International Electronic Journal of Mathematics Education
2026, 21(2), em0878
e-ISSN: 1306-3030
https://www.iejme.com

MODESTUM

Research Article

OPEN ACCESS

# How real analysis enhances the ability to identify errors in high school calculus problems

Jeongwook Chang [1*]

[1] Dankook University, SOUTH KOREA
*Corresponding Author: jchang@dankook.ac.kr

**Citation:** Chang, J. (2026). How real analysis enhances the ability to identify errors in high school calculus problems. *International Electronic Journal of Mathematics Education, 21*(2), em0878. https://doi.org/10.29333/iejme/18182

**ARTICLE INFO**

Received: 22 Mar 2025

Accepted: 08 Feb 2026

**ABSTRACT**

The study investigates how helpful college-level real analysis knowledge is in developing the ability to detect errors in high school calculus problems. Preservice teachers were selected as subjects to solve four calculus problems containing errors, primarily due to insufficient or incorrect conditions. The test subjects were simply asked to solve problems without knowing that there were errors in them. In addition to a quantitative analysis of the subjects' results, in-depth qualitative examinations of how they identified the errors were conducted. The study was designed based on the following two perspectives. Firstly, finding errors in mathematics problems requires a deep understanding of the concepts within the content of the area to which the problem belongs. Secondly, students may focus only on calculations without understanding the concepts in calculus, it is important to find ways to assess their understanding of those concepts. The results of the experiment seem to show, both quantitatively and qualitatively, that preservice teachers use college-level mathematics concepts when they attempt to understand high school mathematics deeply.

**Keywords:** preservice teacher education, content knowledge for mathematics, error detecting abilities, calculus, real analysis

## INTRODUCTION

The content and level of mathematical knowledge that teachers possess are major issues in preservice teacher training, qualification, and recruitment. In particular, as teachers' mathematical knowledge is known to affect student achievement, studies are constantly conducted to analyze and improve teacher knowledge (Charalambous et al., 2020; Creager et al., 2016; Hill et al., 2005; Kleickmann et al., 2013).

Shulman (1986), who raised teachers' knowledge as a key research topic, classified content knowledge that must be developed by teachers into subject matter content knowledge, pedagogical content knowledge (PCK), and curricular knowledge (CK). He specifically described PCK, which is a type of content knowledge, as a "special amalgam of content and pedagogy" (Shulman, 1987), thereby amplifying interest in not only pedagogical attributes of teacher knowledge but also subject matter knowledge. The teacher has special responsibility in relation to content knowledge, serving as the primary source of student understanding of subject matter (Shulman, 1987). Therefore, content knowledge that represents the teacher's understanding of the subject matter has significance in teachers' knowledge, and it is also suggested as a prerequisite for development of PCK (Hill et al., 2005; Kleickmann et al., 2013; Ma, 1999; McEwan & Bull, 1991; Stein et al., 1990).

According to previous research on teachers' content knowledge, mathematically knowledgeable teachers more clearly understand mathematical contents and provide better mathematical explanations based on that understanding (Hill et al., 2005). Teachers with a profound understanding of mathematics connect mathematical topics to conceptually similar or more powerful topics (Ma, 1999). Content knowledge is a major source when making pedagogical decisions as a teacher and is greatly affected by the things learned in college (Grossman, 1990). Therefore, preservice teacher education must include in-depth mathematical training to cultivate content knowledge. This is because mathematics teachers that received in-depth mathematical training had a higher degree of cognitive connectedness between content knowledge and PCK in addition to content knowledge (Krauss et al., 2008). Hurdle and Mogilski (2022) explained that students who took pre-calculus perform better in Calculus I than students who took College Algebra & Trigonometry. Hatisaru et al. (2025) anticipated that teacher training in college has influences on teachers' mathematical practices in doing mathematics by examining the gap between knowing mathematics as a teacher and knowing mathematics as a mathematician.

Topics of calculus are covered by intuitively adopting the concepts in high school, after which they are studied in the form of various mathematical problems. Studies have been constantly conducted on how students understand the concepts of continuity

*Copyright © 2026 by Author/s and Licensed by Modestum. This is an open access article distributed under the Creative Commons Attribution License which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.*

or differentiation of functions (Orton, 1983; Tall, 1990), how this is taught to students (Park, 2015), and how preservice teachers understand these concepts (Duru et al., 2010). However, it is difficult to find studies on how mathematical knowledge obtained by preservice teachers about calculus is applied to actual problems. Considering that the value of knowledge increases when it is adequately used in relation to judgment and action rather than comprehension of certain contents (Shulman, 1987), It is as meaningful to analyze how knowledge about mathematical content is used as it is to analyze how it is explored.

Also, assessment is one of the most important aspects in educational context (Darling-Hammond, 2016; Pastore, 2023; Roberts et al., 2021). In addition to previous research, we focus on teachers' deep understanding of high school mathematics concepts. As part of this, an investigation was designed to assess the ability to identify errors in calculus problems. Efforts were devoted to selecting test problems with errors that are difficult to detect due to the omission of subtle conditions. Problems were created in each of the areas of continuity, differentiation, and integration. The test subjects were asked to solve problems, but they were actually assessed on their ability to detect errors in the problems. While analyzing their answers, we also closely examined whether they could adequately apply the knowledge gained from college-level calculus and real analysis courses in the process of detecting errors caused by insufficient or incorrect conditions.

The research questions are:

**RQ1** What is the proportion of test subjects who found errors in the presented problems?

**RQ2** Is there a difference in discovering errors in problems related to continuity, differentiation, and integration depending on whether college-level calculus and real analysis are studied?

It is hoped that this study will help draw meaningful implications regarding the characteristics and direction of college mathematics courses for preservice teachers.

## THEORETICAL FRAMEWORK

### Theoretical Perspective on the Knowledge for Teaching

In studies on teaching, Shulman (1986, 1987) referred to the absence of focus on subject matter as a "missing paradigm" and argued that scholarship in content disciplines is one of the major sources for the teaching knowledge base. Teachers must have an in-depth understanding about what ideas and skills are most important and fundamental, and what rules and procedures are meaningful in the subject they are about to teach. Even teachers giving student-centered and highly interactive lessons cannot use their teaching techniques adequately unless they fully understand the topics they are about to teach.

Ball et al. (2005) assumed that the quality of mathematics class depends on the mathematical knowledge of teachers. The authors attempted to investigate the nature of mathematical knowledge required by those teaching mathematics at school and find a way to measure that knowledge. Unlike general adults or those with occupations that require high-level mathematical knowledge, they analyzed such knowledge necessary for the aforementioned and conceptualized this as mathematical knowledge for teaching (MKT) (Ball et al., 2005) and stated that MKT contributes to improving mathematical achievement of students (Hill et al., 2005). Furthermore, Ball et al. (2005) proposed Specialized Content Knowledge (SCK) that allows teachers to examine and understand unusual solution methods to problems as well as to provide mathematical explanations for common rules and procedures. Later, in Hill et al. (2008), SCK was described as a newer conceptualization that was not included in the Shulman's original subject matter knowledge, and MKT was conceptualized as a model consisting of six strands, including Common Content Knowledge (CCK), Knowledge at the mathematical horizon, Specialized Content Knowledge (SCK), Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of curriculum, of which KCS, KCT, Knowledge of curriculum are related to Shulman's PCK (Hill et al., 2008). Especially, SCK allows teachers to examine and understand unusual solution methods to problems as well as to provide mathematical explanations for common rules and procedures (Ball et al., 2005). Also, SCK was described as a newer conceptualization that was not included in Shulman's original subject matter knowledge (Hill et al., 2008).

The research findings on MKT can be usefully applied to the exploration of teacher knowledge related to student achievement and the design of support materials for teachers (Ball et al., 2008). As teacher domain-specific approaches to knowledge, the level and the association between PCK and content knowledge in two groups of teachers with different mathematical expertise were examined (Krauss et al., 2008). They conducted a PCK test and content knowledge test on 198 secondary mathematics teachers including academic-track Gymnasium (GY) teachers. The results indicated that GY teachers who had to take more advanced master-level courses on the subject received higher scores in both PCK and content knowledge than non-Gymnasium (NGY) teachers, while also demonstrating superior cognitive connectedness between the two knowledge categories. Therefore, Krauss et al. (2008) revealed that in-depth learning on the subject as preservice teachers affect the cultivation of not only content knowledge but also PCK, thereby once again proving the importance of content knowledge in preservice teacher education.

These results may be discussed in relation to Ma (1999)'s work. In a study on elementary school mathematics teachers, Ma (1999) suggested that mathematical substance of teacher knowledge is a "profound understanding of fundamental mathematics (PUFM)". According to the author, teachers with PUFM have a good conceptual understanding of subject knowledge as well as the ability to teach students so that they can build the foundation of the conceptual structure. Teachers' mathematical knowledge must be PUFM, which is developed via a cyclic process of "schooling-teacher preparation-teaching". Therefore, there must be constant support and attention so that teachers can continue developing PUFM through continuous research not only during preservice teacher education but also after they become teachers. Based on a synthesis of the results from studies Krauss et al.

(2008) and Ma (1999), it can be seen that the importance of content knowledge in preservice teacher education should be emphasized.

Zazkis and Mamolo (2011) also discussed what mathematical knowledge is important and useful in teaching mathematics. They presented how advanced mathematical knowledge is applied in specific teaching situations using the pre-calculus class of Grades 3, 5, and 12. In this process, teachers' advanced mathematical knowledge led students to more systematically organize what they learned, helped provide quicker feedback to students' responses in class or resolve their confusion, and intrigued students by finding elements that may stimulate their curiosity. Thus, teachers' knowledge beyond school curriculum obtained from mathematics courses in college is useful not only for teachers' instructional choice but also students' learning.

Kleickmann et al. (2013) studied the period in which teacher knowledge is formed. They analyzed the difference in PCK and CK depending on the teaching career of preservice and in-service mathematics teachers. The results revealed that teacher knowledge of subject matter is mostly developed in the initial stage of teacher training. The in-service phase slightly contributed to the improvement of PCK, but barely contributed to CK. Therefore, the initial stage of preservice teacher education is a critical period in forming teachers' mathematical content knowledge.

Creager et al. (2016) conducted interviews with 15 preservice elementary school teachers using five MKT items on multi-digit addition and subtraction. Some preservice teachers answered questions about mathematical validity based on pedagogical reasoning. In other words, their pedagogical concern eclipses the characteristics of the items concerning mathematical nature, ultimately making mathematical decisions based on pedagogical concern. Therefore, in assessing teacher knowledge and nurturing preservice teachers, it is necessary to not only develop each category of MKT but also perceive the importance of their integration and find ways to cultivate it.

Using data collected from teachers, Charalambous et al. (2020) examined whether teacher knowledge is multidimensional. Their research utilized surveys with items on advanced common content knowledge (ACCK), specialized content knowledge (SCK), and knowledge of content and teaching (KCT). They concluded that teacher knowledge comprises a single dimension and reported the positive effects of teacher knowledge on student learning through the analysis of student observations. Based on the results, they suggested that mathematically competent individuals should be selected as elementary school teachers and that teacher education programs should provide subject-specific knowledge.

Larsson et al. (2024) investigated the relation between mathematics teachers' beliefs, MKT, and error-handling practices. They showed that teachers with high MKT could handle students' error according to the teachers' beliefs, but teachers with low MKT could not handle students' error effectively.

These studies revealed that mathematical content knowledge of mathematics teachers is a significant part of research on mathematics education and has a great impact not only on instruction but also on learning. Advanced-level mathematical knowledge beyond the school curriculum formed in college is even more important as it enables teachers to provide meaningful lessons and appropriate feedback to student responses.

### Knowledge of Calculus and Real Analysis Content

The definition of "continuity" and "discontinuity" used in high school calculus, college-level calculus, and real analysis in general are outlined in **Table 1**.

**Table 1.** Definition of "continuity" (Hockett & Bock, 2010, p. 155. Stewart, 2012, pp. 118-119, Bartle & Sherbert, 2000, p. 120)

| Courses | Definitions |
|---|---|
| High school calculus | A function $f(x)$ is continuous at the point $x = c$ if:<br>(a) $f(x)$ is defined at $x = c$<br>(b) $\lim_{x\to c} f(x)$ exists<br>(c) $\lim_{x\to c} f(x) = f(c)$<br>If a function is not continuous at $x = c$, we say it is discontinuous there, or has a discontinuity at $x = c$ |
| College-level calculus | A function $f(x)$ is continuous at the point $x = c$ if:<br>1. $f(c)$ is defined (that is, c is in the domain of $f$)<br>2. $\lim_{x\to c} f(x)$ exists<br>3. $\lim_{x\to c} f(x) = f(c)$<br>If $f$ is defined near c (in other words, $f$ is defined on an open interval containing c, except perhaps at c), we say that $f$ is discontinuous at c (or $f$ has a discontinuity at $c$) if $f$ is not continuous at c. |
| Real analysis | Let $A \subseteq \mathrm{R}$, let $f: A \to \mathrm{R}$, and let $c \in A$. We say $f$ is continuous at $c$ if, given any number $\varepsilon > 0$ there exists $\delta > 0$ such that if $x$ is any point of $A$ satisfying $\lvert x - c\rvert < \delta$, then $\lvert f(x) - f(c)\rvert < \varepsilon$.<br>If $f$ fails to be continuous at $c$, then we say that $f$ is discontinuous at $c$. |

As the concept of continuity advances through high school calculus, college-level calculus, and real analysis, the definitions become more stringent. According to them, the understanding of discontinuity also changes. In high school calculus, the function is discontinuous when it is not defined at the given point. In college-level calculus, to consider whether the function is continuous or discontinuous, it must be defined near the given point, even if it is not defined at that point. In real analysis, the given point must be within the domain to determine whether the function is continuous or discontinuous. Thus, it is meaningless to determine continuity or discontinuity at a point that is not included in the domain. Thus, in problems asking whether a function is continuous at a certain point, the answer may differ not in terms of being right or wrong but based on which of the definitions in high school calculus, college-level calculus, or real analysis is followed.

In high school calculus, the students learned that if the derivative of a function is positive on the interval, that function is increasing on the interval. However, high school calculus generally does not cover the phenomenon in which the derivative of a function is positive at one point in detail. It may seem that if the derivative is positive at one point, the function may be increasing at that point. However, as shown in **Figure 1**, even if the derivative is positive at one point, the function may not be an increasing function in any neighborhood of that point.

The differentiable function defined by

$$g(x) := \begin{cases} x^2 + 2x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is such that $g'(0) = 1$, yet it can be shown that $g$ is not increasing in any neighborhood of $x = 0$.

**Figure 1.** Example of non-increasing function on any neighborhood of a point (Bartle & Sherbert, 2000, p. 171)

In real analysis it is emphasized that the fact that the derivative is positive at just one point does not guarantee that the function increases in the neighborhood of that point. Students lacking a background in differentiation topics of real analysis are expected to misconceive the condition that the derivative is positive at a single point, which they are likely to interprete as applying to an interval. Hence Problem 2 confirms the importance of content knowledge.

Not all functions become derivatives of differentiable function. For example, if:

$$g: [-1, 1] \to R \text{ defined by } g(x) := \begin{cases} 1, 0 < x \leq 1 \\ 0, x = 0 \\ -1, -1 \leq x < 0 \end{cases} \quad (1)$$

then a function $f$ such that $f'(x) = g(x)$ for all $x \in [-1, 1]$ does not exist. Therefore, g is not the derivative of any function on [-1, 1] (Bartle & Sherbert, 2000).

Darboux's theorem (cf. **Figure 2**) states that the derivative of a function cannot have a jump discontinuity. Therefore, it provides a useful way to determine that a given function does not have an antiderivative without going through a complex procedure. In many cases, existence of an indefinite integral can be verified through high school-level calculations, but Darboux's theorem allows one to recognize it immediately. That is, instead of performing complex calculations to verify an error in the problem, one can readily identify it by applying an advanced topic in real analysis.

**Darboux's theorem** If $f$ is *differentiable* on $I = [a, b]$ and if $k$ is a number between $f'(a)$ and $f'(b)$, then there is at least one point $c$ in $(a, b)$ such $f'(c) = k$

**Figure 2.** Darboux's theorem (Bartle & Sherbert, 2000, p. 174)

Although this theorem is not covered in high school curriculum, when high school students ask questions related to the above topic, teachers who are familiar with it can offer a clear and accurate explanation. Furthermore, the theorem is expected to assist teachers in responding accurately and appropriately to calculus problems in various contexts from high school students.

## METHODOLOGY

### Participants

The sample for this study consisted of 83 preservice teachers who are undergraduate students of a university in Korea. They were divided into three subgroups: Group A (28 preservice teachers), Group B (28 preservice teachers), and Group C (27 preservice teachers).

The members of Group A, all first-year college students, have completed college-level calculus, including the epsilon-delta definition of limits, but have not undertaken a course in real analysis. The members of Group B, all second-year college students, have studied only the concepts of limits, continuity, and differentiation in real analysis, including Darboux's theorem and the phenomenon that occurs when the derivative is positive at a single point, so it is not expected that they will understand the concepts of Riemann integral in real analysis. The members of Group C, all third-year college students, have completed real analysis course, including limits, continuity, differentiation, and Riemann integral.

The members of Group A have also taken a course in set theory. The members of Group B have also taken a course in number theory in addition to the courses taken by the members of Group A. The members of Group C have also completed courses in topology, complex analysis, and differential geometry for at least one semester in addition to the courses taken by the members of Group B.

The assessments in this study were designed and carried out independently of the courseworks.

### Instrumentation

This study developed four problems involving fundamental concepts of high school calculus – two on continuity (Problem 1-(a), 1-(b)), one on differentiation (Problem 2), and one on integration (Problem 3) - that contain mathematical flaws difficult to detect at the high school level but readily identifiable with knowledge of advanced topics in real analysis, and were presented as instrumental items for assessing participants' understanding. The problems and scoring codes used in this study are presented in **Table 2**.

**Table 2.** Items and rubrics for scoring

| no. | | Items and rubrics |
|---|---|---|
| Problem 1 | | Choose all the right statements and verify your answers.<br>(a) Function $y = \sqrt{x}$ is not continuous at $x = -2$<br>(b) Function $y = \frac{1}{x}$ is not continuous at $x = 0$ |
| Code | 3 | Cannot determine whether it is continuous or discontinuous because the function value cannot be defined. |
| | 2 | Discontinuous (stated that (a) the function is not defined, (b) the function value/limit cannot be defined). |
| | 1 | Stated that the function value cannot be defined, but did not select the given proposition as correct and did not mention that it cannot be determined. |
| | 0 | Missing, other response |
| Problem 2 | | A differentiable function $f : R \to R$ satisfies $f'(0) > 0$,<br>and $f\left(\frac{1}{n}\right) \neq f\left(\frac{1}{n+1}\right)$ for any positive integer $n$.<br>Find the value of $\lim_{n\to\infty} \frac{f\left(\frac{1}{n+1}\right)-f\left(\frac{1}{n}\right)}{\left|f\left(\frac{1}{n+1}\right)-f\left(\frac{1}{n}\right)\right|}$ |
| Code | 3 | Cannot find the answer because it does not increase in any neighborhood of 0. |
| | 2 | Cannot find the answer with incomplete explanation. |
| | 1 | Solved the problem assuming that it is an increasing function. |
| | 0 | Missing, Other response |
| Problem 3 | | A function $f(x)$ satisfies<br>$f'(x) = \begin{cases} 2, & x < -1 \\ 2x, & -1 \leq x \leq 1 \\ -2, & x > 1 \end{cases}$<br>and the graph of $y = f(x)$ passes through the origin.<br>Find the value of $f(2) - f(-2)$. |
| Code | 3 | Cannot find the answer because there is no function that satisfies the condition of the problem. |
| | 2 | Cannot find the answer (incomplete explanation using continuity, constant of integration). |
| | 1 | Solved the problem using integration. |
| | 0 | Missing, Other response |

Problem 1 was designed to assess how students understand and apply the definition of continuity at points where the given functions are not defined. To elaborate, Problem 1-(a) is mathematically inappropriate in any case, and Problem 1-(b) could be either appropriate or inappropriate, depending on the definition of continuity that the subjects know. The examination focused on observing the subjects' reactions to Problem 1-(a) and the types of definitions of continuity they know through Problem 1-(b).

Problem 2 was designed to assess whether students can discover the condition $f'(0) > 0$ cannot imply $f\left(\frac{1}{n}\right) < f\left(\frac{1}{n+1}\right)$. This error can be seen as subtle and difficult to notice unless one has studied this part carefully. As this phenomenon cannot be readily understood within the framework of high school mathematics, it represents an error that is quite difficult to identify without in-depth study of real analysis. One of the interests of this problem was an examination of how many members of Group C, who had completed real analysis course, improved their ability to recognize and explain the error.

Problem 3 was designed to assess whether students understand the antiderivative of a piecewise-defined function. This type of error occasionally arises in high school calculus problems. The inadequacy of the conditions in the problem can be discovered through high school-level calculations, but some of the test takers immediately noticed it through the Darboux theorem. Answers of the problem was analyzed with a focus on the manner in which the objects who discovered the error explained their reasoning.

This study is related to preservice teachers' SCK in that the learning of real analysis provides unusual at the high school level yet efficient methods for identifying errors in high school–level calculus problems. In particular, the descriptions of errors written by the participants are expected to provide meaningful information for identifying how mathematical knowledge learned through college-level calculus and real analysis courses function as PUFM.

Each item was coded by two independently trained assessors, and the final score was determined through discussion only when there was a difference in the scoring results.

## RESULTS

### Quantitative Analysis of Assessment Codes

Problem 1 concerns the continuity at the given point. A total of 49.4% (41 subjects) responded that they "cannot determine" to 1-(a) and 28.9% (24 subjects) responded the same to 1-(b). A total of 24.1% (20 subjects) responded that 1-(a) was discontinuous, and 51.8% (43 subjects) that 1-(b) was discontinuous. In other words, many subjects claimed that they "cannot determine" the answer to 1-(a), and many that 1-(b) was discontinuous. A total of 6.0% (5 subjects) responded that the function cannot be defined and chose the given proposition as the correct statement or did not mention that continuity cannot be determined in 1-(a); 2.4% (2 subjects) responded the same for 1-(b). For 1-(a) and 1-(b), 20.5% (17 subjects) and 16.9% (14 subjects) did not give an answer or gave meaningless answers, respectively.

Problem 2 concerns a non-increasing function on any neighborhood of a point. Only 13.3% (11 subjects) explained that $f'(0) > 0$ does not indicate that $f(x)$ is an increasing function on any neighborhood of 0 and highlighted that the problem does not have sufficient conditions. A total of 6% (5 subjects) responded that the problem has an error but did not include interpretation of $f'(0)$ in their explanation. A total of 28.9% (24 subjects) tried to solve the problem assuming that $f'(0) > 0$ indicates that $f(x)$ is an increasing function on a neighborhood of 0. Meanwhile, in Problem 2, 51.8% (43 subjects) did not give an answer or gave meaningless responses.

Problem 3 concerns determining whether the given function can be an antiderivative of another function. Only 8.4% (7 subjects) indicated that the problem does not have sufficient conditions since there are no antiderivatives in the given function. A total of 32.5% (27 subjects) responded that the problem includes an error, but gave a reason that was not directly related to this problem, such as continuity or constant of integration. A total of 44.6% (37 subjects) solved the problem assuming that the given function can be integrated. A total of 14.5% (12 subjects) did not give an answer or provided meaningless responses.

The results are reported in **Table 3**.

**Table 3.** Results of responses

| no. | Code 3 | Code 2 | Code 1 | Code 0 | Total |
|---|---|---|---|---|---|
| Problem 1-(a) | 41 (49.4%) | 20 (24.1%) | 5 ( 6.0%) | 17 (20.5%) | 83 |
| Problem 1-(b) | 24 (28.9%) | 43 (51.8%) | 2 ( 2.4%) | 14 (16.9%) | 83 |
| Problem 2 | 11 (13.3%) | 5 ( 6.0%) | 24 (28.9%) | 43 (51.8%) | 83 |
| Problem 3 | 7 ( 8.4%) | 27 (32.5%) | 37 (44.6%) | 12 (14.5%) | 83 |

### Quantitative Analysis of Group Responses

This section examines the differences in responses based on the completion of mathematics courses in college by analyzing the response characteristics of subjects in each group. The results of the responses to each item by group are presented in **Table 4**.

**Table 4.** Responses by group

| no. | Group | Code 3 | Code 2 | Code1 | Code 0 | Total |
|---|---|---|---|---|---|---|
| Problem 1-(a) | Group A | 11 | 13 | 2 | 2 | 28 |
| | Group B | 15 | 3 | 3 | 7 | 28 |
| | Group C | 15 | 4 | 0 | 8 | 27 |
| Problem 1-(b) | Group A | 6 | 21 | 1 | 0 | 28 |
| | Group B | 9 | 13 | 1 | 5 | 28 |
| | Group C | 9 | 9 | 0 | 9 | 27 |
| Problem 2 | Group A | 1 | 3 | 7 | 17 | 28 |
| | Group B | 0 | 0 | 12 | 16 | 28 |
| | Group C | 10 | 2 | 5 | 10 | 27 |
| Problem 3 | Group A | 0 | 7 | 12 | 9 | 28 |
| | Group B | 1 | 8 | 18 | 1 | 28 |
| | Group C | 6 | 12 | 7 | 2 | 27 |

A chi-square test of independence revealed a statistically significant association between group membership and response codes such as:

Problem 1-(a): $\chi^2(6, N = 83) = 16.25, p = 0.0125$,

Problem 1-(b): $\chi^2(6, N = 83) = 15.71, p = 0.0154$,

Problem 2: $\chi^2(6, N = 83) = 24.77, p = 0.0004$,

Problem 3: $\chi^2(6, N = 83) = 24.88, p = 0.0004$.

Thus, the between-group differences for each item are statistically significant. A more detailed item-by-item examination is provided below.

In Problem 1, Groups B and C showed different results compared to Group A. This phenomenon seems to be because the subjects in Group A only know the college-level definition of continuity, while the subjects in Groups B and C have the same knowledge regarding continuity in real analysis.

In Problem 2, Group C showed significantly different results compared to Groups A and B. On the other hand, there was not much difference between Groups A and B. This may be because the subjects in Groups A and B had not yet studied differentiation in real analysis. Therefore, only Group C subjects who studied differentiation in real analysis showed a high ratio of response that $f'(0) > 0$ does not indicate that $f(x)$ is an increasing function on the neighborhood of 0.

In Problem 3, only Group C showed different results, similar to Problem 2. In this group, 67% of subjects responded that $f(x)$ could not be obtained (Code 2, Code 3), and 22% stated that there is no function that satisfies the conditions of the problem (Code 3). This indicates that Group C subjects were more aware that the conditions included an error compared to other groups.

Meanwhile, in Groups A and B, 25% and 32% of subjects responded that $f(x)$ could not be obtained, respectively, but only one subject in Group B stated that there is no function that satisfies the conditions of the problem.

### Qualitative Examinations of the Answers to Each Problem

This section analyzes the responses to each item in detail and examines whether preservice teachers properly identified the errors included in the problems, as well as how they explained them.

In Problem 1, applying the definition of "discontinuity" used in real analysis in **Table 1** to Problem 1-(a) and 1-(b) are neither continuous nor discontinuous at the given point. Therefore, subjects who adequately applied the mathematical contents of real analysis responded that the problems are "mathematically incomplete", "cannot determine", or "impossible to judge". A total of 24 subjects responded as shown above in both 1-(a) and 1-(b), and one of the responses is as follows.

> Because the given point should be within the domain of the function, it cannot be determined whether both functions are continuous or discontinuous at the given points.

Different responses are given due to the different situations of "discontinuity" in high school calculus, college-level calculus, and real analysis. Therefore, both of the answers of Codes 2 and 3 are considered correct. According to the definition in college-level calculus, the function must be defined on the neighborhood of the given point to say it is discontinuous at that point, even if it is not defined. Thus, 1-(a) is "mathematically incomplete", and 1-(b) is "discontinuous". A total of 11 subjects gave this response, and the most common response was as follows:

1-(a) It can't be determined whether $y = \sqrt{x}$ is continuous or discontinuous at $x = -2$

$\because$ $y = \sqrt{x}$ is defined on $\{x \geq 0\}$

1-(b) $y = \frac{1}{x}$ is discontinuous at $x = 0$

$\because$ Since $\lim_{x\to 0^-} \frac{1}{x} = -\infty$, $\lim_{x\to 0^+} \frac{1}{x} = \infty$, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

According to the definition in high school calculus, points that are not continuous are all discontinuous, and the function cannot be defined at all given points in 1-(a) and 1-(b) and thus is "discontinuous." Many of those who responded that the function is discontinuous in 1-(a) and 1-(b) gave the following response:

1-(a) $y = \sqrt{x}$ is discontinuous at $x = -2$

$\because$ $y = \sqrt{x}$ is not defined at $x = -2$

1-(b) $y = \frac{1}{x}$ is discontinuous at $x = 0$

$\because$ Since $\lim_{x\to 0^-} \frac{1}{x} = -\infty$, $\lim_{x\to 0^+} \frac{1}{x} = \infty$, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

So $\lim_{x\to 0} f(x)$ does not exist.

According to the definition in high school calculus, students could answer that the function was not continuous at the point in 1-(b) simply because the function could not be defined at the given point. However, many subjects considered the limit, which is noteworthy.

Regarding Problem 2, high school calculus and college-level calculus only explain that the function increases when the derivative is greater than 0 on an interval, whereas real analysis provides that $f'(0) > 0$ does not guarantee that the function increases even on the smallest neighborhood of 0. In fact, 10 members of Group C, who had completed a full course in real analysis, responded that the condition was "mathematically incomplete" in Problem 2, reflecting a precise understanding and application of the mathematical content of real analysis. Some of the correct answers are as follows:

A: Even though $f'(0) > 0$, the function $f$ may not increase on any neighborhood of 0.

B: There is a counterexample to this problem.

It was a somewhat unexpected finding that one student in Group A, without prior study of real analysis, detected the error even though no specific reasoning was provided.

The subjects who were assigned Code 2 tried to solve the problem without appropriately applying the condition $f'(0) > 0$, simply, by dividing $f$ into increasing and decreasing cases and offering no specific explanation. These subjects did not provide accurate reasons, although they may have perceived that the conditions were not sufficient to prove that the given function is an increasing function.

**Figure 3** presents one of the answers coded 2; this subject tried to solve the problem by dividing the type of function into increasing and decreasing cases without considering the condition $f'(0) > 0$.

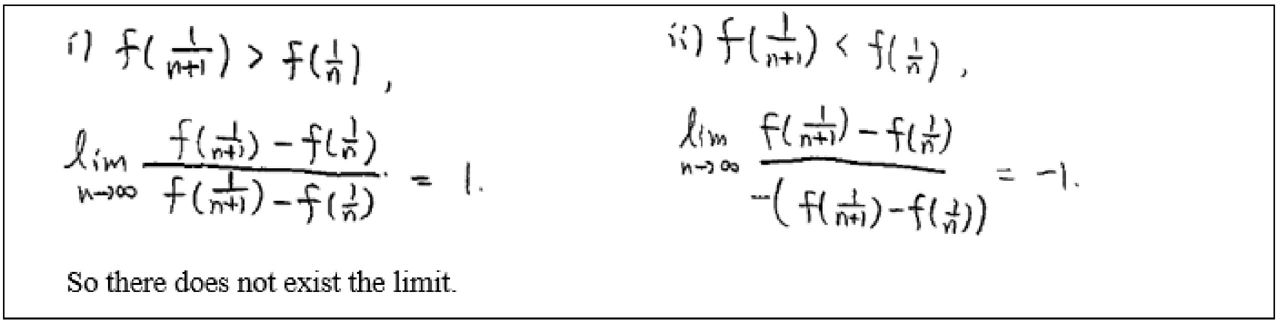


**Figure 3.** Example of an answer coded 2 (Source: Field study)

Meanwhile, subjects who were not taught real analysis have not covered the condition that the derivative is greater than 0 at just one point. Thus, as shown in **Figure 4**, many of them solved the problem misunderstanding that $f\left(\frac{1}{n+1}\right) < f\left(\frac{1}{n}\right)$, and thus were assigned Code 1.

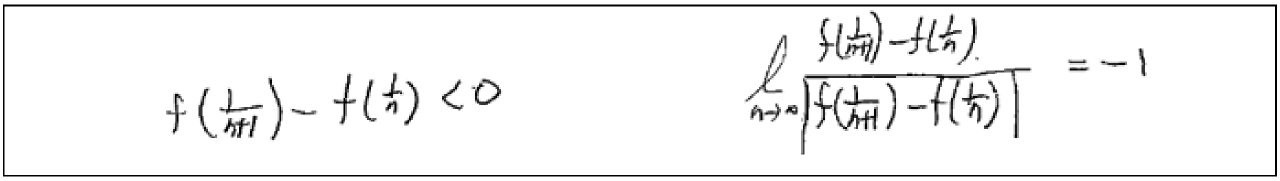


**Figure 4.** Example of an answer coded 1 (Source: Field study)

Problem 3 determines whether subjects know that there is no differentiable function whose derivative is given in the problem. Among subjects who noticed something unusual while solving the problem, many recognized the issue as stemming from the absence of conditions such as continuity. All students in Group C with Code 3 accurately responded that there exists no such $f\,(x)$, referring to Darboux's theorem. **Figure 5** shows an answer demonstrating how the error in Problem 3 can be immediately identified

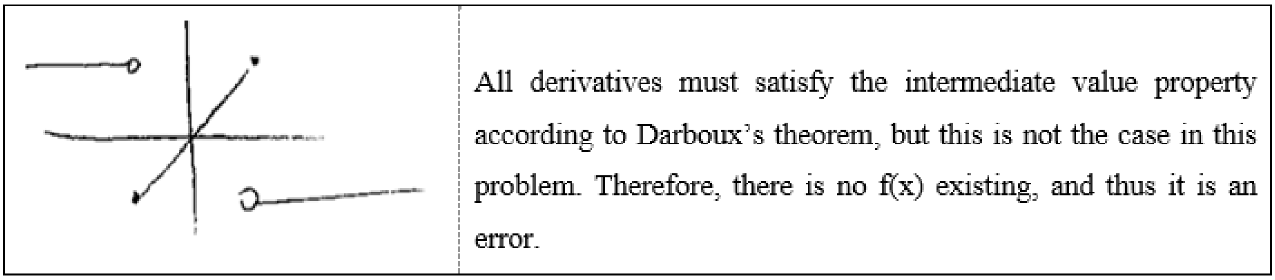


**Figure 5.** Example of an answer coded 3 (Source: Field study)

In fact, high school calculus is enough to show that $f(x)$ cannot exist. Just after integrating $f'(x)$ in each interval, it can be verified that the resulting function is not differentiable. However, not a single answer was based on this method. Among the groups, 6 members from Group A, 4 from Group B, and 10 from Group C responded that $f(x)$ could not be obtained because the given function was not continuous, also, 1 from Group A, 4 from Group B, and 2 from Group C responded the constant of integration could not be confirmed. They indicated that antiderivatives could not be obtained from the problem's conditions but did not provide accurate reasons; thus, they were assigned Code 2.

**Figure 6** presents one of the answers coded 2. This subject tried to express $f(x)$ with integration in each interval and calculate the constant of integration, and stated that more conditions were needed to obtain function $f$. Since even high-achieving high school students are expected to receive Code 2 on this problem, effective and accurate instruction for these students requires that teachers be familiar with an approach to solving the problem using Darboux's theorem, which can be considered part of the content knowledge required of teachers with SCK.

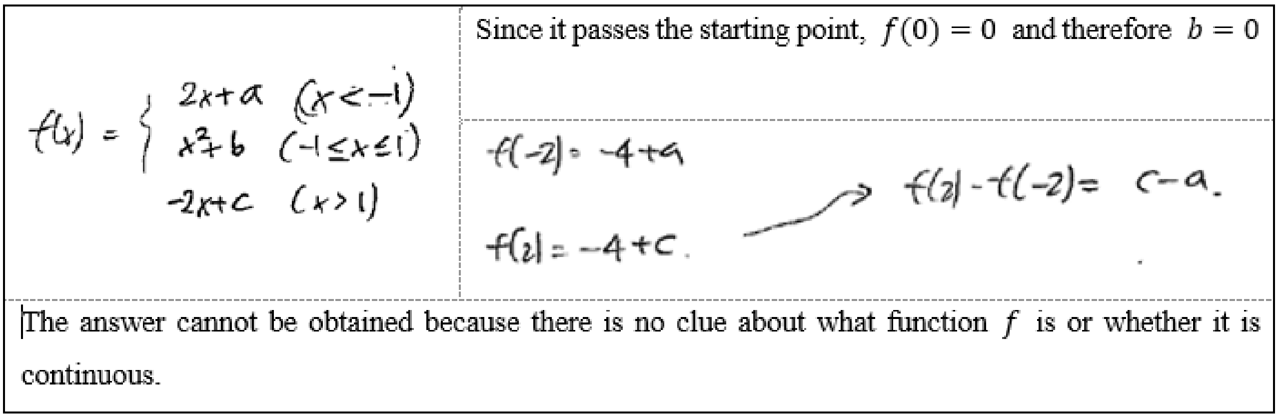


**Figure 6.** Example of an answer coded 2 (Source: Field study)

Meanwhile, the subjects who did not recognize that the given function cannot be the derivative of any function failed to identify the error in this problem. As shown in **Figure 7**, they tried to find the answer by obtaining the function $f(x)$ by integrating $f\,'(x)$ over each of three intervals, and were thus assigned Code 1.

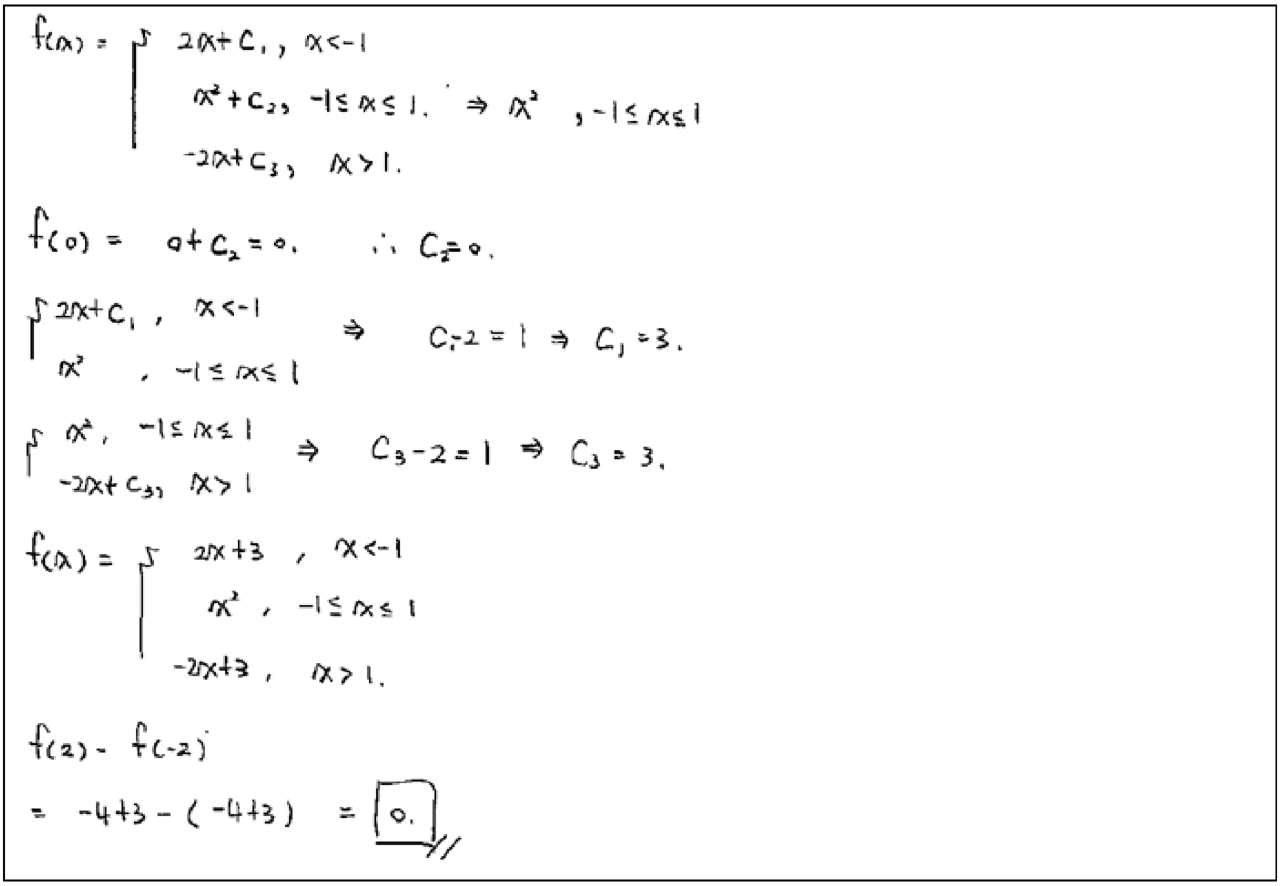

**Figure 7.** Example of an answer coded 1 (Source: Field study)

Among the students who provided answers, none completely misunderstood the problem. The most common misconception was that objects thought they can find a function that satisfies the given conditions once additional continuity conditions are imposed.

## DISCUSSION

It is generally accepted that college-level mathematics knowledge is beneficial for high school teachers, but developing more specific ways to apply it remains an ongoing challenge. It can also be regarded as an obvious fact that, in order to teach mathematics effectively in schools, teachers must have a strong understanding of the mathematical content included in the curriculum. Shulman (1986) asserted teachers should possess subject matter knowledge at least equivalent to that of those with a major in the subject, and Agathangelou and Charalambous (2021) considered that content knowledge could serve as a prerequisite for pedagogical content knowledge. From this perspective, they require knowledge of more advanced-level mathematical contents beyond what students must learn (Zazkis & Mamolo, 2011).

Moreover, some parts of the high school curriculum overlap with what is learned in college, which is why preservice teachers obtain knowledge that can be used by applying knowledge learned from college mathematics to the school curriculum (Krauss et al., 2008). In this context, this study can also be related to exploring whether college-level mathematics knowledge helps enhance teachers' ability to create high-quality, error-free problems for assessing students, as well as their ability to identify errors in the problem-review process, which is one of the important competencies required of teachers. Since the ability to identify errors in a problem relies on a deep understanding of the concepts, it can be considered an SCK of teachers with PUFM.

In this article, it was analyzed whether preservice teachers are using the mathematical knowledge obtained in college to find and explain errors in high school mathematics problems. There was certainly a difference in response rates for each item among the groups divided according to the amount of mathematical knowledge acquired in college. Group C had the highest proportion of members who gave adequate answers to all items, particularly in problem 3. Although the error in the problem can be recognized after performing high school–level calculations and realizing that the constant of integration cannot be determined, 67% of Group C vs. 25% of Group A and 32% of Group B gave 'cannot obtain' responses, and only Group C frequently invoked Darboux's theorem; this suggests that understanding of Darboux's theorem is associated with deeper insight into high school calculus. Such a performance by Group C, having completed real analysis course, can be interpreted in the same context as Krauss et al's. (2008) findings that the cognitive connectedness between PCK and CK in secondary mathematics teachers is related to mathematics expertise. Thus, this study are consistent with the view that preservice teachers with knowledge of real analysis may possess the foundational competencies necessary for cultivating SCK related to high school calculus. These results provide the following implications about preservice teacher training.

Firstly, in the sense that college mathematics courses can play a direct and meaningful role in deeply understanding high school mathematics concepts, it is important and essential for preservice teachers to be provided with a curriculum in college that enables them to acquire and develop the ability to apply sufficient mathematical knowledge that can be directly utilized in teaching students. Such courses should not be just about obtaining the knowledge and methods for doing mathematics (Ball et al., 2008). Since teachers must develop and use mathematical problems to assess students at school, they must cultivate the ability not only to solve the problems but also to determine whether these are mathematically complete. The findings of this study, showing that preservice teachers who had studied real analysis were better at identifying errors in high school calculus problems, are consistent with the potential benefit of including preservice teacher education programs designed to encourage the analysis of high school mathematics problems from a college-level mathematics perspective.

Secondly, teacher training programs designed to ensure that preservice teachers complete advanced studies related to mathematical content and can specifically apply this knowledge to solve challenging high school concepts and problems is also necessary. Efforts to improve mathematical knowledge of teachers through content-focused preservice teacher programs will also lead to improved student achievement (Hill et al., 2005). Furthermore, teachers who studied in-depth mathematics in college but did not receive additional education on teaching mathematics scored high not only in content knowledge but also in PCK. Further, they showed excellent connectivity between these two types of knowledge (Krauss et al., 2008). This study aligns with prior research in showing that subjects who had studied up to real analysis may be better at analyzing and identifying errors in challenging high school calculus problems than subjects in other groups. Therefore, preservice teacher must be given enough opportunities to gain a much deeper mathematical understanding.

As further research, it could be considered to more directly identify the relationship between mathematics teachers' knowledge and student achievement by observing how preservice teachers' knowledge is implemented in actual lessons.

### Limitations of the Study

The present study was based on a one-time assessment of four items on concepts in continuity, differentiation, and integration, which limits the extent to which college-level mathematics content knowledge can be generalized to preservice teachers. Furthermore, as noted in the recommendations for further research, additional studies are needed to examine whether preservice teachers' knowledge for identifying errors in problems effectively contributes to addressing students' misconceptions.

**Funding:** No funding source is reported for this study.

**Ethical statement:** The author stated that the activities in the study were based on standard educational practices. No sensitive personal information was collected and the study involved no risk to participants. Participants were informed that the data would be used solely for academic purposes and handled securely, and that they could withdraw from the study at any time without adverse consequences.

**AI statement:** The author stated that no generative AI or AI-based tools were used in this study.

**Declaration of interest:** No conflict of interest is declared by the author.

**Data sharing statement:** Data supporting the findings and conclusions are available upon request from the author.

## REFERENCES

Agathangelou, S., & Charalambous, C. Y. (2021). Is content knowledge pre-requisite of pedagogical content knowledge? An empirical investigation. *Journal of Mathematics Teacher Education, 24*, 431-458. https://doi.org/10.1007/s10857-020-09466-0

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 29*(1), 14-17, 20-22, 43-46. https://hdl.handle.net/2027.42/65072

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407. https://doi.org/10.1177/0022487108324554

Bartle, R. G., & Sherbert, D. R. (2000). *Introduction to real analysis* (3rd edition). John Wiley & Sams, Inc.

Charalambous, C. Y., Hill, H. C., Chin, M. J., & McGinn, D. (2020). Mathematical content knowledge and knowledge for teaching: Exploring their distinguishability and contribution to student learning. *Journal of Mathematics Teacher Education, 23*(6), 579-613. https://doi.org/10.1007/s10857-019-09443-2

Creager, M. A., Jacobson, E. D., & Aydeniz, F. (2016). Can pedagogical concerns eclipse mathematical knowledge? *For the Learning of Mathematics, 36*(2), 2-7.

Darling-Hammond, L. (2016). Research on teaching and teacher education and its influences on policy and practice. *Educational Research, 45*, 83-91. https://doi.org/10.3102/0013189X16639597

Duru, A., Köklü, Ö, Jakubowski, E. (2010). Pre-service mathematics teachers' conceptions about the relationship between continuity and differentiability of a function. *Scientific Research and Essays, 5*(12), 1519-1529.

Grossman, P. L. (1990). *The making of a teacher: Teacher knowledge and teacher education*. Teachers College, Columbia University.

Hatisaru, V., Richardson, S., & Star, J. R. (2025). Mathematical practices (commonly used by mathematicians) in mathematics teachers' solutions to algebraic problems. *European Journal of Science and Mathematics Education, 13*(1), 41-57. https://doi.org/10.30935/scimath/15889

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406. https://doi.org/10.3102/00028312042002371

Hill, H., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal of Research in Mathematics Education, 39*(4), 372-400. https://doi.org/10.5951/jresematheduc.39.4.0372

Hockett, S. O., & Bock, D. (2010). *Barron's AP calculus*. Baron's Educational Series, Inc.

Hurdle, Z. B., & Mogilski, W. (2022). The impact of prerequisites for undergraduate calculus I performance. *International Electronic Journal of Mathematics Education, 17*(3), Article em0696. https://doi.org/10.29333/iejme/12146

Kleickmann, T., Richter, D., Kunter, M., Elsner, J., Besser, M., Krauss, S., & Baumert, J. (2013). Teachers' content knowledge and pedagogical content knowledge: The role of structural differences in teacher education. *Journal of Teacher Education, 64*(1), 90-106. https://doi.org/10.1177/0022487112460398

Krauss, S., Brunner, M., Kunter, M., Baumert, J., Blum, W., Neubrand, J., & Jordan, A. (2008). Pedagogical content knowledge and content knowledge of secondary mathematics teachers. *Journal of Educational Psychology, 100*(3), 716-725. https://doi.org/10.1037/0022-0663.100.3.716

Larsson, M., Kaufmann, O. T., & Ryve, A. (2024). Investigating the relation between mathematics teachers' beliefs, knowledge and error-handling practices – an exploratory study of 12 teachers in Grades 4–6. *International Journal of Mathematical Education in Science and Technology*, *57*(2), 253-274. https://doi.org/10.1080/0020739X.2024.2438373

Ma, L. (1999). *Knowing and teaching elementaryr mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Lawrence Erlbaum Associates. https://doi.org/10.4324/9781410602589

McEwan, H., & Bull, B. (1991). The pedagogic nature of subject matter knowledge. *American Educational Research Journal, 28*(2), 316-334. https://doi.org/10.3102/00028312028002316

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics, 14*, 235-250. https://doi.org/10.1007/BF00410540

Park, J. (2015). Is the derivative a function? If so, how do we teach it? *Educational Studies in Mathematics, 89*(2), 233-250. https://doi.org/10.1007/s10649-015-9601-7

Pastore, S. (2023). Teacher assessment literacy: A systematic review. *Frontiers in Education, 8*, Article 1217167. https://doi.org/10.3389/feduc.2023.1217167

Roberts, C., Khanna, P., Lane, A. S., Reimann, P., & Schuwirth, L. (2021). Exploring complexities in the reform of assessment practice: A critical realist perspective. *Advences in Health Sciences Education, 26*, 1641-1657. https://doi.org/10.1007/s10459-021-10065-8

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14. https://doi.org/10.3102/0013189X015002004

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Education Review, 57*(1), 1-23. https://doi.org/10.17763/haer.57.1.j463w79r56455411

Stein, M. K., Baxter, J. A., & Leinhardt, G. (1990). Subject-matter knowledge and elementary instruction: A case from functions and graphing. *American Educational Research Journal, 27*(4), 639-663. https://doi.org/10.3102/00028312027004639

Stewart, J. (2012). *Calculus: Early transcendentals* (7th edition). Cengage Learning.

Tall, D. (1990). Inconsistencies in the learning of calculus and analysis. *Focus on Learning Problems in Mathematics, 12*(3-4), 49-63.

Zazkis, R., & Mamolo, A. (2011). Reconceptualizing knowledge at the mathematical horizon. *For the Learning of Mathematics, 31*(2), 8-13.