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From drawing to defining: Cognitive stratification in secondary students' conceptions of line, segment, and parallelism

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ABSTRACT

A persistent challenge in geometry education is students' difficulty coordinating perceptual experience with the theoretical demands of formal definitions. This qualitative exploratory study examines how 9th-grade students (13-14 years old) in Cameroon conceptualise three foundational objects the straight line, the line segment, and parallelism drawing on the *concept image / concept definition* framework. Data from 92 written questionnaires and 18 semi-structured interviews reveal that no student produced a formally adequate definition of either the line or the segment, while approximately 39% offered acceptable definitions of parallelism. Analysis of this differential suggests that semiotic congruence between a mathematical object and its representation, rather than didactic exposure alone, may be a critical factor in the formation of operational concept images. These findings converge with international research on definitional activity, figural reasoning, and epistemological obstacles in geometry. We propose a stratified account of the concept image comprising perceptual, procedural, and theoretical components as a falsifiable theoretical hypothesis that refines Vinner's original construct and generates four testable predictions for comparative and longitudinal investigation. The proposed stratification advances the classical concept image construct by specifying that its components are organised into distinct strata perceptual, procedural, and theoretical each governed by a different mode of validation (ostension, successful action, logical necessity), rather than treating the concept image as an undifferentiated pool of elements from which only an evoked subset is activated. This reframing yields four falsifiable predictions concerning how task format selects which stratum is recruited, offering an analytic tool that is transferable beyond the geometric objects examined here and that can directly inform task design in teacher education and curriculum development.

Keywords: concept image, geometry education, definitional activity, figural reasoning, epistemological obstacle, stratification

INTRODUCTION

The role of definitions in geometry is both foundational and pedagogically contested. A definition is not simply a formula to be memorised; it is, from a didactic standpoint, an intellectual construction that mediates the shift from perceptual engagement with figures to deductive reasoning about ideal objects (De Villiers, 1998; Durand-Guerrier & Herault, 2006; Freudenthal, 2012; Shir & Zaslavsky, 2002). Yet research across several national contexts suggests that this shift is neither automatic nor well-supported by standard instructional approaches. Students typically encounter definitions as fixed statements to be reproduced, rather than as tools for structuring geometric thought a pattern documented in French, Québécois, and sub-Saharan school systems alike.

The difficulty is not incidental. Geometric objects such as the straight line and the line segment occupy an ambiguous position: they are simultaneously primitive in axiomatic terms and unavoidably represented through bounded, imperfect drawings. The resulting tension between the ideal mathematical object and its semiotic representation (Duval, 1995; Parzysz, 1988) constitutes a structural obstacle one that Brousseau (2002) would characterise as epistemological rather than merely pedagogical. When the line can only be depicted as a finite stroke, and when classroom routines conflate "line AB" with the segment connecting A and B, students have good perceptual reasons to construct concept images that systematically diverge from formal definitions.

Within the Tall and Vinner (1981) framework, a *concept image* encompasses the entire cognitive structure associated with a concept: mental images, properties, procedures, examples accumulated through experience. It is not governed by logical coherence and may harbour internally contradictory elements that coexist without the learner's awareness. The *concept definition*, by contrast, designates whatever form of words the individual uses to specify the concept, whether or not it aligns with the institutionally recognised definition. The gap between the two is diagnostic: it reveals which properties have been integrated into the learner's representation and which remain absent or distorted.

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The present study uses a Cameroonian 9th-grade cohort as an observation site for examining this gap, not to characterise Cameroonian education per se, but because this institutional context provides a particularly informative case. Definitions are typically transmitted axiomatically, with few opportunities for students to reconstruct their meaning, while the five surveyed textbooks show considerable heterogeneity in their treatment of the line. Together, these features offer a relatively unmediated view of the conceptions that develop when definitional scaffolding is limited. The research questions are: (1) What are the dominant components of students' concept images of the straight line, the line segment, and parallelism? (2) How do these representations relate to the formal definitions expected? (3) What cognitive and semiotic mechanisms might account for the observed patterns? To avoid any implication that comparable studies have already characterised the cognitive structures of students in other national systems, we underscore that Cameroon is treated here as a single instructive case rather than as a basis for cross-national claims ; any resonance with patterns documented elsewhere is addressed comparatively in the Discussion, where the relevant international literature is cited explicitly.

THEORETICAL FRAMEWORK

Defining in Mathematics and in the Classroom

A formal mathematical definition specifies a minimal set of necessary and sufficient properties that distinguish an object within a given domain (Durand-Guerrier et al., 2015). This logical criterion, however, does not capture the full complexity of definition in a learning context. De Villiers (1998), Durand-Guerrier et al. (2015); Tanguay and Geeraerts (2012), and Freudenthal (2012) share the view that mathematical definitions are not pre-given statements to be transmitted but objects to be constructed and justified by learners. We adopt this shared view as such, without gathering it under a named position : existing '-isms' in mathematics education carry specific theoretical commitments constructionism, in particular, is associated with Papert's programme, and the delicacy of such labels is well documented (Confrey & Kazak, 2006). From this standpoint, the elaboration of a definition is itself a mathematical activity one that requires understanding why a particular property is necessary, what happens if it is dropped, and how different equivalent formulations relate to one another. Vinner (1991) demonstrated that memorising a definition guarantees neither conceptual understanding nor appropriate deployment in problem-solving: what governs behaviour in most task situations is the concept image, not the formally stated definition.

The straight line presents a particular difficulty: in Euclidean geometry, it is a primitive notion, characterised axiomatically rather than defined. Its material representation a bounded stroke on paper or board inevitably truncates what is, mathematically, an infinite object. This semiotic incongruence (Duval, 1995) between representation and referent is a known source of epistemological obstacle (Bachelard, 1938). Gobert (2007) has shown that drawings actively induce interpretations incompatible with the formal properties of objects they are meant to represent, and Parzysz's (1988) analysis of the knowing/seeing tension in geometry points to the same structural difficulty. The segment, in turn, is typically defined relationally as a portion of a line so that its conceptualisation inherits whatever instability exists at the level of the line concept.

Concept Image and Concept Definition

Tall and Vinner's (1981) framework was developed precisely to account for the recurrent gap between formal mathematical structure and actual cognitive acquisition. The concept image is characterised by four properties relevant to the present study: it may harbour internal contradictions; in any given task situation, only a subset of it is activated (the *evoked* concept image); it is resistant to change even in the face of formal instruction; and it is historically constituted, shaped by the sequence and variety of examples a learner has encountered.

Several frameworks address related phenomena but from distinct vantage points. Within the concept image tradition itself, two recent refinements are central to the account developed in this paper. Tsamir and Tirosh (2022) characterise the boundary of a concept image in terms of mis-in and mis-out attributes: formally irrelevant properties admitted into the concept, and criterial properties excluded from it. Harel and Sowder (2005) argue that learners' proof schemes, what they accept as ascertaining truth for themselves and persuading others, develop across ages and content domains, and that epistemological and didactical sources of difficulty interact rather than exclude one another. Section *A Tentative Stratified Account of the Concept Image* builds both refinements into the theoretical model itself. Houdement and Kuzniak's (2006) geometric paradigms natural, natural axiomatised, and axiomatic characterise institutional environments rather than individual cognitive structures. Duval's (1993, 1995) semiotic register theory foregrounds the coordination of representational systems (figural, verbal, algebraic) as the central challenge in mathematical learning. These frameworks are complementary: while Duval illuminates the semiotic conditions for concept formation, and Houdement and Kuzniak describe the institutional geometry the learner inhabits, Tall and Vinner provide the cognitive vocabulary for analysing what the learner has internalised. Our analytical refinement operates within this latter register.Throughout this paper we reserve concept image and concept definition for Tall and Vinner's (1981) constructs as defined above ; we use conceptualisation as a neutral cover term for the broader process by which a student comes to understand an object, and cognitive structure only when referring, non-technically, to the general organisation of a learner's knowledge, without implying commitment to any architecture beyond the stratified account proposed in Section *A Tentative Stratified Account of the Concept Image*.

METHODOLOGY

Design and Participants

The study adopts a qualitative exploratory design aimed at characterising the nature and diversity of concept images rather than measuring performance quantitatively. This orientation is consistent with the Tall and Vinner framework, which calls for interpretive analysis of student productions rather than their quantification. The study should be read as generative rather than confirmatory: its purpose is to develop analytically grounded hypotheses that subsequent research with larger, more diverse samples and complementary task types can test more rigorously.

Participants were 92 volunteer students (13-14 years old) from two urban secondary schools in Yaoundé, Cameroon, enrolled in the 9th grade a pivotal transition year in which deductive geometry is formally introduced and definitions become operational tools for proof. Sampling was limited to two urban institutions, restricting generalisability to rural and private school populations. Students' individual levels of geometric reasoning were not independently assessed; at this grade level, deductive justification is only beginning to be institutionalised, and considerable individual variation must be assumed.

Institutional Context

The Cameroonian curriculum transmits the notions of the line, segment, and parallelism without requiring students to reconstruct definitions. Analysis of five official textbooks (Njomgang Ngansop & Tchonang Youkap, 2018; Tchonang Youkap et al., 2020) reveals substantial heterogeneity: the line is explicitly defined in only two of the five; elsewhere it is introduced by ostension. The segment is systematically defined, but formulations vary. This variability means that students at different schools may have encountered different framings of the same objects, which likely contributes to the diversity of conceptions observed a finding with potential relevance beyond this particular context.

Data Collection

Data were gathered through an individual written questionnaire administered to all 92 students under identical conditions, followed by semi-structured interviews with 18 purposively selected students. The questionnaire contained a single instruction: "If you were to write a mathematics textbook, what definition would you give for: (1) a straight line, (2) a line segment, (3) two parallel lines?" This wording positioned students as authors rather than recallers, reducing the likelihood of mechanical reproduction. The questionnaire was administered in French (the medium of instruction) over one hour; students were informed their responses would not be graded. All quotations in this paper are translated by the author.

Interview participants were selected via maximum variation sampling (Mason, 2002) to represent the range of response types: six students who produced incomplete but coherent definitions, eight displaying major conceptual confusions (line/segment, line/right angle), and four who had provided no written response. Interviews (15-25 minutes each) proceeded in three phases: clarification of the written response; confrontation with drawn examples and counterexamples in varied orientations; and exploration of the origins of the student's conception. Sessions were audio-recorded with consent and partially transcribed. Triangulation across written and oral data enabled a working distinction between formulation difficulties (where conceptual understanding was present but poorly verbalised) and deeper conceptual gaps (where the concept image appeared genuinely incompatible with the formal definition). This distinction is interpretive, and the inferences drawn should be treated accordingly.

Analysis

The 92 questionnaires were transcribed verbatim, preserving exact wording. Each student was assigned an anonymous code (E1-E92). Responses were classified into three mutually exclusive categories: formal, incomplete, and other definitions. To maximize coding replicability, classification boundaries were operationalized a priori based on a strict matrix of logical completeness (identifying minimal necessary and sufficient properties), thereby minimizing interpretative ambiguity in borderline cases; this a priori matrix also served as the shared framework for the independent double coding described below. Within each category, qualitative analysis identified recurrent formulations, confusions, and additional elements, which were then interpreted in terms of their probable conceptual origins perceptual, didactic, or epistemological. The Spearman correlation ($\rho=0.68$, $p<0.001$) is computed on the ordinal classification (formal > incomplete > other), where the ordering reflects increasing conceptual distance from the formal definition; it serves as a descriptive index of intra-individual systemic coherence, quantifying the structural co-instability of the line and segment concepts.

For clarity, the formal definitions used as coding benchmarks were as follows: a straight line was counted as formal only if the response specified both infinite extension and straightness (absence of deviation from a fixed direction) without relying on a diagram to convey either property in place of stating it; a line segment was counted as formal only if it was characterised as a bounded portion of a straight line, explicitly naming straightness as inherited from the line rather than merely stating the presence of two endpoints; and two lines were counted as formally parallel if the response specified either non-intersection under indefinite extension or constant distance (equidistance) the operative criterion being disambiguation with respect to indefinite extension, not the particular lexical marker (e.g. ‘never’) used to express it. The classification of the 92 responses was subjected to independent double coding. A second coder independently classified the complete dataset using the a priori coding framework described above and without access to the initial coding. Inter-rater reliability was assessed using Cohen's kappa and indicated substantial agreement ($\kappa = 0.79$). The few remaining discrepancies were subsequently discussed and resolved by consensus before proceeding with the qualitative analyses.The quantitative counts reported in **Tables 1-3** are not intended as an inferential test but as a descriptive scaffold situating the subsequent qualitative analysis: percentages index the relative frequency of response types within this sample, while the Spearman correlation reported below quantifies the co-occurrence of definitional (in)adequacy

across the line and segment tasks for the same respondents an internal-consistency check that supports the conceptual-contamination interpretation developed in Section *Epistemological Obstacles and Conceptual Contamination*.

A methodological caveat bears stating explicitly. A definition-writing task captures only the portion of a concept image that a student can articulate verbally in that moment. The concept image is broader than what any single task activates; complementary recognition and construction tasks would access procedural and perceptual strata more directly. The interpretations below should be read as tentative inferences about cognitive structures, grounded in discursive traces rather than direct observation of those structures.

RESULTS AND ANALYSIS

We present results for each of the three target notions, beginning with a quantitative overview before moving to qualitative analysis. The aim is not to describe all response types exhaustively but to identify the analytically richest patterns those that shed light on the mechanisms generating observed concept images. Throughout this section, interview excerpts are used not merely as illustrations of questionnaire-based categories but, where indicated, to refine or challenge the written classification: E47's and E52's transcripts (see Section *The Straight Line*) show that the same ‘incomplete’ written classification can mask two distinct cognitive configurations, and E74's transcript (see Section *Parallelism*) shows a written ‘incomplete’ response coexisting with a theoretically adequate but unarticulated understanding; these instances are flagged explicitly at each occurrence below.

The Straight Line

No student produced a formally adequate definition of the straight line. Among the 36 incomplete definitions, the overwhelming majority characterised the line solely by its infiniteness "a line without endpoints," "a line unlimited on both sides" while omitting straightness entirely. This pattern is explicable on at least two counts. First, teachers characteristically emphasise extension to distinguish the line from its bounded representation; infiniteness thereby becomes the most cognitively salient property. Second, verbalising straightness without circularity is genuinely difficult: formulations such as "the shortest path between two points" are abstract and non-obvious for early secondary students, and the textbooks analysed offer little guidance. The result is a concept image centred on a necessary but insufficient condition.

Table 1. Distribution of responses for the straight line

Category	Formal definitions	Incomplete definitions	Other definitions	Total
Number	0	36	56	92
Percentage	0%	39.1%	60.9%	100%

Ten students attempted to integrate straightness but produced tautological formulations: "a straight line is a straight line" (E12); "a straight line is a straight stroke without end" (E19). Although logically inadequate, these responses may signal awareness that infiniteness alone is insufficient the student perceives a missing property but lacks the vocabulary to articulate it. This is better characterised as a verbalisation obstacle than as conceptual absence.

More diagnostically significant is the confusion between line and segment, present in 24 responses: "a straight line is a straight line with two endpoints" (E3); "a straight line is the distance between two points" (E4). These formulations suggest concept images in which the line is bounded in direct tension with its mathematical nature. The origin appears semiotic: the only available depiction of a straight line is a finite stroke, and the graphic convention of unmarked endpoints (signalling continuation) proves insufficient to override perceptual evidence of boundedness. E4's formulation introduces a further ontological conflation, equating the geometric object with its scalar measure a confusion likely reinforced by classroom notations in which "AB" denotes both segment and length interchangeably.

Interview evidence: Cognitive configurations underlying the line/segment confusion

Interviews revealed that this confusion is not cognitively uniform. The following exchange with E47 is instructive. Asked to explain why he had written that a line "*has two endpoints*," he replied: "When we draw a straight line, it always has a beginning and an end." When invited to draw the line through two given points, he produced a segment and labelled it "line AB." Confronted with a figure showing a line with arrows at both ends, he hesitated: "*That is a line too, but it continues... it doesn't actually end.*" The contradiction became visible to him only when made explicit: "*Ah! What I drew is a segment! The line continues, but the segment stops*." Asked why he had nevertheless written that a line has endpoints, he offered a remarkably precise diagnosis of his own difficulty: "*Because when they say 'draw line AB', you only draw between A and B. I was confused*."

What this exchange makes legible is not ignorance but dissociation. E47 held, in the same cognitive system, a theoretically adequate representation of the infinite line activated by the arrows and a perceptually driven one shaped by years of classroom drawing practice. The definition-writing task had recruited the second without the first. Student E52 presents a different picture. Having written that a straight line is "the distance between two points," he maintained under questioning that the line "stops at the points." When shown the same figure with arrows, he described it as a "big line" an improvised subcategory that preserved his original concept image while accommodating the anomalous drawing. Unlike E47, he did not revise; he added a label. His own explanation was candid: "The teacher sometimes says the line continues, but when we draw it, it always stops." Perception, here, has simply crowded out the theoretical statement rather than coexisting uneasily with it. These two configurations what might be called *practical/theoretical dissociation* and *exclusive perceptual anchoring* are not interchangeable. The first may respond to tasks designed to surface dormant knowledge by creating productive cognitive conflict; the second appears to require a more sustained reconstruction, one in which the theoretical dimension is built from scratch rather than merely activated. We note that this

dissociation is inferred from a single interview exchange and is offered as one plausible account consistent with the transcript, rather than as a directly observed cognitive mechanism.

A third recurring pattern involved confusion between the straight line and the right angle (12 responses): "*a straight line is a geometric figure that has a right angle*" (E5). This appears to stem from lexical interference: in French, *droite* (straight line) and *droit* (right, as in right angle) share a morphological root, and both notions are typically introduced in proximity within early geometry sequences. The resulting polysemy can generate persistent terminological interference for students whose geometric vocabulary is not yet fully stabilised.

The Line Segment

Results for the segment are, perhaps counter-intuitively, worse than for the line. The segment is more frequently used and systematically defined in textbooks; one might expect better performance. The inversion demands explanation.

Table 2. Distribution of responses for the line segment

Category	Formal definitions	Incomplete definitions	Other definitions	Total
Number	0	14	78	92
Percentage	0%	15.2%	84.8%	100%

The 14 incomplete definitions share a common structure: limitation is specified ("a line with two endpoints"), but straightness is absent. As with the line, the perceptually most accessible attribute bounded endpoints, marked by graphical conventions dominates the concept image, while straightness, which cannot be directly observed (any short line appears straight to the eye), is omitted.

The 78 responses classified as other predominantly define the segment by reference to the line but in a way that conflates the two: "a segment is a line with two endpoints" (E30); "a segment is a small line" (E14); "a segment is a part of a line" (E33). This confusion is the mirror image of the line/segment confusion observed in Section *The Straight Line*, and it points to a structural mechanism: the standard textbook definition "a portion of a line bounded by two points" presupposes an adequate concept image of the line. Given that no student produced such a concept image, the relational definition of the segment is deprived of its conceptual foundation. The observed Spearman correlation ($\rho = 0.68$) between definition quality for lines and segments is consistent with this interpretation: conceptual instability at the level of the line appears to propagate systematically to the segment. Isolated responses such as "a segment is like a ruler" (E27) further suggest that, for some students, the concept image is anchored in the material instrument rather than the abstract object.

Parallelism

The contrast with the preceding results is sharp. Thirty-six students (39.1%) produced formally acceptable definitions of parallelism, clustering around three mathematically equivalent formulations: non-intersection ("lines that can never meet," "do not intersect even if extended": 22 students); equidistance ("always keep the same distance between them": 10 students); and common perpendicularity ("perpendicular to the same line": 2 students). The equidistance formulation is noteworthy in that it is not the standard textbook phrasing, suggesting these students have constructed a concept image that incorporates the spatial regularity of the relationship through perceptual experience railway tracks, notebook lines rather than purely through institutional transmission. The common perpendicularity formulation, by contrast, indicates a degree of geometric maturity: the student has grasped that multiple distinct properties can equivalently characterise the same relation.

Table 3. Distribution of responses for parallelism

Category	Formal definitions	Incomplete definitions	Other definitions	Total
Number	36	22	34	92
Percentage	39.1%	23.9%	37.0%	100%

The 22 incomplete definitions are mainly of the type "lines that do not touch," which differ from the formal definitions above primarily in the absence of an explicit quantifier ("never") or reference to infinite extension. Whether these reflect a superficial perceptual understanding or a tacit grasp of infinite non-intersection is difficult to determine from the written response alone a genuine interpretive limit of the instrument. The operative distinction here is not the presence of the lexical item ‘never’ as such, but whether the response commits to non-intersection (or constant distance) holding under indefinite extension of the lines ; formulations using ‘never’ typically signalled this commitment explicitly, whereas the incomplete responses left the extension condition implicit it is this substantive commitment, not the wording, that is coded.

Among inadequate definitions, prototype-restriction is the most common pattern: "two horizontal lines" (E13); "lines that go in the same direction" (E24); "lines that do not touch and have the same length" (E43). The horizontal prototype reflects the dominance of horizontally oriented examples in classroom drawing; the "same length" attribution reveals a residual confusion between parallel lines and parallel segments, since a line's length is undefined.

Interview evidence: The transition from perceptual to theoretical

Student E74's interview illustrates what a mediated conceptual transition can look like. Having written "parallel lines do not touch," E74 was shown two gradually converging lines and asked whether they were parallel. He observed without hesitation: "No, they will touch if we continue." When the researcher noted that he was imagining the lines extending beyond the drawing, he confirmed: "Yes, because a line always continues." Pressed to explain how one could be certain two lines would never touch even far away, he worked through the problem aloud: "They must... they must go in the same direction and always keep the same distance between them." Told that this constituted a mathematically valid definition, he was visibly surprised "Really? But the

teacher always says 'lines that do not intersect', not 'constant distance'" and then, after a moment: "Ah, I understand. It's the same thing, just said differently."

Several things are worth noting here. The theoretical layer of E74's concept image the idea that a line always continues was present before the interview; it had simply not been mobilised in the written task. What the interview provided was not new knowledge but an occasion for articulation: a sequence of questions that connected perceptual observation ("they will touch"), causal inference ("if the spacing changes"), and formal criterion ("constant distance") into a coherent chain of reasoning. The surprise at discovering that his own formulation was mathematically legitimate reveals something about the implicit epistemology students bring to definitions: the definition is the teacher's sentence, not a criterion that can be expressed in multiple equivalent ways. Recognising that equivalence moving from a ritual to a tool use of definition, in De Villiers' (1998) sense is perhaps the more significant transition than any single conceptual acquisition. Whether such transitions persist beyond the interview situation is a question this study cannot answer.

DISCUSSION

The Perceptual-Theoretical Tension in Geometric Conceptualisation

Across all three concepts, the data converge on a familiar but analytically tractable difficulty: students' concept images are organised primarily around perceptually accessible properties rather than the idealised attributes that formal definitions require. Observable features limitation, orientation, constant spacing dominate over properties that must be inferred or theoretically constructed infiniteness, straightness, invariance under rotation. This predominance of the perceptual is consistent with learning anchored in what Houdement and Kuzniak (2006) call the *natural geometry* paradigm, where validation relies on visual evidence. Duval's (1993, 1995) framework sheds complementary light: when the semiotic registers of figural, verbal, and theoretical representation are not coordinated, the visual register tends to dominate, and concept formation remains anchored in what can be seen rather than in what must be reasoned.

Building on this predominance of the perceptual and distinct from the strata analysis developed in Section *A Tentative Stratified Account of the Concept Image* a further consequence, which we term lexical masquerade, is that students deploy the institutional vocabulary "line," "segment," "parallel" without commanding the conceptual conditions that govern its use. The gap between mathematical lexicon and conceptual understanding resembles what Skemp (1976) termed instrumental versus relational understanding, and it may be particularly durable in contexts where definitions function as declarative knowledge rather than as working tools for reasoning.

A Tentative Stratified Account of the Concept Image

Our data suggest that the concept image in elementary geometry may be usefully analysed as comprising three simultaneously present components, which we describe as strata to emphasise their coexistence rather than sequential replacement.

The perceptual stratum is constituted through direct experience with graphical representations. It is characterised by ostensive validation and by the prominence of visible attributes: the line as a bounded stroke, the segment as a "small line," parallels as horizontal. This component appears to dominate in approximately 60% of responses for the line and segment in our data.

The procedural stratum stems from drawing and measuring routines. While the written questionnaire captures only its discursive traces, the interviews illuminate how geometric action ('joining points') yields operational, non-theorized knowledge aligned with instrumented schemes (Laborde & Capponi, 1994). This stratum, discursively manifested, underlies the rationalizations of approximately 25% of the sample.

The theoretical stratum integrates formal properties and logical relations: infiniteness as an axiomatically stipulated attribute, straightness as a non-perceptual property, the segment as hierarchically dependent on the line. Only approximately 15% of responses concentrated in the parallelism task appear to mobilise this stratum, and even then it is often incomplete.

What the interview data suggest, beyond this descriptive taxonomy, is that these strata coexist within a single cognitive system and are differentially activated by task demands. E47 possessed a theoretically adequate representation of the infinite line but activated his perceptually grounded one when writing a definition. E74 could reason his way to an equidistance formulation for parallelism through scaffolded questioning, despite having initially produced a perceptually based written response. These observations are compatible with Vinner's notion of the *evoked* concept image: what a task activates may not represent the full extent of what the learner possesses. The stratification hypothesis extends this idea by proposing that the structure of the concept image rather than its content alone may determine which aspects are accessible in which conditions. As with the E47/E52 contrast above, this reading of E74's transcript is an interpretive inference offered to illustrate the stratification hypothesis, not an independent confirmation of it.

The strata are not developmental stages to be traversed sequentially: a student at any point of schooling may have perceptual, procedural, and theoretical components simultaneously available, though unevenly elaborated. The account differs from Houdement and Kuzniak's geometric paradigms in that it describes the internal organisation of an individual learner's representations rather than the institutional geometry of the classroom. The analytical value of the stratification hypothesis lies in directing attention to the conditions under which learners can coordinate across strata a question that current diagnostic instruments are not well-equipped to address and in generating the following predictions: (1) tasks varying in cognitive demand

will systematically activate different strata in the same student; (2) perceptual-stratum responses will predominate in recognition tasks, procedural-stratum responses in construction tasks, and theoretical-stratum responses in proof tasks; (3) targeted instruction on definitional reconstruction will increase the proportion of theoretical-stratum responses without suppressing the perceptual stratum. (4) a student's dominant proof scheme (Harel & Sowder, 2005) will covary with the stratum recruited in definitional tasks: external-conviction and empirical proof schemes will predict perceptually and procedurally anchored definitions, whereas deductive proof schemes will predict theoretical-stratum definitions.

This hypothesis is grounded in a limited and interpretive dataset and should be treated accordingly; its value is generative rather than confirmatory. The four predictions stated above constitute the empirical programme it opens, and each can be evaluated through appropriately designed follow-up studies.

We are conscious that the term ‘stratum’ could suggest a hierarchy of increasingly sophisticated levels to be traversed, which is not our intended meaning. Following Ely's (2010) demonstration that learners can hold multiple, situationally activated, and even mutually incompatible conceptions of the same object, we understand the three strata as coexisting registers rather than ordered stages, and the analysis above is restricted to the concept images made visible within the specific context created by the definition-writing task, rather than claimed as an exhaustive account of what each student ‘has’. In the same spirit, we do not treat the theoretical stratum as a normatively correct concept image meant to replace the others : following Pinto and Tall (2002), a partial or perceptually anchored concept image can itself be a productive resource if it supports reconstruction of the formal definition when the task requires it, so the pedagogical target implied by our data is coordination and reconstructive capacity, not replacement of the perceptual by the theoretical.

The evidentiary standard appropriate to this claim should be made explicit. Case-based qualitative traditions that build emergent theoretical categories from richly analysed exemplars rather than from an unweighted majority pattern across an entire sample, treat such exemplars as adequate grounds for generating a framework, while reserving empirical confirmation for purpose-built subsequent designs (Charmaz, 2006). The four predictions listed above, rather than the present dataset, constitute the confirmatory step this standard calls for.

This is where the stratified model departs from, rather than merely restates, Vinner's (1991) evoked concept image and Duval's (1993, 1995) register theory: Vinner's construct explains that only part of a concept image is activated in a given task but does not specify an internal principle of organisation for the unactivated remainder, whereas Duval's registers concern the coordination of external representational systems (figural, verbal, symbolic) rather than the internal structure of an individual learner's representation. The stratified model proposes a substantive internal principle: that the concept image is organised around distinct modes of validation (ostension, successful action, logical necessity). This principle is falsifiable in the specific sense that it predicts systematic covariation between task type and the stratum recruited (predictions 1–4 above); a finding that perceptual-stratum responses occur equally often across recognition, construction, and proof-type tasks would count as evidence against it. We report this as a theoretical model generated by, and illustrated through, the interview data rather than as a claim already validated by them; the empirical validation called for is precisely the object of these predictions and of the follow-up designs outlined in Section *Limitations*.

The mis-in/mis-out framework of Tsamir and Tirosh (2022) can now be located within the model rather than alongside it. Because each stratum is governed by its own mode of validation, the boundary of the concept is stratum-relative: at the perceptual stratum, validation by ostension admits as mis-in whatever is visibly present, the thickness and boundedness of the drawn stroke that interviewees treated as unproblematic attributes of 'a line' (Section *The Straight Line*), and excludes as mis-out what cannot be shown, most critically infinitude, which becomes criterial only at the theoretical stratum. On this reading, mis-in and mis-out attributions are not idiosyncratic errors but systematic products of the stratum a task recruits a claim our data illustrate without having been designed to test it, and which prediction (4) renders directly examinable. In the same movement, the modes of validation that individuate the strata correspond to Harel and Sowder's (2005) taxonomy of proof schemes: ostension aligns with external and perceptual sources of conviction, successful action with empirical proof schemes, and logical necessity with deductive ones. The stratification hypothesis thereby extends proof-scheme theory from proving to defining, the epistemic criteria by which a learner judges an argument convincing are the same criteria by which a definition is judged adequate at each stratum. **Figure 1** records these two correspondences.

One clarification is worth making explicit, since a reader familiar with Vinner's framework may reasonably ask what the stratification hypothesis adds. The notion of the *evoked* concept image already captures the observation that only a subset of a learner's cognitive structure is activated in any given task. The stratification hypothesis proposes something more specific: that this subset is not drawn randomly from a homogeneous pool of elements, but that the concept image is itself internally layered, with each stratum organised around a distinct logic of validation. The perceptual stratum is validated by ostension "I see that." The procedural stratum is validated by successful action "it works when I do it this way." The theoretical stratum is validated by logical necessity "it must be so, because." These are not merely different pieces of knowledge; they are, in a loose sense, different epistemological registers nested within a single cognitive structure. The consequence is that task format does not merely select which content is activated it selects which *kind* of knowing is recruited. A definition-writing task tends to engage the theoretical stratum; a construction task tends to engage the procedural; a recognition task may engage the perceptual or the theoretical depending on the salience of the figures presented. This is why E47 could "know" that a line is infinite in one sense while drawing it as a segment in another: the two representations belong to different strata, each internally coherent, but not yet coordinated with one another. Whether this coordination can be deliberately fostered through instructional design and what forms of task sequencing might best achieve it remains, in our view, the central open question this study raises.

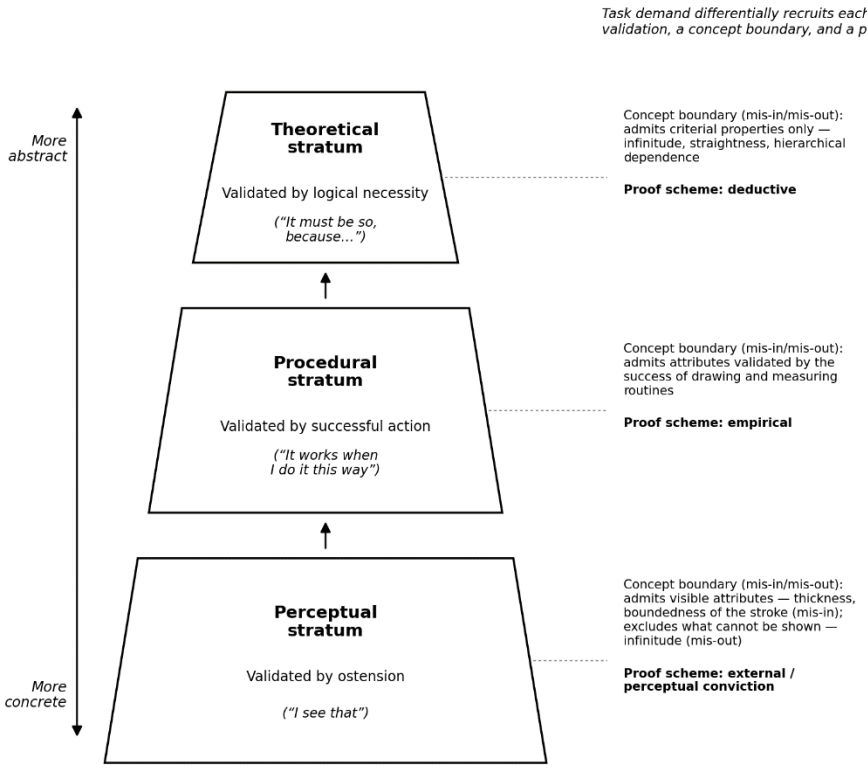


Figure 1. The stratified model of the concept image: modes of validation, stratum-relative concept boundaries (mis-in/mis-out; Tsamir & Tirosh, 2022), and corresponding proof schemes (Harel & Sowder, 2005)

The Differential Conceptualisation of Parallelism

The stark asymmetry between results for the line and segment (0% formal definitions) and for parallelism (39.1%) calls for explanation. Three mechanisms, individually plausible and mutually reinforcing, appear to be at work.

First, *semiotic congruence* (Duval, 1995): parallelism is a relation that can be perceptually apprehended two lines visibly do not intersect and maintain a constant spacing whereas the infiniteness of a line and the straightness of both line and segment are properties that must be inferred beyond what is represented. Parallelism thus benefits from direct alignment between what is visible and what is mathematically relevant, a condition that the line and segment definitions do not enjoy.

Second, *definitional autonomy*: conceptualising parallelism does not require a formally adequate concept image of the line. A minimal operational understanding suffices, and the definition can stand independently. The segment definition, by contrast, presupposes the line and the strong correlation between definition quality for lines and segments ($\rho = 0.68$) suggests that this dependence is not merely logical but cognitive.

Third, *didactic and social salience*: parallelism is explicitly taught across multiple instructional sequences (perpendicularity, parallelograms, Thales' theorem), frequently assessed, and reinforced by everyday visual experience (railway tracks, ruled paper). The line, despite its theoretical primacy, functions as a didactic background object continuously used, rarely thematised. This invisibility may explain why repeated exposure does not generate conceptualisation: there is no instructional moment at which the concept image of the line is treated as an object of explicit reconstruction.

This convergent explanation has comparative implications. It predicts that concepts characterised by semiotic congruence, definitional autonomy, and high didactic salience will generally yield better conceptualisation outcomes a hypothesis that could be tested for perpendicularity, angles, or polygons in future studies.

Epistemological Obstacles and Conceptual Contamination

Two structural mechanisms, beyond individual cognitive profiles, appear to sustain the observed confusions. The first is what Brousseau (2002) would identify as an epistemological obstacle: the infinite straight line, irreducible to perceptual experience, conflicts with its only available material representation a bounded stroke. No amount of teacher explanation that "the line continues beyond the board" can fully override the perceptual evidence that the drawn stroke ends. This contradiction between perception and theory is not a pedagogical accident; it is constitutive of the object itself, and it explains the durability of the line/segment confusion across different instructional contexts (Gobert, 2007; Parzysz, 1988). We take seriously the possibility that what we describe reflects a didactical rather than a strictly epistemological obstacle in Brousseau's (2002) sense, since a representational convention such as systematically marking segment endpoints might in principle resolve much of the confusion. Following Harel and Sowder's (2005) observation that epistemological and didactical obstacles need not be mutually exclusive,

we treat the line/segment confusion as having a didactical entry point (the representational convention) layered onto a more resistant epistemological core (the impossibility of depicting an unbounded object at all) a distinction that follows from, rather than contradicts, the stratified account: the didactical lever operates most directly on the procedural stratum, while the epistemological difficulty concerns the theoretical stratum's dependence on a property that cannot be ostended.

The second mechanism is what the data suggest calling *conceptual contamination:* the confusion is bidirectional (see Section *The Straight Line - The Line Segment*): a substantial minority define the line as a segment, and the large majority define the segment as a "limited line." This symmetrical entanglement indicates that the two objects are not cognitively independent a partial or distorted concept image of one propagates instability to the other. From a Duvalian perspective, this may reflect the failure to coordinate figural and verbal registers for each object: students share graphical representations across line and segment without the semiotic resources to mark the distinction within their concept images. The systemic character of this interdependence suggests that remedial instruction targeting one concept in isolation may be insufficient; the line–segment distinction may need to be explicitly thematised as a relational pair.

These findings resonate with international research on proof and definitional activity. As Hadas et al. (2002) and Sowder and Harel (2003) have shown, the persistence of perceptual concept images, uncoordinated with formal definitions, directly impedes the transition from empirical to deductive reasoning a transition that is central to the geometry curriculum at this level. The specific obstacles documented here, though observed in a Cameroonian setting, appear to reflect mechanisms documented across Francophone educational contexts (France, Québec) and likely beyond, pointing to a genuinely epistemological rather than locally pedagogical difficulty.

Limitations

Three limitations should be held in view when reading these findings. As already noted (see Section *Design and Participants*), the two-school urban sample restricts generalisability to rural, private, and differently resourced populations. The cross-sectional design provides a snapshot rather than a trajectory; whether targeted instruction on definitional reconstruction produces sustained conceptual change requires longitudinal investigation. Finally, the definition-writing task activates primarily the verbal layer of the concept image; recognition and construction tasks are needed to access procedural and perceptual strata more directly. These limitations define the agenda for follow-up studies rather than undermining the exploratory value of the present results.

These findings are presented as analytically, not statistically, generalisable : they indicate mechanisms (semiotic congruence, definitional autonomy, didactic salience, and the perceptual/procedural/theoretical stratification) that may inform analysis in other settings, rather than results expected to replicate numerically elsewhere. Comparative statements elsewhere in the Discussion regarding resonance with research from other Francophone contexts are offered as hypotheses for future cross-national work, not as evidence of an already-established universal phenomenon. A further limitation concerns the significance attached to the absence of formal definitions of the line and segment. Because the definition-writing task did not, unlike Vinner and Dreyfus (1989), pair definition elicitation with tasks whose correct solution required deploying the formal definition, we cannot determine from this design alone whether the observed gap reflects a genuine conceptual limitation or simply the limited functional need for these definitions in typical secondary-level problem-solving; this methodological asymmetry with the classical concept-image literature should be borne in mind when interpreting the results.

Didactic Implications

The findings point towards several directions for instructional design, though these should be understood as hypotheses awaiting empirical evaluation rather than prescriptions. The most direct implication is that definitional reconstruction allowing students to formulate, contest, and refine definitions collaboratively may be more effective than transmission in building the theoretical stratum of the concept image. De Villiers (1998) argues that the cognitive necessity of a definition only emerges through confrontation with examples and counterexamples that reveal the inadequacy of working formulations; our data are consistent with this view.

The explicit thematisation of the drawing–figure distinction what can be inferred from a representation versus what must be stipulated by definition appears particularly necessary for the line and segment, where semiotic incongruence is most severe. Varying the orientation, scale, and medium of representations may help deconstruct visual prototypes (horizontal parallels, bounded strokes) that currently anchor concept images to contingent perceptual features. In teacher education, attention to lexical precision and to the multiple meanings of terms such as *droite* in French mathematical discourse seems warranted.

More concretely, three types of activity appear directly indicated by the data. First, definitional-reconstruction tasks in which students test candidate definitions some formally adequate, some perceptually plausible but incomplete, comparable to those produced by our respondents against systematically varied drawings (bounded strokes, arrows, unusual orientations), directly targeting the perceptual/theoretical coordination gap documented in Section *The Straight Line*. Second, for the line–segment confusion specifically, paired tasks requiring students to justify in writing why a given figure is a segment and not a line (and vice versa) before any measurement is taken, addressing the conceptual contamination described in Section *Epistemological Obstacles and Conceptual Contamination*. Third, for teacher education, transcript excerpts such as those from E47 and E74 (Sections *The Straight Line* and *Parallelism*) could themselves serve as case-study prompts in methods courses, since they concretely illustrate how a definitionally adequate representation can coexist, unrecognised by the student, with an incompatible perceptual one.

CONCLUSION

This study investigates how students construct concept images of foundational geometric objects when definitional reconstruction remains weakly institutionalized in classroom practice. The results confirm a pervasive and analytically structured difficulty: students' representations are organised around perceptually accessible properties, and the theoretical layer of the concept image which formal definitions require remains largely inaccessible in standard task conditions. The differential between parallelism (39.1% formal definitions) and the line or segment (0%) appears attributable to semiotic congruence, definitional autonomy, and didactic salience rather than to inherent cognitive difficulty, suggesting that these factors are promising targets for instructional intervention.

The findings suggest that students' concept images may be analytically described through partially differentiated perceptual, procedural, and theoretical layers. An important implication of these findings is that students may mobilize different dimensions of their concept images according to task conditions, representational formats, and institutional expectations. Unlike Houdement and Kuzniak's institutional paradigms, it addresses the internal organisation of an individual learner's representations. The value of this refinement is both analytic and empirical: it generates four testable predictions about the conditions under which different cognitive layers become accessible, and it directs diagnostic approaches beyond the correct/incorrect dichotomy toward the qualitative structure of learners' representations.

Whether these cognitive patterns are peculiar to students with limited exposure to definitional activity, or whether they characterise a broader developmental trajectory that more intensive instruction can redirect, remains an open question. Longitudinal studies comparing students who engage in systematic definitional reconstruction with those following a transmissive approach would be the most direct way to answer it. The findings suggest that difficulties in coordinating perceptual and definitional dimensions of geometric objects may not be specific to this context, but could reflect broader tensions inherent in the teaching of deductive geometry. In sum, the principal contribution is empirical (a documented differential across three related geometric objects), theoretical (the stratified account and its four testable predictions), and methodological (a definition-writing protocol combined with confrontation interviews); the sample size, cross-sectional design, and single-task focus discussed in Section *Limitations* bound the strength of these claims, and the four predictions define the most direct route to testing them further.

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AI statement: The author stated that DeepL translation was used solely to translate selected excerpts from French into English. No AI-generated content was used beyond translation.

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REFERENCES

Bachelard, G. (1938). *La formation de l'esprit scientifique: Contribution à une psychanalyse de la connaissance objective* [The formation of the scientific mind: Contribution to a psychoanalysis of objective knowledge]. J. Vrin.

Brousseau, G. (2002). *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970–1990*. Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47211-2

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative analysis*. Sage.

Confrey, J., & Kazak, S. (2006). A thirty-year reflection on constructivism in mathematics education in PME. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 305–345). Sense Publishers.

De Villiers, M. (1998). To teach definitions in geometry or teach to define? In *Proceedings of the Twenty-Second International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 248–255).

Durand-Guerrier, V., & Herault, J. L. (2006). Définition et règle, le mythe de la transparence en géométrie [Definition and rule: The myth of transparency in geometry]. In *Interactions verbales, didactiques et apprentissage* (pp. 139–167).

Durand-Guerrier, V., Hausberger, T., & Spitalas, C. (2015). Définitions et exemples: Prérequis pour l'apprentissage de l'algèbre moderne [Definitions and examples: Prerequisites for learning modern algebra]. *Annales de Didactique et de Sciences Cognitives, 20*, 101–148.

Duval, R. (1993). Registres de représentation sémiotique [Registers of semiotic representation]. *Annales de Didactique et de Sciences Cognitives, 5*, 37–65.

Duval, R. (1995). *Sémiosis et pensée humaine: Registres sémiotiques et apprentissages intellectuels* [Semiosis and human thought: Semiotic registers and intellectual learning] (Vol. 4). Peter Lang.

Ely, R. (2010). Nonstandard student conceptions about infinitesimals. *Journal for Research in Mathematics Education, 41*(2), 117–146.

Freudenthal, H. (2012). *Mathematics as an educational task*. Springer. https://doi.org/10.1007/978-94-010-2903-2

Gobert, S. (2007). Conditions nécessaires à l'usage des dessins en géométrie déductive [Conditions necessary for the use of drawings in deductive geometry]. *Petit x, 74*, 34–59.

Hadas, N., Hershkowitz, R., & Schwarz, B. B. (2002). Analyses of activity design in geometry in the light of student actions. *Canadian Journal of Science, Mathematics and Technology Education, 2*(4), 529–552. https://doi.org/10.1080/14926150209556539

Harel, G., & Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical Thinking and Learning, 7*(1), 27–50. https://doi.org/10.1207/s15327833mtl0701_3

Houdement, C., & Kuzniak, A. (2006). Paradigmes géométriques et enseignement de la géométrie [Geometric paradigms and the teaching of geometry]. *Annales de Didactique et de Sciences Cognitives, 11*, 175–193.

Laborde, C., & Capponi, B. (1994). Cabri-géomètre constituant d'un milieu pour l'apprentissage de la notion de figure géométrique [Cabri-Géomètre as a component of an environment for learning the concept of a geometric figure]. *Recherches en Didactique des Mathématiques, 14*(1–2), 165–210.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Routledge.

Njomgang Ngansop, J., & Tchonang Youkap, P. (2018). La ligne droite, un objet d'étude au début du secondaire : Une analyse institutionnelle des manuels [The straight line as an object of study in early secondary education: An institutional analysis of textbooks]. *Revue de Mathématiques pour l'école, 230*, 23–29. https://doi.org/10.26034/vd.rm.2018.1777

Parzysz, B. (1988). "Knowing" vs. "seeing": Problems of the plane representation of space geometry figures. *Educational Studies in Mathematics, 19*(1), 79–92. https://doi.org/10.1007/BF00428386

Pinto, M., & Tall, D. (2002). Building formal mathematics on visual imagery: A case study and a theory. *For the Learning of Mathematics, 22*(1), 2–10.

Shir, K., & Zaslavsky, O. (2002). Students' conceptions of an acceptable geometric definition. In *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 201–208).

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20–26.

Sowder, L., & Harel, G. (2003). Case studies of mathematics majors' proof understanding, production, and appreciation. *Canadian Journal of Science, Mathematics and Technology Education, 3*(2), 251–267.

Tall, D., & Vinner, S. (1981). Concept images and concept definitions in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169. https://doi.org/10.1007/BF00305619

Tanguay, D., & Geeraerts, L. (2012). D'une géométrie du perceptible à une géométrie déductive: À la recherche du paradigme manquant [From a geometry of the perceptible to a deductive geometry: In search of the missing paradigm]. *Petit x, 88*, 5–24.

Tchonang Youkap, P., Njomgang Ngansop, J., Tieudjo, D., & Pedemonte, B. (2020). The introduction of proof at the secondary school in Cameroun: A first approach trough the study of quadrilaterals and triangles in the textbook. *International Electronic Journal of Mathematics Education, 15*(3), Article em0599. https://doi.org/10.29333/iejme/8404

Tsamir, P., & Tirosh, D. (2022). Mis-in and mis-out concept images: The case of even numbers. *Educational Studies in Mathematics, 112*(2), 207–224. https://doi.org/10.1007/s10649-022-10183-z

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–81). Kluwer Academic Publishers.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356–366. https://doi.org/10.2307/749441