With discussions targeting the teaching practices of the mathematics teacher, how can a class that is based on the proof of a theorem be conducted? To this end, the mathematical object used to illustrate the answer to this question was the ordering in the set of rational numbers, aiming at the didactic session in the mathematics undergraduate course. Faced with the problem, one noted that Fontenele (2018) carried out a survey of works that guide the teaching of mathematics in higher education that used the FS as a teaching methodology, as shown in **Table 1**.

These papers comment on the importance of proof in mathematics classes at the higher education level, but there was no detail on how this could be done in class. One believes that there was no such approach because it escapes the research objective of each one presented in **Table 1**. In addition to these works in **Table 1**, there are the doctoral theses of Fontenele (2018) and Menezes (2018). Entitled "Contribuições da sequência Fedathi para o desenvolvimento do pensamento matemático avançado: Uma análise da mediação docente em aulas de álgebra linear [Contributions of the Fedathi sequence to the development of advanced mathematical thought: An analysis of teaching mediation in linear algebra classes]", Fontenele (2018) states that he did not perform mathematical proof in the didactic sessions contained in the thesis, characterizing a limitation of his research.

Addressing related rates, in differential and integral calculus, Menezes (2018), in turn, with the thesis "O ensino de cálculo diferencial e integral na perspectiva da sequência Fedathi: Caracterização do comportamento de um bom professor [The teaching of differential and integral calculus from the Fedathi sequence perspective: Characterization of a good teacher's behavior]", carried out mediations to solve problems concerning the content of derivatives. However, Menezes (2018) does not mention mathematical proof in his work.

From the study of **Table 1** and the works of Fontenele (2018) and Menezes (2018), one finds that there are no published works with the content of proof of theorems (or propositions) mediated by the FS teaching methodology. With this, the study is justified by the absence of works that relate mathematical proof and FS; thus, this article consists of a brief explanation, of a theoretical nature, in common with these themes, resulting in a proposal for a didactic session designed by means of a generalizable situation.

**Table 1.** State of the art: Fedathi sequence and mathematics in higher education (Fontenele, 2018, p. 30-31)

| Author/year | Title | General objective |
|---|---|---|
| Barroso (2009) Thesis | A teaching model of calculation concepts for engineering courses based on a historical epistemology and based on didactic engineering methodology: Validation through the concept of integral | To propose a model for the introduction of (differential and integral) calculus key concepts in the classroom, recalling ideas that contributed to the formation of these concepts, both as a form of motivation for its learning and as a link between students' old and new knowledge. |
| Souza (2010) Thesis | Applications of Fedathi sequence in geometry teaching and learning mediated by digital technologies | To analyze influences of the Fedathi sequence in geometry teaching and learning, with *Cabri-Géomètre* software. |
| Alves (2011) Thesis | Applications of the Fedathi sequence in the promotion of the categories of intuitive reasoning in calculus to several variables | To describe/identify the categories of intuitive reasoning along the levels of the Fedathi sequence. |
| Fontenele (2013) Dissertation | The Fedathi sequence in the teaching of linear algebra: The case of the basis notion of a vector space | To verify if the use of the Fedathi sequence, in classes about the basis concept, provides resources that can become a leverage goal for the students. |
| Moreira (2014) Dissertation | Analysis of the professor-tutor's view on the appropriateness of mathematics didactic material in the light of the Fedathi sequence: The case of mathematics licentiate of the Federal Institute of Education, Science and Technology of Ceará (IFCE) | To analyze the vision of the professor-tutor about the adequacy of mathematics didactic material in the semi-presential modality of higher education. |
| Nasserala (2014) Dissertation | Elaboration and description of didactic situations with support in the Fedathi sequence: The case of the improper integral | To describe didactic situations with the help of *GeoGebra* software, supported by the Fedathi sequence on improper integrals with emphasis on visualization. |
| Bezerra (2015) Dissertation | Proposal of an approach for integration techniques using *Geogebra* software | To structure and propose teaching situations related to integration techniques, exploring graphic-geometric patterns related to integrated functions and their primitives, using *Geogebra* software. |
| Macedo (2015) Dissertation | Geometric manifestation of the indeterminate forms of functions: Didactic situations supported by technology | To present a teaching proposal for the indeterminate forms and the L'Hôspital rule by means of didactic sequences structured based on the Fedathi sequence and the exploitation of the *Geogebra* software. |

**Table 2.** State of the art for mathematical proof articles (Elaborated by the authors)

| Journals | Quantity |
|---|---|
| BOLEMA | 2 |
| Educational Studies in Mathematics | 102 |
| Journal For Research in Mathematics Education | 17 |
| **Total** | **121** |

On the other hand, a state of the art was carried out concerning works that guided the mathematical proof in higher education, which had as a parameter for the initial search the Sucupira Platform, which indicates the qualis[1] of academic journals. For that, in the "evento de classificação" [classification event] bar was assigned the value "classificações de periódicos quadriênio 2013-2016" [classifications of periodicals 2013-2016], with "área de avaliação" [evaluation area] in "educação" [education]. To delimit these journals in mathematics and that they are of qualis A1, in the "título" [title] the keyword "mat" has been inserted, with the purpose of covering both the Portuguese language and English, for the term "matemática" (or in English, "mathematics").

These criteria were thus chosen for the reasons of the MM being integrated into FACED and collaborating in research in post-graduate. And since the faculty offers master's and doctorate courses in the education area, it was considered necessary this delimitation.

Another factor was the "qualis", which the best evaluated, for CAPES[2], is the A1. And the work refers to the area of research mathematical education, so a term was placed in the bar of "title" in order to limit the search to this field of study.

Thus, the platform presented three journals and was approached in each one of them, in its search bar, the keyword "demonstração" [proof] (in journals of predominance in the Portuguese language) and "proof" (in journals of predominance in the English language), between the years 2016 and 2020 (survey conducted on May 20, 2020). The result of the sample of journals and the number of articles can be seen in **Table 2**.

It is noted the high quantity for the search performed, and in order to improve the selection criteria of the works, there was the reading of abstracts of the 121 works found. After the analysis, using as a parameter the approach in mathematical proof classes in higher education, three articles were selected, which follow in **Table 3**.

These three works were selected because they explicitly and objectively approach mathematical proof in classes for higher education. With this, it is possible to observe how mathematical proof is worked in the classroom, and thus, to observe the possibilities that the teacher may have if he/she uses the FS in the classroom. The work of Lockwood et al. (2020) presents types of representation systems that teachers and students can address in mathematical proof. However, the work lacks a teaching methodology to work on "how to prove in the classroom". With presentations of theorems and their respective proof, the research of Azrou and Khelladi (2019) do not show how the teacher can come to mediate the mathematical proof for the student.

[1] Brazilian journal ranking system.
[2] Coordenação de aperfeiçoamento de pessoal de nível superior [Coordination for the improvement of higher personel].

**Table 3.** Titles of the selected articles (Elaborated by the authors)

| Journal | Title | Authors |
|---|---|---|
| Educational Studies in Mathematics | An essay on proof, conviction, and explanation: Multiple representation systems in combinatorics | Lockwood et al. (2020) |
| Educational Studies in Mathematics | Why do students write poor proof texts? A case study on undergraduates' proof writing | Azrou (2019) |
| Journal for Research in Mathematics Education | Lectures in Aavanced mathematics: Why students might not understand what the mathematics professor is trying to convey | Lew et al. (2016) |

On the other hand, Lew et al. (2016) point to future researches, studies that deal with the "how" the student may come to understand what the teacher says at the time of a proof, as well as the comments to interact, redefine and reorganize their mathematical thoughts. To this end, the authors suggest the use of a methodology that seeks to solve these problems. In addition, Doruk (2019) found that undergraduates, idealization of the target audience of this theoretical study, have difficulties in proofs by contradiction and proofs by contraposition.

In the face of these gaps presented in these works that guide the mathematical proof in teaching, one has the following guiding question: Is it possible to use the FS as the teaching methodology in a teaching session led by the mathematical proof of theorems or propositions? Therefore, the objective is to develop a model of approach to mathematical demonstration based on the SF methodological proposal. In addition, in view of Doruk's (2019) gap, this work presents an exercise that addresses proof by contradiction. In addition, Doruk (2019) found that undergraduates, idealization of the target audience of this theoretical study, have difficulties in proofs by contradiction and proofs by contraposition.

For this, the intention is to develop a model of mathematical proof approach based on the FS methodological proposal. The choice of the verb "develop" is attributed by the Bloom taxonomy, which, in the application category, has as description

> "ability to use information, methods and contents learned in new concrete situations. This may include applications of rules, methods, models, concepts, principles, laws and theories" (Ferraz & Belhot, 2010, p. 426).

This paper is then divided into the sessions of introduction, theoretical basis, research methodology, mediation of mathematical knowledge, and the final considerations. Regarding the methodological processes, the FS is used as a research methodology to design the investigative actions and the didactic session proposed in this work aims to use the FS in a class that treats the proof of a proposition.

## THEORETICAL BASIS

To provide guidance on how to carry out the proof, this section focuses on the presentation of the FS teaching methodology as well as on what the previous works (**Table 3**) have dealt with this theme and how this methodological proposal can contribute to filling these gaps found in the aforementioned literature.

### Fedathi Sequence: Teaching Methodology

In **Table 1**, one can see a chain of research that occurs about the FS, in which is the teaching in higher education. According to Borges Neto (2016), the FS can be defined as the scientific method transposed to the teaching environment.

Regarding the scientific method, Descartes (1983) presents it with four stages: clarity, analysis, synthesis, and listing (or revising), and these stages start from doubt, from questions. To understand the statement of Borges Neto (2016), in which he refers to the FS being the scientific method transposed to teaching, this brief cut by Descartes (1983) clarifies one of the essences of the FS, which is to follow the steps of the scientific method.

So, there is equivalence in the FS with the scientific method as far as the stages are concerned, because this teaching methodology also has four stages, and they are called positioning, maturation, solution, and proof. In relation to Descartes (1983), one must attribute the course of the method, starting from questions, and it can be seen in the FS. This visualization of questions is given, for example, in the presentation of a problem that is generalizable for the student.

However, the FS is not just about going through these stages during the teaching session, as it works on how the teacher can come to provide his teaching in class. This way of conducting the class is guided by the fundamentals of the methodology.

These steps of the FS can be defined as follows: the positioning, which is the moment the teacher presents a problem that is generalizable; the maturation, which is the student's action in developing hypotheses, conjecturing ideas on how to solve the problem; when the student writes a solution and presents it, it is characterized as a solution; and finally, the proof stage, in which the teacher systematizes the solution and formalizes the content that was behind the problem and which is the subject of the teaching session.

### Mathematical Proof

As an example of the development of a mathematical proof in the light of the FS methodological proposal, an exercise (proposition) which addresses the order relation in the set of rational numbers was resolved.

Regarding the purpose of making proof, Polya (1973, p. 71) states that the justification for proving something in mathematics is to

> "[...] satisfy himself that his proposition is true [...]."

Thus, the way to test the veracity of a mathematical affirmation is the proof. If it is proved, the result is valid. If it is refuted, it is "forgotten", and if it cannot reach a conclusion (if it is true or false), it is called conjecture (it can be proved or disproved).

With regard to learning, the proof helps in the construction and verification of ideas. Such thoughts can be transposed to the resolution of exercises, based on the understanding of how the mathematical tool works or also on what pattern occurs in the mathematical field under study.

For a justification on how to approach classroom proof, Polya (1973, p. 71) is again used, who stresses that

> "it is not so desirable that the teacher should present many proofs in the pure Euclidean manner, although the Euclidean presentation may be very useful after a discussion in which, [...] the students guided by the teacher discover the main idea of the solution as independently as possible."

Thus, Polya (1973) has a concern that the student should not be passive and that he should follow his path of discovery through the guidance of the teacher. With regard to proof, one notes the interest in not only presenting this task in a finished manner, but also in providing an environment in which the student discovers by his own means how to carry out this action.

The idea of not having a passive student in the classroom is also shared by FS, since the teacher mediates and the student "goes out of his way"[3] to solve the problem, starting from his own ideas, doubts and mistakes in building the solution.

The need to approach proof in classroom is seen by Polya (1973, p. 217), when he makes the following reflection:

> "[...] if general education intends to bestow on the student the ideas of intuitive evidence and logical reasoning, it must reserve a place for geometric proofs."

In this case, the author bases himself on the logic that proof show and that this is seen in the geometry study. However, in the development of the mathematical proof based here, besides considering this logic in the arguments, the justification is also generated jointly with the FS, considering that this teaching methodology works with generalizable problems, raising the reasoning used in the proof and in the solution of several other problems.

However, one must be careful about the difference between generalizable and generalization. The generalizable is something, whose solution has an idea that can be transposed to another set of related problems or that need such a mathematical artifice. The generalization is the content formalization, as an example, a theorem, a motto, a definition.

Conversely, and that should be taken into consideration in mathematics teaching when proving results, the teacher should provide didactic time for the student to

(i) try to prove a theorem before reading the demonstration,

(ii) identify the structure of the proof being used,

(iii) divide the proof into parts,

(iv) illustrate statements of the proof with examples, and

(v) compare the method used in the proof with the approach itself.

These points are listed by Weber (2015) and are embraced in this theoretical proposal for use in classroom, mediated by FS.

In Mowahed et al. (2020), the authors attribute these considerations of Weber (2015) and conclude that these strategies for understanding mathematical proof helped students in their studies in modern algebra. Another point to highlight from the work of Mowahed et al. (2020) is the fact that the authors stress the importance of mathematics education to be concerned with the approach to mathematical proof in the classroom, in which the focus should be on making the subject matter meaningful to the student, and understandable. Considering this, one expects that the proposal presented here, following the FS fundamentals, when applied in the classroom, contributes to the students' understanding of the mathematical theme addressed.

## RESEARCH METHODOLOGY

This session is based on the methodological description of the work (application of the FS as a research methodology) and the procedures of the model developed for mathematical proof in the light of the FS teaching methodology.

### Fedathi Sequence: Research Methodology

This research is qualitative, with a design that aims to explain with an example an idealization of the Fedathian act in mathematics classes in which the teaching of proof is necessary, starting from a generalizable situation, as foreseen in FS.

To produce this work, the FS research methodology was used, which is divided into the stages of problem, modeling, validation, and results. For Menezes (2018, p. 25-26), this methodology represents

[3] "In Portuguese: "colocar a mão na massa". Term used in the Multimedia Research Laboratory for building knowledge, where the student is active in his/her learning, where he/she learns by doing, making mistakes, getting it right" (Xavier, 2020, p. 40).

Show that Q is not well-ordered, that is, there are in Q non-empty subsets, lower bounded that have no minimum element.

**Figure 1.** Problem situation (Ferreira, 2013, p. 63)

> "[...] a scientific research methodology with support in the expansion of studies and reflection on the Fedathi Sequence and, also, the need to create in the Multimedia Laboratory of the Faculty of Education of the Federal University of Ceará a research identity, both for the students and advisors."

In his doctoral thesis, Menezes (2018) uses the FS research methodology as a basis for the construction of his methodological procedures, besides aligning this concept with participant research.

Another work using this methodological proposal was the one by Felício et al. (2020). The authors present a proposal for teacher training in multi-area, with theoretical input from the FS, as a teaching and research methodology.

The application of this research methodology was also based on Barbosa (2020) and Xavier (2020). The authors, in their respective master's dissertations, addressed the first two stages (problem and modeling) to model a digital educational object, the raízes [roots].

In this work, the justification is based on an open field in the literature on the "how" to combine a teaching methodology with the "mathematical proof" of theorems and propositions in the classroom (Lew et al., 2016). The act of developing a mathematical proof in the light of a teaching methodology aims to provide an opportunity for applications in classes for Mathematics Licentiate, as well as in the bachelor of mathematics.

It is common sense in mathematics courses, the difficulty that students have in understanding, as well as in producing the mathematical proof. However, this activity intrinsic to the mathematician is the validation criterion for results in mathematics.

In order to elucidate the use of the FS as a teaching methodology, intervening in the teaching of mathematical proof, an exercise contained in Ferreira (2013) in which the set of rational numbers is reported as "not being well ordered" is addressed. The proposition was chosen because it has a possible applicability in basic education and for being a theme studied in mathematics higher education, which are the properties of numerical sets.

To develop the teaching session model, it was necessary to choose the mathematical proposition to be proved, as well as the act of doing the proof. After making the proof, understanding the steps to be used for the validation of the proposition, the didactic processing was carried out. In other words, one thought was how to carry out the teaching of this proof in the light of the FS teaching methodology proposal.

As the work is theoretical, that is, a development of mathematical proof from the use of the FS, the modeling stage consisted in verifying if there was the insertion of the FS assumptions, referring to teaching, in dealing with the modeling of the didactic session. This concern also occurred in the works analyzed in relation to the mathematical proof, where there was no use of a teaching methodology.

The last stage, results, is the writing of the work, since Menezes (2018) suggests that in this stage should be carried out the analysis of the results, which is done in the final considerations of this article, and in the dissemination of the results obtained in the research. For this purpose, the analysis will be done by using the assumptions of the FS in dealing with teaching, that is, if the use of the fundamentals of the methodology was predicted, and in the search for an understanding of when the class is seen in the steps of positioning, maturation, solution, and proof.

### Development of Didactic Session in Action with the Teaching Methodology

The FS, in terms of teaching methodology, is concerned with how the teacher will act in the classroom, both in dealing with the students and in didactically addressing the content studied. Regarding the content, the Fedathian teacher must have his attention focused on understanding the subject well and have the idea that the activity in the classroom should be a generalizable situation. In dealing with the student, a priori, the teacher should be aware of his role as mediator.

Regarding the planning of a didactic session based on the FS, it must have the didactic objective for which the generalizable activity will be based, that is, which mathematical object (skill) will be the focus of the activity. So, for this teaching model, a mathematical proposition should be verified, which will guide the ordering relation in the set of rational numbers.

The positioning will be a proposition contained in Ferreira (2013), which addresses in its fundament the fact that the set of rational numbers is not well ordered, in accordance with **Figure 1**.

The need for positioning to be generalizable is covered in this example of a proof addressed in the work discussed, since, according to Davis and Hersh (1981, p. 147),

> "what makes mathematical proof more than mere pedantry is its application in situations, where the statements made are far less transparent?"

With that, the proof presented here has as a generalizable situation the well-ordering principle (WOP)[4], as this tool can be used for proofs of the properties of the order relationship between natural numbers (Lima, 2013a).

[4] "Every non-empty subset $A \subset \mathbb{N}$ has a least element" (Lima, 2013b, p. 39).

**Table 4.** Essential prerequisites for the teaching session

| Content | Definition |
|---|---|
| Bounded set | A set is limited if it is both upper and lower bound (Muniz Neto, 2013). |
| WOP | Every non-empty subset of the natural numbers set has a least element (Ferreira, 2013). |
| Ordered field | An ordered field is a K field, in which a subset has been highlighted $P \subset K$, called the set of positive elements of K, such that the following conditions are met: P1. The sum and the product of positive elements are positive. That is, $x, y \in P \Rightarrow x + y \in P$ and $x \cdot y \in P$. P2. Given $x \in K$, exactly one of the following three alternatives occurs: or $x = 0$, or $x \in P$ or $-x \in P$ (Lima, 2013b, p. 65). |
| Order relation in the set of rational numbers | Let r and s be rationals; we say that r is strictly smaller than r (or that s is strictly greater than r) and write $r<s$ (respectively $r>s$) if there is a strictly positive rational t such that $s=r+t$ (Guidorizzi, 2001, p. 2). |

The presentation of the problem to the class takes place at a moment after the teacher's plateau. For the FS, this stage is an action in which the teacher seeks the leveling among the prerequisites and previous mathematical knowledge needed by the students to solve the activity that will guide the teaching session. For the proof of the proposition presented in **Figure 1**, the prerequisites are the definitions presented in **Table 4**.

These prerequisites for the realization of the activities were defined through the formulation of the hypotheses of the problem, since they are the premises for achieving the "new" result. As the problem asks to show that the set of rational numbers is not well-ordered, one has to understand what an ordered set is.

Another hypothesis of the problem is the idea of a limited set. Therefore, there is a need to classify a prerequisite for the development of the problem. The assertion of having no minimum element refers to WOP, and as one talks about order, the order relationship of the set of rational numbers has been characterized as another prerequisite, as can be seen in **Table 4**.

The definition of Muniz Neto (2013) for a bounded set, contained in **Table 4**, only states that it should have the lower and upper bound. However, an upper bound is a set in which there is a number that is greater than all the elements of that set, and not necessarily that number belongs to the set. Similarly, the lower bound is a set in which there is a number which is smaller than all the elements of that set, and not necessarily that number belongs to the set.

Regarding the student's previous knowledge, it is reinforced that he must have a "mathematical background"[5] in which he knows the concepts of sets seen in basic education, for example. If the student does not have this previous knowledge at this time, the teacher, knowing the student's reality, seeks to perform mediations to optimize the understanding of the mathematical object.

If the teacher has no knowledge of the student's reality in this context, then he/she should include the idea of sets in the prerequisites. It is worth mentioning that the plateau is an action in which the teacher tries to equate the previous knowledge of the students with the prerequisites that the activity requires. For this reason, the relevance in having a period of the didactic session reserved for the realization of this action.

The measurement of such prior knowledge may be through a pre-test or a debate on such topics. In this way, the teacher will acquire feedback on how mature the student is in relation to these topics which will serve as a foundation for the new knowledge.

If the class does not follow the plateau, that is, feels impeded to remember or understand the concepts necessary for that didactic session, it is advisable to hold didactic sessions on the pre-requirements in order to then seek to make them previous knowledge of the student.

If perhaps there is no time for such applications, the teacher in the search of clearing up possible doubts of the students regarding the contents present in the basic education curriculum and the time in the classroom is not sufficient, the use of extracurricular activities is suggested (and in this case it can associate with flipped classroom[6] or opt for hybrid teaching).

After presenting the problem, the FS foresees the maturation stage, in which the student will seek to understand what the problem provides and what must be solved. In front of this, the student will create hypotheses. If the student does not think of strategies to prove the proposition, the teacher, through the FS, will make enquiries for which (s)he does not give the answer, but points out a path. These questions will enable the student to reflect on what (s)he knows to achieve the new knowledge (use of the path).

The student may have the following question: "Teacher, the set of rational numbers is ordered, so how to prove it is not well-ordered?"; or "Teacher, the set of rational numbers is ordered, so why prove it is not well-ordered?" In the face of this, the teacher will use two fundamentals of the FS: the question and the *hand-in-pocket* pedagogy.

The question

> "[...] refers to a situation in which the teacher asks, interrogates, urges the student to think about the problem proposed as a challenge for his learning [...]" (Sousa, 2015, p. 47).

And for our problem, a question the teacher can ask for such a student's doubt is: "what do we know about not being well-ordered?" This may lead the student to reread the sentence in order to understand what is meant by the expression "not being well-ordered".

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[5] Topics of mathematics that citizens should have (Papert, 1986).

[6] "[...] The concept of a flipped classroom is as follows: What is traditionally done in the classroom is now done at home, and what is traditionally done as homework is now done in the classroom?" (Bergmann & Sams, 2018, p. 31).

These questions can be categorized, according to Souza (2013), as clarifying, guiding and challenging. The clarifying questions address the how the student relates his/her knowledge to which he/she is learning, being a form of *feedback* for the teacher during pedagogical mediation.

The stimulating questions are defined by Souza (2013) as those that have the objective of making the student discover on his/her own how to do it, to provide the creativity of the student in the face of the problem, being a way of exemplifying the hard working. The classification of a question as a guiding question covers the moment when the teacher, in the face of the students' obstacles, points out a path for the student to reflect on the debate and create a way of solving the problem.

The teacher must be attentive to asking the questions and to all mediation with the use of the *hand-in-pocket* pedagogy, considering that for Santana (2019, p. 219)

> "[...] it presupposes attention, security and boldness from the educator, to know when to intervene, and if he should do so."

Thus, the teacher will not decide for the student, neither will he give tips on how to do it.

This way of acting in the classroom was seen in a way not yet systematized by Barroso (2009). The author, in differential and integral calculus classes, used the FS teaching methodology and, during the teaching sessions, considered that the teacher had "[...] the role of a mentor [...]" (Barroso, 2009, p. 108), that is, there was no intervention and no ready-made response for the student, but the debate in order for the student to build the path to the solution autonomously.

In agreement with this, Souza (2013, p. 54) stresses that

> "the teacher cannot take charge of a series of decisions that should be taken by the student."

Thus, the FS seeks to provide an environment in which the student can develop his/her autonomy.

Another obstacle that may appear in the class is the student trying to prove, starting from the set of rational numbers, using the definition of this set and its properties of order relationship. The teacher, then, must question what is "not being well-ordered", because the definition, according to the statement of the problem, falls on a subset of the rational numbers.

Once the moment of making hypotheses has passed, the student must draft a solution to the problem. If the student claims that the proof is based on saying that the open interval between 0 and 1 has no minimum element, the idea is valid, but without the formal resources that a mathematical proof requires (mathematical rigor).

The teacher, in order to induce the student to write the formal proof, may ask the student what are the other ways of writing the open interval between 0 and 1. The student may rewrite the interval in either the number line or tabular form. The teacher then asks which of the two situations would help the proof. The teacher asks the student without giving any clues on how to do it, not falling on the *topaze effect*[7].

Thus, the tabular representation is seen in Eq. (1), and then the teacher can inquire what conclusions have been drawn from this set. The student can say that the set is non-empty, since fraction 1 over 2 belongs to the S-set and is lower bounded by zero and upper bounded by one, as can be seen in the following representation:

$$S = \left\{\frac{a}{b} \in \mathbb{Q} \mid 0 < \frac{a}{b} < 1\right\}. \tag{1}$$

Hence, the teacher may question again what a "not well-ordered" set is and, with the statement, the student may manifest that the set must be non-empty, lower bounded and not have a minimum element. In view of this, the student realizes that, for S, there is a lack of evidence of not having the minimum element, in order to prove the proposition of **Figure 1**.

Knowing the WOP, he will seek the proof for a contradiction, because the hypothesis does not provide enough information to deduct the thesis (de Morais Filho, 2016). And, thus, he will suppose that S has such a minimal element (the contradiction of the proposition hypothesis). In the face of this, the student provides the following proof: Let Eq. (1) be, S is lower bounded by zero and not empty, because ½ ∈ Q. Supposing that S has a minimum element and can be represented without loss of generality, as c/d, therefore, c/d ≤ a/b, for all a/b ∈ S. As 0 is a lower bounding of S, one concludes that Eq. (2):

$$0 < \frac{c}{d} \Rightarrow \frac{c}{d} < \frac{c}{d} + \frac{c}{d} \Rightarrow \frac{c}{d} < \frac{2c}{d} \Rightarrow 0 < \frac{c}{d} \cdot \frac{1}{2} < \frac{c}{d}. \tag{2}$$

There is a contradiction with the minimality of c/d. Hence, S has no minimum element. One notices here that the generalizable is the use of the WOP, because it is a mathematical tool that solves so many other problems correlated with the minimality of a set.

Finally, the proof stage consists of the teacher systematizing and formalizing the content. The teacher already has the formal treatment of the problem, as it was a proof exercise solved by the student, however, he can systematize the proof. This form is given by "tighten up" steps, as for example the premise of 0< c/d implies Eq. (3), as follows:

$$0 < \frac{c}{d} \cdot \frac{1}{2d} < \frac{c}{d}. \tag{3}$$

[7] "When the teacher solves a situation that should be the student's job, then it falls on the "*topaze effect* [...]" (Menezes, 2018, p. 53).

One notices that all the treatment of the teacher with the student and the content studied is guided by the FS with a foundation called mediation. To this end, Sousa (2015, p. 46) infers that

> "[...] the teacher must provide situations in which the student is a researcher, from his/her mediating action between him/her and the knowledge."

With this, the teacher is a bridge between the student and knowledge and his way of acting should provide this student's journey towards knowledge.

This mediation can also address use of counterexamples, since it is a foundation provided by the FS, in which the teacher has

> "[...] the objective of unbalancing the student in order to make him reflect on something that affects a wrong situation [...]" (Menezes, 2018, p. 52).

However, the use may start from a student's mistake, which is not seen as something to be discarded, for the Fedathian teacher. For the FS, the error is an instrument that can provide learning, because it is only an unwanted result, but one that deserves to be highlighted.

Nevertheless, if the teacher has the perception that the class is not yet able to prove a theorem, that is, to work on the problem in question, starting from the theorem and requiring the proof, it will lead to difficulties of resolution not mentioned here, which may be the object of a new study. To this end, the teacher, seeking to provide this mathematical maturity, which is the proof, for the class, can start the problem with the set S mentioned here and, at the same time, ask the students to find out about the minimum element of the set, if it exists or not and, if it exists, which would be the candidate.

In this way, the teacher could "break" the proof into "smaller" parts that give objectivity to mathematical activity. And so, after the realization of these items characterized as minor parts of the proof, the teacher would synthesize and formalize the activity with the proposition, making it clear that the activities carried out there would form the proof of a mathematical result.

An example of FS in practice in relation to mathematical proof is seen in de Araújo et al. (2020). However, the authors carried out the didactic session with participants of the Multimedia Mathematics Education Group, who are different subjects from the proposed here, since the target audience are undergraduate students in mathematics, characterizing subjects in initial training. In de Araújo et al. (2020), the research subjects have experience in mathematics, since most participants have an undergraduate degree in mathematics.

## MEDIATION OF MATHEMATICAL KNOWLEDGE

This work took care to insert a teaching methodology to mediate the construction of a mathematical proof, which suggests the doubt: is the way treated here mediation or transmission? This doubt arises both from the foundation of the FS, mediation, and from the transmissibility of mathematical knowledge seen in Becker (2019).

Before seeking the answer to the present doubt, some considerations are necessary, since the teaching methodology approached seeks to guide the teacher's posture in order to provide more meaningful learning for the student, making him/her autonomous and creative, as well as the initial process of creating a mathematician, as seen in Araújo et al. (2020).

One notices in the FS, the concern in the positioning to be generalizable, because the idea is to make the student create a concept, or theorem, starting from a problem situation. For Becker (2019, p. 970), mathematical knowledge

> "[...] was built to solve a problem and, when it is built, it begins to solve a multitude of problems; it has become generalized."

One realizes that the generalizable will be generalized and this is the didactic task of the teacher to carry out in the Proof stage, according to the FS.

In dealing with this general situation, Menezes (2018, p. 43) considers

> "[...] that its way of performing can also solve numerous other situations."

In this way, the WOP is the generalizable situation, since it solves problems in countless other situations that address a minimum element of a set. This idea of the generalizable can also be seen in Davis and Hersh (1981, p. 81), when the authors comment that

> "application of theory A to theory B within mathematics means, then, that the materials, the structure, the techniques, the insights of A are used to cast light or to derive inferences with regard to the materials and the structures of B."

In the face of this, it can transpose the application of A to other theories (B, C, D, etc.), turning it, in this line of thought, into a generalizable mathematical object.

In order to propose this activity of a general nature, it is necessary for the teacher to have the expertise on the mathematical prerequisites necessary to carry out the activity, and to be aware of the previous knowledge of his student, which requires from him observation, and approximation, as didactically strategic conducts.

Becker (2019, p. 968), in the interview with teachers of higher education and basic education, raised points about the

> "[...] conception of mathematical knowledge, [...] transmissibility of mathematical knowledge and the [...] formation of the genesis of this knowledge by levels of progressive complexity. The author wonders whether it is enough for the student to have previous knowledge to learn, and one subject of his research states that "previous knowledge [...] is part of it, but for the child to learn, "*it is necessary to have a good methodology, a good working environment, a teacher who dominates his subjects*."

This corroborates with the FS, whereby the teacher should be the most experienced subject in the theme, that is, the one who dominates the subject being mediated. On the other hand, the FS does not exist if the student does not take action. Thus, there is an

> "action that can count on the substantial contributions of the teaching staff whose actions should never replace the action of the learning subject" (Becker, 2019, p. 986).

It should be noted that there is no point in the teacher mastering the mathematical prerequisites, being familiar with the student's previous knowledge and mastering the content in which he/she intends to mediate, if the student is not in action. The FS seeks to motivate the student to have an active participation in class, by giving importance to his previous knowledge for the construction of the new, meeting the necessary prerequisites, characterizing the action of the *plateau.*

Corroborating with the *plateau* and the didactic mediation envisaged in the FS is the Vygotskynian perspective. The development of the subject is through his previous knowledge and with the mediation of another person more experienced in the topic for then, the subject to achieve new potential. In this way, this methodological proposal gets closer to the zone of proximal development.

In the mediation model presented, one started from themes defined as pre-requisites and which should constitute the student's previous knowledge. If the student does not know any of the themes or has any obstacles, the teacher will carry out the plateau. With this, the potential to achieve new experiences and learning through didactic actions involving the mediated content arises.

## FINAL CONSIDERATIONS

This work had as a guiding question: *is it possible to use the FS as the teaching methodology in a teaching session led by the mathematical proof of theorems or propositions?* In view of the modeling presented here, which oriented the proof of a mathematical proposition and based on the FS as a teaching methodology, one notices that the answer to the guiding question is given as positive.

Starting from this question, the objective was defined, which was *the intention to develop a model of mathematical proof approach based on the FS methodological proposal*. With the proof of the proposition through the mediation approach, the question, the *hand-in-pocket* posture and, in a certain way, making use of every moment of the construction of the proof in the stages of the FS, one believes that the objective has been achieved.

Thus, given the works in **Table 1**, which dealt with mathematics in higher education, but not with mathematical proof as an object of research, it is possible to see this interweaving of the intrinsic work of the mathematician (the proof) with pedagogical mediation (a way of dealing with the subject in a teaching methodology).

It briefs, therefore, the proof in central themes of mathematics, for example, regarding the squeeze theorem (seen in differential and integral calculus), but with the pedagogical mediation based on FS, to then visualize this model in theorems.

However, although this study does not deal with central theorems seen in a licentiate's or bachelor's degree in mathematics, it does have a model of how to work pedagogical mediation in a classroom mathematical proof, based on the FS.

Therefore, one observes as a gap the non-application in the classroom of the model built and discussed here, being in charge of future research that addresses the FS methodological proposal and its mathematical proof.

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