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Fostering mathematical thinking through a computer algebra system in a differential equation course

Fereshteh Zeynivandnezhad [1*], Ramón Emilio Fernández [2], Yudariah binti Mohammad Yusof [3], Zaleha binti Ismail [4]

[1] Organization for Educational Research and Planning, Bureau of Compiling TVE and Kar-Danesh Textbook, IRAN
[2] Department of Mathematics, Pace University, USA
[3] Universiti Teknologi Malaysia, Faculty of Science, Johor, MALAYSIA
[4] Universiti Teknologi Malaysia, Faculty of Education, Johor, MALAYSIA

***Corresponding Author:** fzeynivand@gmail.com

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ABSTRACT

This study explores the effects of a computer algebra system on students' mathematical thinking. Mathematical thinking is identified with mathematical thinking powers and structures. We define mathematical thinking as students' capacity to specialize and generalize their previous knowledge to solve new mathematical problems. The study was conducted in three phases: a preliminary study, a teaching experiment (main study), and task-based interviews (follow-up study). In the first phase, we intended to get to know students' levels of mathematical thinking; the second phase sought to promote students' mathematical thinking; and the final phase was designed to help us identify the enhancement of students' usage of their mathematical thinking powers. A test was conducted at the preliminary study, a teaching experiment was run at the main study, and task-based interviews, like those in the main study, were conducted in the follow up phase. The main study's participants were part of an undergraduate differential equations class in Malaysia. The worksheets used in the main and follow up studies were designed by the researchers, based on the instrumental genesis, prompts, and questions to be used in the teaching experiment sessions. Qualitative data analyses showed that using a computer algebra system for learning differential equations had a positive impact on the development, identification, and usage of students' mathematical thinking. Moreover, it was revealed that the students applied specializing powers, imagining, expressing, changing, varying, comparing, sorting, organizing, and checking the calculation in general to make sense of mathematical structures. The findings could be incorporated not only in the mathematics curriculum at the tertiary level but could also be extended to k-12 schools.

Keywords: computer algebra system, differential equations, mathematical structures, mathematical thinking, mathematical thinking powers

INTRODUCTION

Mastering both new mathematical content and processes, such as mathematical thinking, is crucial (Breen & O'Shea, 2011; Devlin, 2012; NCTM, 2000). Prior research shows that using information technologies such as a computer algebra system in courses like differential equations can facilitate the conceptual learning and visualizations of a variety of mathematical problems (Paraskakis, 2003; Zeynivandnezhad & Bates, 2018; Zeynivandnezhad et al., 2013). In mathematics classroom, particularly when working on differential equations, the knowledge of sequential steps or actions dominate the problem-solving strategies employed by students (Zeynivandnezhad & Bates, 2018). Nonetheless, conceptual learning emphasizes the comprehension of mathematical concepts, operations and relationships. While the traditional (chalk and board) teaching of differential equations requires strict procedural practice, computer algebra system can help instructors build interactive, dynamic, student-centered technologically enriched curricula (Artigue, 1992; Kwon, 2020). Furthermore, mathematical processes and powers can be promoted and enhanced by CASs capabilities, such as the integration of prompts and probing questions, to provide an exploratory environment for students, while they work on tasks (Dubinsky & Tall, 2002; Soboleva et al., 2020). Nonetheless, the literature on the integration of mathematical thinking into the teaching of differential equations using computer software such as CAS is at best, scant (Raychaudhuri, 2008; Zeynivandnezhad & Bates, 2018). Based on our research, to date, there is no manuscript detailing students' mathematical thinking powers based on the integration of information technologies such as CASs in the classroom.

CASs capabilities can promote mathematical thinking power usages (Zeynivandnezhad & Bates, 2018). Despite consistent calls to incorporate digital technologies into our educational systems (Lockwood & Mørken, 2021), there is little evidence or consensus of whether or how CAS enhances students' mathematical thinking processes in detail, or about how CAS can foster Mathematical Thinking (MT). In the present study, the authors seek to take advantage of the dynamism and speed of CAS in calculating, visualizing, and exploring differential equations. The primary aim is to uncover how CAS can help instructors to enhance the mathematical thinking power of students through the following question:

RQ To what extent can mathematical thinking processes be fostered while students work with a computer algebra system in differential equations courses?

Mathematical Thinking

Philosophies regarding mathematical thinking are several, and diverse. *The Nature of Mathematical Thinking*, by Sternberg and Ben-Zeev (1996) is a reference that describes these views. However, within mathematics education, Mason et al. (2010) identify mathematical thinking as processes and structures. They believe that people use mathematical thinking powers to make sense of mathematical structures (Roselainy et al., 2012; Zeynivandnezhad, 2016; Zeynivandnezhad & Bates, 2018). Within the context of this study, *Mathematical Thinking* refers to the general use of mathematical themes and mathematical powers or processes, including Habits of Mind (Cuoco et al., 1996) and Concept Images (Tall & Vinner, 1981). As presented by Mason (2014), mathematical thinking is underlined by students' capacity to specialize, generalize, conjecture, and convince first themselves, as well as later others. They include imagining and expressing, specializing & generalizing, conjecturing & convincing, organizing & characterizing. As is explained in the theoretical framework, mathematical themes include invariance in the midst of change; freedom & constraint; doing & undoing mathematical structures, or mathematical relationships, including, for example, group axioms & group theory; rings etc.; but also relationships like $F(n+1) = F(n) + F(n-1)$ for Fibonacci numbers, $Sn = \left(\frac{a(r^n-1)}{r-1}\right)$ for geometric progressions (Mason, 2024). It is worth noting that the role of technology in mathematical thinking is also a challenge for mathematics education (Hansson, 2020; Sam & Yong, 2006; Zeynivandnezhad et al., 2023). Technology, particularly digital technologies, have some characteristics that can be applied to promote mathematical thinking.

Teaching and Learning Differential Equations Through a Computer Algebra System

Studies have addressed several issues related to students' mathematical thinking aspects in differential equations, such as understanding equilibrium solution functions (Kwon, 2020; Kwon et al., 2005), students' reasoning for solutions to Differential equations (Artigue, 1992), the DE classroom and learning in a social environment (Allen, 2006), the abilities of students in translating information from symbolic into graphical (Arslan, 2010; Habre, 2012; Kwon et al., 2005), and using technology (Habre, 2012; Maat & Zakaria, 2011). Habre (2012) believed that the emphasis on procedural techniques in DE classrooms fails to provide students with opportunities to experiment with alternative techniques to solve non-textbook, real-world differential equations. Another area of struggle is students' understanding of ordinary differential equations using traditional and computer-assisted environments (Arslan, 2010; Habre, 2012; Kwon et al., 2005). According to Watson (2001), most mathematical thinking processes are related to conceptual understanding in mathematics. Several scholars indicate that using computers for teaching mathematical concepts has ample potential in supporting mathematical thinking (Santos-Trigo et al., 2021; Soboleva et al., 2020; Trouche, 2016; Zeynivandnezhad & Bates, 2018; Zeynivandnezhad et al., 2020). Studies have indicated that computers are useful at any of the mathematical thinking stages, especially in conjecturing (Dubinsky & Tall, 2002; Zeynivandnezhad & Bates, 2018). For example, students could demonstrate and expand upon various problem-solving strategies including simpler cases, dragging orderly objects, measuring the attributes of objects, and identifying the loci of certain objects, which influence their reasoning and problem-solving methods (Santos-Trigo et al., 2021).

THEORETICAL FRAMEWORK OF THE RESEARCH

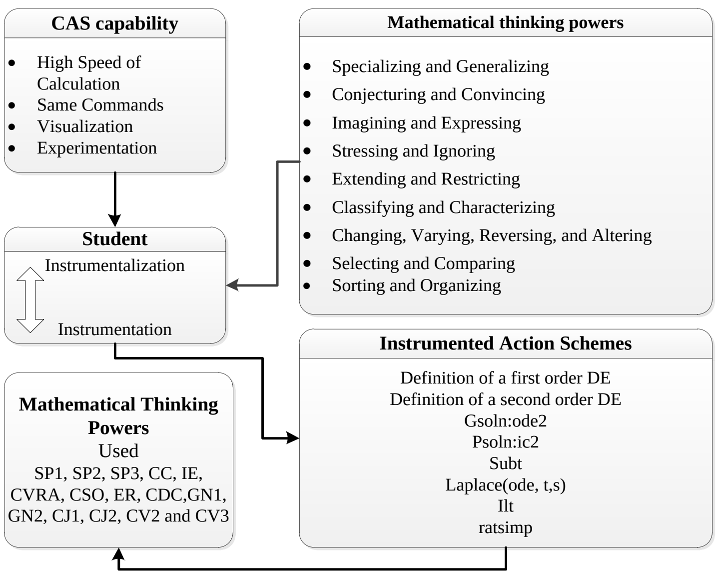


Figure 1. Conceptual framework of the study (Zeynivandnezhad & Bates, 2018)

Table 1. Mathematical questions and prompts (Adapted from Mason, 2000)

Element for asking	Type	Examples of Prompts and Questions
Mathematical themes	Invariance amidst change	What is variable and what is constant as you employ the technique? What are the most important characteristics of general solutions? What are the differences and similarities between the general solution and the solution? (Compare and contrast the general and more specific solutions).
	(Direct and backwards building) Doing and undoing	The solution to a DE is given, what is the corresponding DE to this solution?
	Freedom and constraints	Find the general solution of a DE with initial value $y'(0) = 1$, and $y(0) = 1$.
Mathematics powers	Stressing and ignoring; specializing and generalizing; conjecturing and convincing;	Show that every member of the family of functions $y = \frac{c}{x} + 2$ is a solution of the first-order DE $\frac{dy}{dx} = \frac{1}{x}(2 - y)$.
Mathematical heuristics	A process of scaffolding and fading, reflecting on the prompts, questions, and actions taken to find the solution.	(What are the similarities and differences between given Differential Equations and others not given in terms of how we solve them? Justify it. What is same and what is different between the given Differential Equations and others in terms of solving Differential Equations? Are you sure your solution is correct? Justify it.

A version of this table appears on Zeynivandnezhad and Bates (2018) and was adapted from Mason (2000)

Figure 1 shows the theoretical framework of the research in relation to how to create prompts and questions within a computer algebra system's environment and capabilities such as high speed of calculation and visualization. These were considered through instrumentation theory. Postulating questions, conjectures, and well-informed argumentative justifications are essential to the creation of an environment that allows for the display of students' mathematical thinking powers (Mason, 2014; Wiggins & McTighe, 2005). We follow (Zeynivandnezhad & Bates, 2018) in the creation of prompts and questions employed to understand and explain students' mathematical thinking, anchored on various guidance, which include action themes and heuristics, as well as mathematical themes outlined by Mason (2000). Mathematical heuristics are learned through a process of scaffolding and fading. This involves reflecting on prompts, questions and actions that enable finding solutions. Reinforcement occurs by imagining oneself acting similarly in the future. By promoting the actions of specializing and generalizing, students can better utilize their own mathematical reasoning abilities by means of their prior mathematical knowledge. This helps them make sense of new mathematical concepts and to understand new mathematical situations. The research team hypothesized that using prompts and questions would help identify students' use of mathematical thinking (MT) powers. Additionally, we conjectured that connecting a Computer Algebra System (CAS) environment during intentionally designed activities would enhance students' MT. One of the driving forces behind the creation of prompts and questions was computer algebra system's capabilities such as high-speed calculation and visualization. For instance, the prompts in few tables were grounded on high-speed calculations through the same commands in Maxima environment. Students were also allowed to graph solutions to visualize the behaviours of differential equations. Overall, the prompts and questions aided students to comfortably shift between graphical and symbolic forms of solutions (see **Table 1**).

As summarized in Zeynivandnezhad and Bates (2018), invariance is a general topic that interconnects various important theorems in mathematics. For example, a cardinal number is invariant under a finite countable set. In the world of Differential Equations, invariant is an important concept; for example, differentiability is invariant under both, addition, and scalar multiplication. Direct and backwards building can be very challenging and rewarding. It creates a space where mathematical thinking flourishes by having the student wonder, for example, can the answer lead me back to the original problem? Exercises of this kind are integral to the fostering of MT in problem solving (Mason, 2000).

Research suggests that graphically embedded materials aid students in the modelling, remodelling, and deciphering of mathematical problems (Marshall et al., 2012; Salleh & Zakaria, 2011). Following this consensus, we placed questions and prompts (**Figure 1**) in boxes as suited, separately, or embedded, based on the activity (digital in a lab, or handwritten) (Zeynivandnezhad et al.., 2013). The duality between instrumentalization and instrumentation (**Figure 1**) allows us to assess the building schemes through which students solve differential equations and justify their solutions. The process of *instrumentation* consists of the development and evolution of schemes used by individuals to execute specific tasks. In the instrumentalization process, one tries to shape tools and their functionalities to adapt them or shape them for use. In tandem, these lead to instrumental genesis, the co-evolving technical and conceptual components of mental schemes (Drijvers & Trouche, 2008). The technical skills, coupled with the required understanding to utilize CAS for tasks, are known as a utilization scheme (Drijvers & Trouche, 2008).

Two types of utilization schemes are identified in connection with an artefact assigned to a specific task: usage scheme and instrumented action schemes (Drijvers & Trouche, 2008). The usage scheme of an artefact deals with how the artefact is managed. For instance, switching on a calculator, adjusting screen contrast, and selecting a specific key. The instrumented action scheme involves performing a particular task, such as calculating a function's derivative (Guin, 2005). For example, the drag mode in a dynamic geometry setting can be seen as a tool for recognizing the geometric properties of a shape. Students must grasp both how to move points (instrumentalization) and why it's essential to do so (instrumentation) to fully comprehend the conceptualization of geometric properties (Goos et al., 2010). While detailed examples of schemes are infrequent (Drijvers, 2015; Jupri et al., 2016) outlined two schemes.

METHODOLOGY

We used a qualitative approach to better understand the complex phenomenon that is mathematical thinking. **Figure 2** shows the research procedures, including *preliminary investigations*, the *main study* (the teaching experiment sessions in Differential Equations

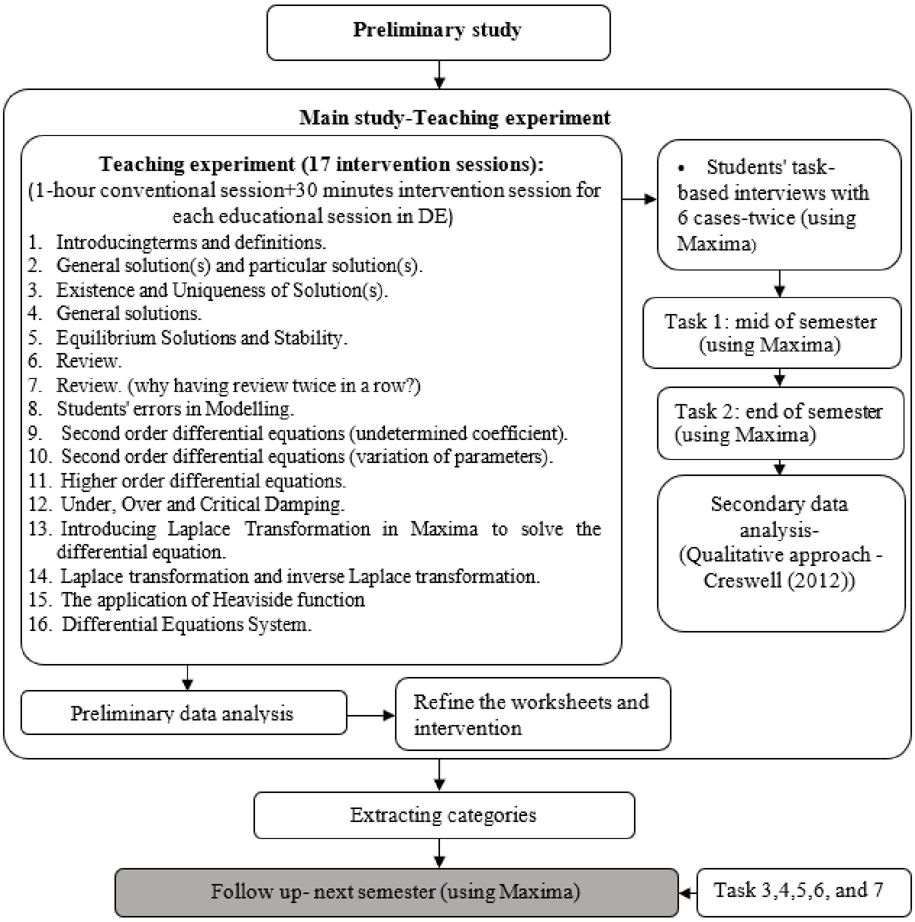


Figure 2. Research procedure (Zeynivandnezhad, 2014)

using Maxima as a CAS), and *follow-up interviews*. In the preliminary study, the level of students' mathematical thinking powers was identified. To do so, we recruited students who had recently passed a DE course and had them complete a survey. In this stage of the study, 51 students, pre-service teachers, took part in the study and answered the questions.

The main study, the teaching experiment method (Cobb, 2000), took place in a Chemical Engineering classroom; it allowed us to experience students' mathematical learning and reasoning. Because of her familiarity with the mathematical thinking approach, we chose the class for a member of the research team. We covered the concepts of undergraduate ordinary differential equations during the 17 teaching experiment sessions using a mathematical thinking approach aided with a computer algebra system. We then followed up by means of interviews. In the follow up stage, 6 participants who participated in the main study were chosen to take part in task-based interviews. Therefore, the participants of the preliminary study were different from those of the main study, as well as those from the follow-up study.

The preliminary study's aim was to identify the mathematical thinking powers displayed by students in taking differential equations courses in traditional environments, e.g., not using technologies such as computer algebra systems. The set of items was distributed to 51 students (pre-service teachers) to be answered in an hour and 10 minutes. We employed a test with the aim of determining some aspects of students' mathematical thinking powers (Mason, 2014) in DE courses. The test has three questions, all of which have several items which have their selection based on thinking mathematically by Mason et al. (2010). The specifics about this process are found in Zeynivandnezhad et al. (2013). The main study (teaching experiments) consisted of seventeen intervention sessions, covering various differential equations topics such as first order differential equations, second order differential equations with constant coefficients, and Laplace transformations. The teaching experiment served as an experiment to help us inquire about teaching-researching questions, such as the nature of mathematical learning, how mathematical thinking develops, and other compelling matters related to pedagogy and learning (Czarnocha & Maj, 2008). All the interventions were observed by the same member of the research team. Based on the nature of this experiment, our study is intrinsically qualitative, and we characterize it as a natural inquiry (Creswell, 2012; Moschkovich & Brenner, 2000).

Participants in Main and Follow up Studies

We selected a differential equations class of 37 Chemical Engineering students (the same students in the main study). According to their performance, six students were chosen to participate in task-based interviews during the main study and the follow-up study. They took part in task-based interviews for the follow-up study after finishing the teaching experiment study the following semester. The researchers wanted to see the development of students' mathematical thinking without the students worrying about exams. The in-depth interview participants were chosen based on their scores in the exam, assignments, quizzes, and communication during the class intervention (i.e., teaching experiments). Task-based interview participants (**Table 2**) were classified as follows: John and Ckin, lower

Table 2. Student's demography in the main study

No	Student	Gender	Engineering Mathematics I	Engineering Mathematics II
1	Ade	Female	A	A+
2	Ckin	Female	B+	B+
3	John	Male	B-	C+
4	Philip	Male	A+	A+
5	Una	Female	B	A
6	Wendy	Female	A+	A

achievers; Philip and Ade, higher achievers; and Una and Wendy, middle achievers. The researcher-teacher, based on the reactions in the teaching experiments sessions, classified the students as higher, middle and lower achievers. The distinction between middle and higher achievers was based more on a qualitative than a quantitative measure, such as reactions to the researcher-teacher's questions; as such, Ade was classified as a high achiever with grades of A and A+, but Wendy was classified as a middle achiever with grades A+ and A.

Worksheets Used in the Main Study

The conceptual framework of the study yielded eleven worksheets (See **Appendix A**) to help us foster students' mathematical thinking (Zeynivandnezhad et al., 2013). Based on this framework, we hypothesized that prompts and questions could lead students to use their mathematical thinking powers explicitly, and that these worksheets could, linked to computer algebra systems' environments, promote mathematical thinking (Zeynivandnezhad & Bates, 2018). Below is a list of some of the prompts and questions we employed, hoping they would make students aware of their mathematical thinking powers:

> Show me an example of a first order/second order/third-order differential equation.
>
> What must be added to change the first order differential equation into second order?
>
> What are the similarities and what are the differences between the three Differential Equations?
>
> How do you associate the differential equation with the graphical solution?
>
> How are you sure that your solution is correct?

For example, the following question tries to explain changing, varying, reversing, and altering mathematical thinking powers.

> Show that every member of the family of functions $y = \frac{c}{x} + 2$ is a solution of the first-order differential equation $\frac{dy}{dx} = \frac{1}{x}(2 - y)$.

In sum, the worksheets had two parts: a written (pen/pencil) and a computer lab activity within each problem (Zeynivandnezhad & Bates, 2018). Furthermore, students' written activities were aided by questions and prompts, applied as mathematical heuristics. In contrast, computer lab activities were less direct (e.g., see task 2 in **Appendix A**).

Computer Software: Maxima (wxMaxima)

Maxima (wxMaxima) is an open-source software, computer algebra system, that can be easily installed on any computer software environment. Maxima, the language of which is close to that of mathematics in differential equations, can be used to promote mathematical thinking (Zeynivandnezhad & Bates, 2018). The capabilities of Maxima (symbolical, graphical, and numerical) are like those of other computer algebra systems, e.g., Maple, Mathematica, for solving and drawing Differential Equations functions.

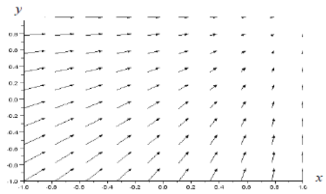
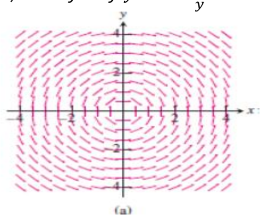
Tasks Used in the Follow-up Study

Table 3 shows the tasks used in the main study and in the follow up study. We used several tasks in the follow-up study to identify the enhancement of students' usage of their mathematical thinking powers; namely, Task 3, Task 4, Task 5, Task 6, and Task 7 (See **Table 3**). Task 3, in the follow-up study, emphasized the specializing powers to identify the first-order differential equation's facts. It was intended to have students use Maxima's capabilities, such as high calculation speed, to see the changes without any intermediate steps. The expectation in Task 4 was to observe how students make conjectures to solve higher-order differential equations while using Maxima.

Table 3. Tasks used in the main study and follow up study

	Session	No. of Task	Task description	Topic
Task based interview in main study	First session: 8^{th} week of the semester	1	II) Computer Lab activity II-1*: Draw the graph* of the solutions to the given differential equation in written activity (I), using Maxima, in which $x(0) = 1, x(0) = 0$. II-2: Using the graph, *identify the under*, over, and critical damping. II- *How are you sure your solutions are correct?* I-1: $x'' + x' + 3x = 0$ I-2: $x'' + 4x' + 3x = 0$ I-3: $x'' + 4x' + 4x = 0$ II-1-a: *What do you know* about the command to get the solutions of a differential equation? II-1-b: *What do you want* to do in the Maxima environment? II-1-c: *How do you* introduce the $x(0) = 1. x'(0) = 0$	Damping situation

Table 3 (Continued). Tasks used in the main study and follow up study

	Session	No. of Task	Task description	Topic
Task based interview in main study	Second session: 12^{th} week of the semester	2	(I) Written activity and Computer Lab activity simultaneously: I-Find a general solution of the following differential equation: $y'' + 5y' + 4y = t, y(0) = 0, y'(0) = 0$ I-1: *Is it a linear differential equation*? If so, *why*? I-2: *What is the order* of the differential equation? I-3: *How do you want to use* Laplace transformation to get the solution to the differential equation? I-4: *How can Maxima* help you to find the solution to the given differential equation? I-5: *How are you sure* your solutions are correct?	Laplace transformation
Task based	First	3	a) *Give me an example* of a first order differential equation. b) *How do you* want to solve your example using Maxima? c) *Change your* first order differential equation to a second order differential equation. d) *How do you solve* the second order differential equation using Maxima?	First -order differential equation
	Second	4	a) Could *you please give an example* of a second order differential equation? *What can be added* to your example to convert it into the following differential equation, $y^{(3)} - 5y^{(2)} + 8y' - 4y = 1$? b) *Could you please solve* the differential equation using Maxima?	Using Laplace second order differential equations
	Third	5	a) *Could you write a real-life problem* in chemical engineering that includes a differential equation? b) *How do you solve* your example using Maxima? *Explain how your* findings are in accordance with real life phenomena in chemical engineering.	Real life problem
		6	 a) *Describe possible solutions* for the following slope field:	Slope fields
		7	a*) Justify why* $y' = -\frac{x}{y}$ has the slope fields are shown in the diagram? 	Slope fields

In task 5, students were asked to give an example of their area of study to inquire into how they see the development of generalizing and conjecturing to find the general equation, then solve it and interpret it using Maxima's capabilities. In task 6, we used undoing themes designed to help students use their mathematical thinking powers to provide a possible graphical solution. In task 7, students were asked to interpret the behaviour of the graph using Maxima. All students' activities were collected for analysis.

Data Analysis in the Main Study and in the Follow up Study

In addition to analyzing the main study's data to modify the worksheets, we used Creswell's procedures (2012) to analyze the qualitative data from interviews. We accomplished this by extracting data (transcribing) from the interviews, as well as the videotaping of the classroom's interventions. All interview's transcriptions were coded line by line according to our codebook (See **Appendix B**). The data were coded through two theories, mathematical thinking powers by Mason et al. (2010) and Instrumental genesis (Drijvers & Trouche, 2008). However, the focus of this experiment is mathematical thinking powers' usage. Therefore, the categories developed by prior studies can support the accumulation and comparison of research findings across multiple studies (Zhang & Wildemuth, 2009). We acquired our coding categories from previous studies, which were then applied deductively to classify the raw data into the categories (See **Appendix B**). Initially, the codebook was developed with Mathematical thinking powers by Mason et al. (2010), based on the book *Thinking Mathematically*. The codebook included the mathematical thinking powers and mathematical structures. This codebook was sent several times to renowned experts in mathematical thinking and was edited to ensure its proper adaptability in using it to analyze the interviews coded (Zeynivandnezhad & Bates, 2018). As a result, these categories were inductively modified within the course of the analysis. The number of all powers used in the follow-up study were counted and summed up to show the students' enhancement of the mathematical thinking powers, as illustrated in the tables for each participant in the interviews.

Trustworthiness

We spent 17 sessions on the main study's research site, and conducted six interviews in the main, and follow up, study. This was done in accordance with Lesh's et al. (2000) credibility enhancement, for us to maximize the quality of our research. The codebook (typical interviews coded) findings were sent to world-renowned experts and two authors of a textbook on differential equations and mathematical thinking; we employed a videotape camera and screen capture (Camasia Studio 7) to save all commands that participants used in the Maxima environment, to improve the dependability technique. Following Lesh et al. (2000), we chose certain strategies that included the four aspects of trustworthiness –credibility, transferability, dependability, and confirmability-, as shown in **Table 4**.

Table 4. Techniques to enhance trustworthiness

Criteria	Technique	Procedure
Credibility	Prolonged engagement	Spending 17 sessions on the research site, conducting 12 interviews in the main study, 18 interviews in the follow up study. Conducting peer debriefing.
Transferability	Persistent observation	Focus on mathematical thinking processes.
	Triangulation	Employing multiple methods for collecting data like interviews, observations, students' written activities, computer activities, and the researcher's notes.
	Member checking	Showing the transcription to respondents.
	Thick description	Explanation to participants, specification of the table for task to solve, difficulty that participants may confront.
	Purposeful sampling	Higher achiever, middle achiever, lower achiever.
Dependability	Audit trail	Sending codebook (typical interviews coded) to a world-renowned expert and two authors of texts on differential equations and mathematical thinking, respectively.
	Multiple researcher	Distinct research members observed the intervention sessions.
	Recording device	Videotape camera, voice recorder, softcopies of students' Maxima files, Camtasia studio 7 trial version software to capture all commands that participants used in Maxima environment
Confirmability	Defining the role of the researcher	The role of the researcher is defined as a teacher-researcher in intervention sessions.

Table 5. An excerpt of double coding procedures to enhance trustworthiness

		Protocol	Coding	Reason for coding	Agree Disagree	Percentage of agreement	Comments of disagreement
		exponential minus equals zero					
175	Researcher	Now could you please solve it					
176	Una	- **Una could not remember the command to** solve at the beginning of the session (searching in Internet frequently)					
177	Researcher	What does the solution of differential equation mean?					
178	Una	Finding of u is the meaning of solution of a differential equation for me	SP2 (facts)	Identify the fact			
179	Researcher	Ok, now please solve it?					
180	Una	(Una after searching in help sheet- copy and paste then change the equation to desire one) `(%i1) de:'diff(u,t)-exp(-1*t)-u=0;` Una pressed Shift and Enter (%o2) $u=\left(\%c-\frac{\%e^{-2t}}{2}\right)\%e^{t}$	SP2 (facts)	Identify the fact			
			SP3 (technique)	Identify the technique to solve			
			IF (representation)	Introducing the symbols			
			CDC (representation)	Selecting appropriate syntax and adding needed information a deleting unnecessary, steps or symbols, completing, correcting the wrong parts			

For example, for checking the inter-coder reliability**,** 385 interview coded transcriptions were randomly chosen (See **Table 5**) and sent to two experts to calculate Cohen's Kappa. The measured Cohen's Kappa of the two coders was .80, which shows a very high strength of agreement (Landis & Koch, 1977). Confirmability was promoted by defining the first author's role as a teacher-researcher in the intervention sessions.

RESULTS

Preliminary Investigation

The test, including three questions and sub-questions (Zeynivandnezhad et al.., 2013) based on Mason et al. (2010), was distributed to both engineering and pre-service teachers in the DE course. The proportion of answers to the differential equation problems were reported in three types of answers - *Right answer, Wrong answer,* and *No answer*. The answers of most participants were related to mathematical thinking levels *exemplifying* and *specializing*. Most unanswered questions were related to higher levels of mathematical thinking, such as *generalizing, conjecturing*, and *convincing* (Zeynivandnezhad et al., 2013).

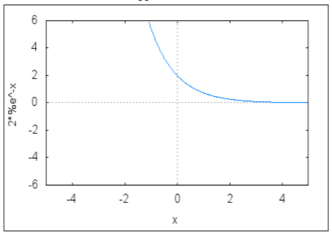
The Main Study

According to previous data (Zeynivandnezhad & Bates, 2018), students used mathematical thinking powers in a zigzag way. It means that they used specializing powers then used generalizing powers visa versa. The results suggest that students were better able to employ mathematical thinking in real-life circumstances rather than in procedural problems. They seemed preoccupied with convincing the researcher that they were able to solve specific differential equations (specializing powers), to later check their calculations to ensure they used correct techniques. The more familiarity they had with Maxima commands, the higher the odds of them being able to use the mathematical thinking powers in the Maxima environment. The computer algebra system environment, through its possibilities, provided opportunities to use mathematical thinking powers more frequently than in the pen and paper environment. The use of mathematical thinking powers to answer research questions is presented below. Details of findings and related argumentation can be found in Zeynivandnezhad and Bates (2018).

Enhancements of Mathematical Thinking Powers

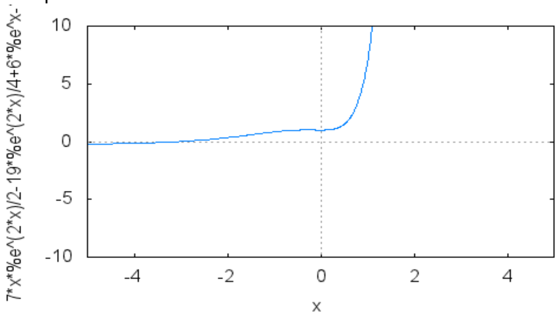
To demonstrate students' mathematical thinking enhancements, follow-up tasks (tasks 3, 4, 5, 6, and 7) were considered. An excerpt of Ade' use of mathematical thinking powers is shown in line 5 (**Table 6**).

Table 6. Ade's use of mathematical thinking powers during the follow-up study

No.		Interview extracts		
1	Researcher	Could you please give me an example of a first order differential equation?		
2	Ade	Can I use the equation from the help sheet?		
3	Researcher	Ok, you can use it.		
4	Researcher	Ade flipped through the help sheet and typed 'diff		
5	Ade	(Ade looked at the help sheet and found the first order differential equation), she copied the command in the help sheet and pasted it in Maxima. `(%i1) eq:'diff(y,x)=-y;` Ade pressed shift and entered. (%o1) $\frac{d}{dx}y=-y$	SP2 (facts) SP3 (technique)	Identify the facts Identify the techniques to solve
6	Researcher	How are you sure this is the first order differential equation?		
7	Ade	Because of the power of d/dx, is one.	CV2 (explanation)	Explain the reason
8	Researcher	Ok, now, could you please solve it using Maxima?		
9	Ade	Ade copied the command that she found in the help sheet and then pasted it onto the Maxima screen. `(%i2) ode2(eq,y,x);` Ade pressed shift and entered. (%o2) $y=\%c\ \%e^{-x}$	SP2 (facts) SP3 (technique)	Identify the facts Identify the techniques to solve
10	Researcher	Now, draw the graph of the solution. (The student has a problem with the range of y to get the solution. Before starting to draw, the researcher showed the student how to draw the graph using selecting the solution of the differential equation and then using the menu bar, plot2d, to change the range of the independent and dependent variables and finally click on ok. The student can use plot2d to draw the graph).		
11	Ade	`(%i3) wxplot2d([2*%e^(-x)], [x,-5,5], [y,-6,6])$` *plot2d: some values were clipped.* (%t3)  (Ade dragged ce^{-x}, used the menu bar, opened the Plot2d, and introduced the value to c and assigned the range to the variables, she replaced c by 2).	SP5 (representation)	Introducing the image and graph
12	Researcher	Now, how are you sure this is the solution of the differential equation? Could you please convince me that this is the solution of a differential equation?		
13	Ade	If you substitute 0 in the equation, y will be 1. It is like an exponential function. If I differentiate this function and put it in the equation, it will satisfy the equation.	CV1 (technique) CV2 (explanation)	Having evidence to get solution Explain the reason for solution or method is correct
14	Researcher	How? For example, differentiation?		
15	Ade	Ade dragged (selected) the general solution that she found using Maxima, then opened Calculus in the menu bar, and clicked on Differentiate and assigned 1 (to order the differentiation) to get the derivative of the general solution `(%i4) diff(%c*%e^(-x),x,1);` Ade clicked on ok. (%o4) $-\%c\ \%e^{-x}$	SP2 (facts) SP3 (technique) SP6 (representation)	Identify the facts. Identify the technique. Introduce the symbols.
16	Researcher	Then?		
17	Ade	I must put in the equation to be held, but I forgot the command for that.	CV2 (explanation)	Explain and communicate the reason about correctness.
18	Researcher	I will help you; it is *ratsimp*, you can explore more in the help sheet.		

Task-follow up study	SP1	SP2	SP3	CVRA	IE	CC	ER	CDC	CSO	GN1	GN2	CJ1	CJ2	CV1	CV2	CV3
Ade	0	24	13	4	28	2	2	16	2	8	2	3	0	5	11	0

Table 7. Ckin' s use of mathematical thinking powers during the follow-up study

No.		Interview extracts		
19	Researcher	Draw the graph		
20	Ckin	Ckin selected the solution and opened the menu and entered the range for the variables `--> wxplot2d([(7*x*%e^(2*x))/2-(19*%e^(2*x))/4+6*%e^x-1/4], [x,-5,5], [y,-10,10])$` Ckin pressed Shift and Enter.	SP6 (representation)	Introducing symbols.
			SP5 (representation)	Introducing image.
21	Researcher	Are you sure the graph is correct?		
22	Ckin	I am quite sure.		
23	Researcher	Why?		
24	Ckin	If I substitute 0, y is $-1/4$.	CV1 (links)	Have evidence.

Task-follow up study	SP1	SP2	SP3	CVRA	IE	CC	ER	CDC	CSO	GN1	GN2	CJ1	CJ2	CV1	CV2	CV3
Ckin	0	44	17	1	39	2	3	7	4	12	0	2	1	8	3	0

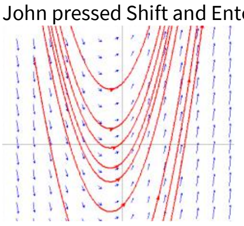
Ade used specializing powers to identify the facts related to the independent, dependent, linear, or nonlinear differential equations in the tasks. She used *to imagine* and *express* and *specializing* powers more than other powers. While her dominant powers are *specializing*, *imagining*, and *expressing*, Ade struggled cognitively to increase her use of *convincing* powers during the interviews. Ade did not read the question loudly. She promoted the use of the *imagining* and *expressing* powers, specifically using a graph for solving and interpreting the solution for a differential equation in the tasks of the follow-up study (IE = 28). She also promoted the use of specializing powers, such as identifying the facts related to the specific differential equation, including the independent and dependent variables (SP2 = 24), and the way to obtain the solutions (SP3 = 13). Comparing the first and second interviews, Ade improved the use of Maxima commands to get the solution (CDC = 16). She could find the appropriate commands and then used *completing*, *deleting*, and *correcting* (CDC) powers to get the solution using Maxima. She took advantage of Maxima's capabilities, such as visualization, to use convincing powers (CV1 = 5). She also developed the use of *characterization* and *classification* (CC = 2). Therefore, she enhanced her use of Maxima's capabilities to obtain evidence to show that the solutions were correct. Consequently, she developed her explanation and communication of the correctness of the solution (CV2 = 11). CJ2 and CV3 were 0 since Ade did not present any check the consequences and justify them to prove the conjectures.

Ckin behaved similarly to Ade; however, her frequency of introducing symbols in the Maxima and pen-paper environments was higher compared with other mathematical thinking powers (IE = 39) (**Table 7**). Tasks 6 and 7 focused on mathematical themes, such as *doing* and *undoing*, in which Ckin was able to use Maxima to justify the behavior of the solutions; this rarely happens in pen and paper. She applied Maxima's capabilities, such as high speed and visualization, to obtain the similarities and differences (CSO = 4) of different cases (CVRA = 1). The use of the graphs was more fluent than in the teaching experiments. It means that students use the graphs with more ease than other activities in Maxima environment. She tried to use Maxima's capabilities, such as visualization and high speed of calculation, to see whether the solution of the differential equation in the follow-up study was correct. Identifying the techniques required to get the solution using the menu or the help sheet was much easier than during the teaching experiments (the number of mathematical thinking powers were more than those on the teaching experiments). The frequencies for extending and restricting (ER = 3) and comparing, sorting, and organizing (CSO = 4), and finding evidence to convince the correctness of the solution showed the student's cognitive struggle.

Although John's final score in differential equations was a C+; he was able to use his *specializing* and *generalizing* powers during the follow-up study (**Table 8**). He had difficulty in convincing the researcher about the characteristics of the solutions of the graphs, as well as associating them with the symbolic solution. He identified the facts related to the specific differential equation and their application to the real-life problem (SP2 = 31). He used the capabilities of Maxima, such as the high speed of calculations to try some specific cases (CVRA = 4), through which he had to identify the facts and introduce the appropriate symbols (IE = 25) to get the solution (SP3 = 8). John performed better than on the pen and paper environment in using his mathematical thinking powers. He tried to examine several differential equations using Maxima's capabilities to get an idea of the problem. Once he had the solution, he was able to explain the solution or graphs' characteristics during the interviews. John's core strength was his insistence on finding the solution using Maxima, even though he lacked knowledge in finding differential equations in the pen and paper environment. His development and use of mathematical thinking powers, e.g., *convincing* (CV1 = 5, CV2 = 6) was evident during the follow-up, in which the frequencies of mathematical structures were higher than those during the main study.

Table 8. John's use of mathematical thinking powers during the follow-up study

No.		Interview extracts		
25	Researcher	Could you please give me the differential equation?		
26	Researcher	How do you want to use Maxima?		
27	John	Simplification.		
28	John	John typed `(%i1) ode: 'diff(y,x) = m;` John pressed Shift and Enter. (%o1) $\frac{d}{d\,x}\,y=m$	SP2 (facts) SP6 (representation)	Identify the facts. Introducing symbols.

29	John	Let me check and examine some cases
		John typed.
		`(%i3) plotdf(x+1);`
		`--> plotdf(1);`
		`--> plotdf(m);`
		`--> plotdf(100);`
		John pressed Shift and Enter
		
		(Only the first one was working)
		`(%i5) plotdf(m);`
		`plotdf: expression(s) given can only depend on x and y`
		`Found extra variable m`
		`-- an error. To debug this try: debugmode(true);`
			SP7 (links)
			SP2 (facts)
			SP6 (representation)
			SP5 (representation)
				Trying some specific cases.
				Identify the facts.
				Introducing symbols.
				Introducing image.

Task-follow up study	SP1	SP2	SP3	CVRA	IE	CC	ER	CDC	CSO	GN1	GN2	CJ1	CJ2	CV1	CV2	CV3
John	1	31	8	4	25	1	1	7	2	11	0	4	0	5	6	0

Philip also answered the tasks during the follow up study and the use of mathematical thinking powers can be seen in lines 32 and 34 (**Table 9**).

Philip used *to imagine* and *expressing* (IE = 23) and *specializing* powers (SP2 = 18, and SP3 = 12) more than other mathematical thinking powers and identified the facts related to a specific differential equation to define it (SP2=18) into Maxima to obtain a solution (SP3 = 12). He used Maxima's capability to convince the researcher that his solution was correct (CV1 = 16). Compared to other participants, Philip was able to use almost all the mathematical thinking powers. He used *convincing* powers more than any others. His high level of technical and conceptual knowledge helped him apply Maxima's capabilities to show that the solutions of differential equations in the follow-up study were correct.

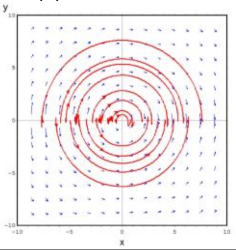
Table 9. Philip's use of mathematical thinking powers during the follow-up study

No.		Interview extracts		
30	Researcher	In the Class, I distributed this test, most of the student did not answer it. Now, could you please explain it for me? What does it mean?		
31		If I explain the equation before drawing the graph, there is no gradient at $y = 0$. If y increases, the slope of the tangent line will be decreased, and it will be zero. But to get the graph, I must solve it then get the solution.	CV1 (links) CV2 (explanation)	Have evidence. Explain the reason.

32	Philip	`(%i1) eq:'diff(y,x)+x/y=0;`
		Philip pressed Shift and Enter.
		(%o1) $\frac{d}{d\,x}\,y+\frac{x}{y}=0$
		Philip typed
		`(%i2) gsoln: ode2(eq,y,x);`
		Philip pressed Shift and Enter.
		(%o2) $-\frac{y^2}{2}=\frac{x^2}{2}+\%c$
		Philip typed
		`(%i3) -y^2/2=x^2/2+%c;`
			SP4 (link)
			SP2 (facts)
			SP3 (technique)
			SP6 (representation)
				Organizing.
				Identify the fact.
				Identify the techniques.
				Introducing the symbols.

Table 9 (Continued). Philip's use of mathematical thinking powers during the follow-up study

No.		Interview extracts		

32	Philip	Philip pressed Shift and Enter.
		(%o3) $-\frac{y^2}{2}=\frac{x^2}{2}+\%c$
		Philip typed
		`(%i4) solve([-y^2/2=x^2/2+%c], [y]);`
		Philip pressed Shift and Enter.
		(%o4) $[y=-\sqrt{-x^2-2\,\%c}, y=\sqrt{-x^2-2\,\%c}]$
		Philip typed
		`(%i5) plotdf(-x/y);`
		Philip pressed Shift and Enter.
		
			SP6 (representation)
			GN1 (technique)
			SP5 (representation)
				Introducing the symbols.
				Check the correctness of calculation.
				Introducing graph to get idea of the solution.

33	Researcher	Why does the function behave like this?	GN2 (links)	Checking the argument (here by clicking on the screen see the circles are in accordance with the symbolic solution).
34	Philip	The semi-circle is due to y' not being defined at $y = 0$.	CV1 (technique) (%04)	Have evidence.
35	Researcher	Is it interesting for you?		
36	Philip	Yes, sure.		
37	Researcher	What are the specific characteristics of these kinds of interviews for you?		
38	Philip	Calculation and visualization.		
39	Philip	Thank you for involving me in this project, I have learned to solve and interpret. I can extend this knowledge to other courses. Thank you so much.		

Task-follow up study	SP1	SP2	SP3	CVRA	IE	CC	ER	CDC	CSO	GN1	GN2	CJ1	CJ2	CV1	CV2	CV3
Philip	0	18	12	1	23	3	2	1	2	5	2	5	0	16	12	1

Una struggled with mathematical themes, such as doing and undoing, which were the focus of tasks 6 and 7. Una used the mathematical thinking power *identifying* the facts related to a specific differential equation (SP2 = 36) and *justifying* the solution using mathematics symbols (**Table 10**). During the interviews, she was able to identify the technique to get the solution to the differential

Table 10. Una' s use of mathematical thinking powers during the follow-up study

No.		Interview extracts		
40	Researcher	In the previous session, you used Maxima perfectly.		
41	Una	Una forgets to give the name to the equation. However, she remembers immediately. She forgets to write minus (-).		
42	Researcher	Execute it.		
43	Una	Una looked at the help sheet. Una typed `(%i1) eq1:'diff(T,r)=-q/(2*%pi*r*L*k);` Una pressed Shift and Enter. (%o1) $\frac{d}{d\,r}\,T=-\frac{q}{2\,\pi\,k\,r\,L}$	SP2 (facts) SP6 (representation)	Identify the fact. Introducing the symbols.
44	Una	Could you please solve it? Use the help sheet.		
45		Una looked at the help sheet, then copied, pasted and modified the name of the variables `(%i2) gsoln:ode2(eq1,T,r);` Una pressed Shift and Enter. (%o2) $T=\%c-\frac{q\log(r)}{2\,\pi\,k\,L}$ (Una checked the calculation and compared the results to what she found manually).	SP2 (facts) SP3 (technique) SP6 (representation) GN1 (technique)	Identify the fact. Identify the technique to get the solution. Introducing the symbols Check the calculation.

Table 10 (Continued). Una' s use of mathematical thinking powers during the follow-up study

Task-follow up study	SP1	SP2	SP3	CVRA	IE	CC	ER	CDC	CSO	GN1	GN2	CJ1	CJ2	CV1	CV2	CV3
Una	1	36	12	0	15	2	0	11	4	7	0	1	0	6	3	0

Table 11. Wendy' s use of mathematical thinking powers during the follow-up study

Task- follow up study	SP1	SP2	SP3	CVRA	IE	CC	ER	CDC	CSO	GN1	GN2	CJ1	CJ2	CV1	CV2	CV3
Una	1	36	12	0	15	2	0	11	4	7	0	1	0	6	3	0

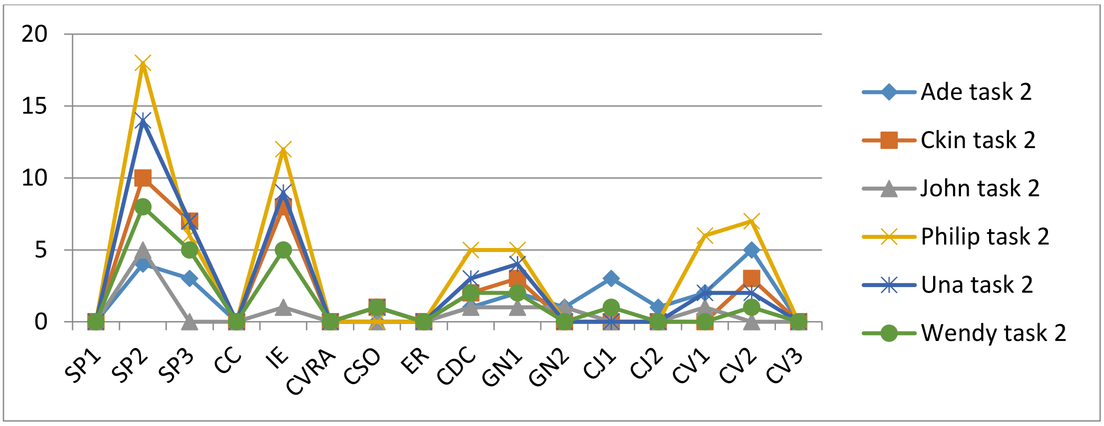


Figure 3. The most frequent mathematical structures and mathematical thinking powers during the interviews (Source: Authors' own elaboration)

equation, including the tasks for the follow-up study (SP3 = 12). She made conjectures using Maxima's capabilities such as visualization and high-speed calculations (CJ1 = 1). Una was able to use Maxima's capacities to show that the solution was correct (CV1 = 6). She also used mathematical thinking powers by Checking answers. She was also able to explain the reason for what she introduced or found in the Maxima environment (CV2 = 3). She improved in correcting, deleting, and completing the previous commands in new situations (CDC = 11). In the follow-up study, Una needed more attention to use Maxima's capabilities because she had forgotten some techniques that she learned in the differential equations classroom.

Table 11 shows the numbers of the mathematical thinking powers Wendy used during the interviews. Wendy was able to identify the facts related to the specific differential equation, such as the independent and dependent variables (SP2 = 15). She used Maxima's capabilities to examine some specific cases (CVRA = 2) to obtain the similarities and differences in their solution (CSO = 2). She also used Maxima's capabilities such as using graphs to convince the researcher that the tasks' solution was correct (CV1 = 4), as well as to communicate why the solution was correct (CV2 = 7). The frequencies show that she tried to use most of her mathematical thinking powers, which implies that there was an improvement to the first and second interviews in the main study. *Identifying* the facts and techniques were based on posing numerous prompts and questions. She tried to use Maxima's syntax to express mathematical concepts. She developed the use of graphs to see the behavior and interpretation of the solutions, compared to the interview in the main study.

As shown in **Figure 3**, to use mathematical thinking powers for making sense of mathematical structures in task 2, the cases used SP2 (some of the participants did not read the question correctly, SP1 or just jump to write the syntax in Maxima environment) and SP3 (specializing powers), IE (imagining and expressing), CVRA (changing, varying, reversing, and altering), CSO (comparing, sorting, and organizing), GN1 (checking the calculation in general), and CV1 and CV2 (convincing powers). However, promoting generalization and conjecturing still needs to be considered. Students' patterns, using mathematical thinking powers to make sense of mathematical structures, were like those used for their mathematical thinking powers in Task 2 (**Appendix C**).

As frequently observed in task 2, **Figure 4** illustrates facts, techniques, and representation as the most frequent mathematical structures that students made sense of in this research. It is worth noting that some of the mathematical thinking powers were not used. There are several reasons for this, such as students' lack of knowledge to generalize and make conjectures. In some cases, distinguishing

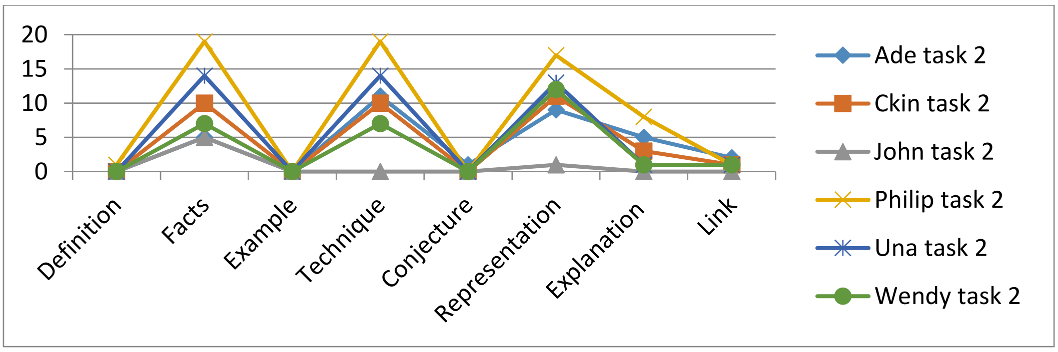


Figure 4. Most acquired mathematical structures, based on frequency in the interview process (Source: Authors' own elaboration)

between techniques and conjectures according to the theoretical framework was quite difficult for the students. For instance, in **Figure 4**, some mathematical structures received zero usage. This finding could be considered for further investigation.

DISCUSSION AND CONCLUSION

Our findings show that identifying the facts, techniques, representation, explanation, and links were the most frequent mathematical structures displayed by students during this study. The sense-making of the mathematical structures differed from task to task. Generally, in the pen and pencil environment, students were able to identify the links and explanations emanating from the presented facts, techniques, and representations; these appeared frequently in the procedural tasks (line 34, 35).

In the following parts, each finding is presented and compared to previous research. We discuss, and contextualize, findings that are aligned, as well as unaligned, with previous research. We also highlight which of our results are novel contributions to the body of the research in this area.

Our findings indicate that maximizing students' mathematical thinking powers through prompts and questions, coupled with the nature of the problems, played a role in making sense of mathematical structures (line 38, 40). Small (2017) showed that when and where students need prompts during teaching and learning mathematics. Parallel to the findings of this research, Breen and O'Shea (2011) showed that the types of tasks, such as generating example, analyzing reasoning, evaluating mathematical statements, conjecturing or generalizing, visualizing and using definition, affect students' learning. Even O'Sullivan et al. (2024) considered tasks analysis as a lens for curriculum reform. Wu and Yang (2022) indicate that computational thinking assists student to develop and apply mathematical concepts. Moreover, they addressed that a reciprocal relationship between computational thinking and mathematical thinking embeds computational thinking into mathematics learning, where CT is involved in problem-solving, and MT is developed to improve student performance on debugging or reflection. They confirmed that tasks rarely involve critical thinking for constructing viable arguments, as well as in critiquing the reasoning of others in mathematical thinking practices.

We found that compared to pen and pencil environments (Wilson et al. 2024), the computer algebra system environment, with its instantaneous potential, provided more opportunities for students to use and maximize their mathematical thinking abilities. Whilst this is not a comparative study, by contrasting pen and paper and a computer algebra system environment, our results seem to suggest that the high speed of calculations and visualization capabilities of a computer algebra system environment afford students more mathematical thinking opportunities than pen and paper environments. Students' use of mathematical thinking powers was not primarily sequential but zigzagged, particularly when tackling unfamiliar topics or themes. These findings have implications for curriculum developers and teachers. They also have applications in assessment, since they provide instant feedback (Olsher et al, 2024; Sangwin, 2004).

In previous works (Balacheff & Kaput, 1996; Lavicza, 2010; Marshall et al., 2012; Martinez & Pedemonte, 2014; Pea, 1987; Zeynivandnezhad & Bates, 2018), particularly in the main study (Zeynivandnezhad & Bates, 2018), it is suggested that students' experimentation with Maxima led them to conjecture. A conjecture is a quest or an assertion as to what might be true, leading to trying to justify it. Having tried several equations seeking to uncover patterns in damping situations, and having been asked to fit together their generalization, students arrived at specializing. However, in the follow up study, there was little evidence of conjecturing because there are some zeros in the tables. Encountering mathematical structures involves recognizing relationships (in specific instances) and conjecturing that these relationships are in fact properties that hold more generally. Working out the scope of that generality is often the hardest part. Examples and Conjectures did not come to the surface (along with definitions, explanations and links). Maxima's speed afforded students an environment in which they were able to switch steadily and swiftly back and forth, from the symbolic and graphical versions of differential equations. Nonetheless, from **Appendix C**, we can infer that students still need planning to promote conjecturing and convincing powers. Nevertheless, there are experimental tools, such as GeoGebra Discovery, that provide user-friendly graphical interfaces and the Tarski/QEPCAD B system to conjecture and prove geometric inequalities through the translation of geometric constructions into semi-algebraic systems; these features may have a lot of potential for computed assisted classroom courses (Brown et al., 2021). Therefore, future research to learn which teaching and learning activities and modalities best promote these mathematical thinking powers is warranted.

During the teaching experiments, pen and paper environment, students often solved differential equations with coefficients and Laplace transforms. This indicates that before students moved on to the computer algebra system, they had gained some sort of prior knowledge about solving differential equations at an intermediate to advanced level, such as solving via Laplace transforms (Zeynivandnezhad & Bates, 2018). Hence, for this cohort of students, Maxima was utilized as one would use any regular calculator (Karadag, 2009).

The most frequent way in which students' mathematical powers were shown via Maxima took place through the introduction of symbols to arrive at solutions. For example, they defined differential equations in Maxima. We found that methodological or procedural knowledge, as well as instrumentalization (technical knowledge) were required to arrive at solutions of differential equations, which is in line with the work of (Kadijevich, 2014). Furthermore, there was a correlation between instrumentalization and mathematical thinking power. Specifically, the high achieving student, Philip, performed tasks 1 and 2 within the main and followed up study with little to no difficulty (e.g., line 32 and 34) while using commands that were more complex than those used by his counterparts, such as *subst* as opposed to *atvalue.*

New Findings to Expand the Field

Our research fills a gap identified by Monaghan (2007), their request for the undertaking of research to determine the kinds of doing, talking, and observing techniques that aid students to move, from knowing, into mastering conceptual understanding (Blume, 2007).

Through the follow up phase of this research, students' use of organization, sorting, characterizing, and comparison was expanded, when compared to all other tasks. During the interpretation of solutions' behavior, participants had to classify both symbolic and graphical solutions. The absence of these powers was very apparent on tasks that were more inherently procedural. Our findings indicate that students used their mathematical thinking powers in complex situations, such as solving the differential equations in their specific area, e.g., a chemistry problem. They modeled the phenomena, found the general equation, and interpreted it based on the graph and their specific content-area knowledge, as well as their differential equations knowledge.

Still, students lack conjecturing and convincing powers; we propose that future work is needed to include the development of teaching and learning activities to effectively promote these mathematical thinking powers. This research was creating instruments for promoting students mathematical thinking. However, similar students could focus on developing pedagogical strategies to foster mathematical thinking powers.

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APPENDIX A

A Typical Worksheet Used in This Study

Worksheet 2

General and particular solutions

- **General solution**
- **Particular solution**
- **Initial value problem**

Name: .... Matric No. ....

(I) Written activity:

I-1 Find the general and particular solution of given differential equation:

$y' = yx \quad , \quad y(1) = 1$

....

....

....

....

....

....

I-1-a: What is the most important characteristic of general solutions?

I-1-b: Describe a particular solution of a differential equation.

I-1-c: What does it mean by particular solution of a differential equation?

I-1-d: What is the same and what is the different of general and particular solutions?

I-1-e: how are you sure your solution is correct?

(II) Computer Lab activity

II- 1 Use a computer algebra system to compute the solution of given differential equations and plot their solution.

$y' = yx \; , \quad y(1) = 1$

....

....

II-1-a: Describe the features of graphic solution of a differential equation?

II-1-b: What is the same and what is the different between exercise I-1 and exercise II-1?

II-1-c: What do you know about the command?

II-1-d: What do you want to do in Maxima environment?

II-1-f: How do you introduce y(1)=1 to Maxima?

II-1-g: How are you sure your solution is correct?

1

(Zeynivandnezhad, 2014)

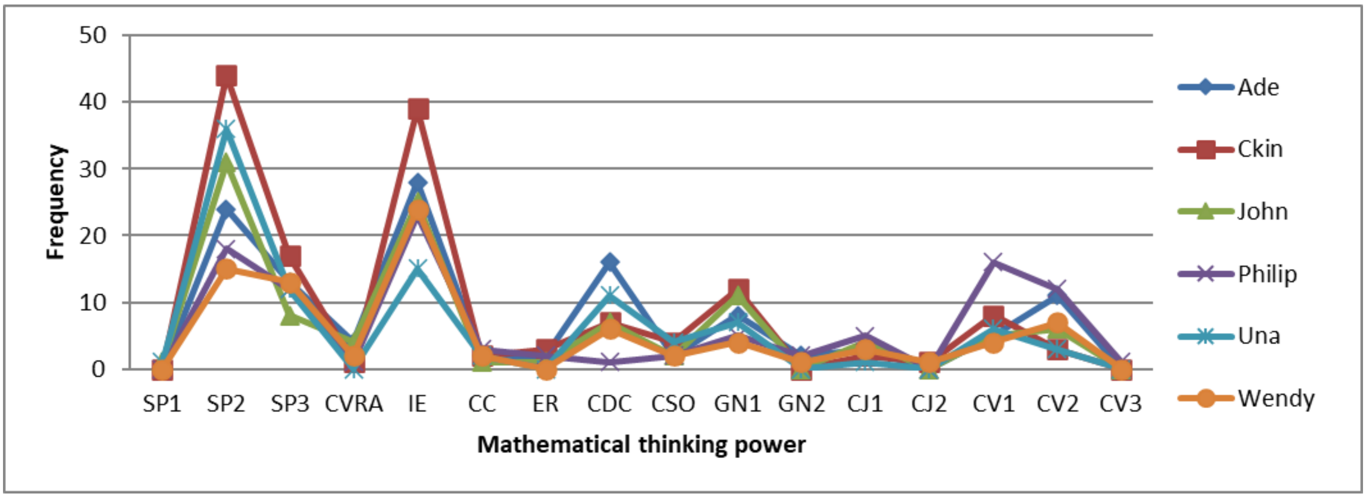
APPENDIX B

Table B1. Groups of mathematics processes associated with mathematical thinking

Constructs	Code	Description	
Specializing	SP1	Reading the question correctly.	
	SP2	Identifying the features of the facts that make it an example of a specific differential equation (identify the facts to solve the equation). Recognizing the specific as an instance of a (familiar) class.	
	SP3	Identifying the relevant facts, theorems and properties, the techniques to solve.	
	CV	(Changing, varying, reversing and altering).	
	RA	Trying some specific cases to get an idea of what the answer might be. Extending by altering some of the constraints.	
Imagining and expressing	IE	Introducing images, diagrams to understand the problem (mental imagery-recognizing a mathematical relationship - to use material objects, diagrams and pictures, voice tones and gestures, words and symbols to express discerned objects, recognized relationships and perceived properties).	
Completing, deleting and correcting	CDC	Selecting appropriate syntax and adding needed information, deleting unnecessary steps or symbols, completing, correcting the wrong parts, completing missing parts.	
Stressing and ignoring	SI	Stressing some features and consequently ignoring others in which generalization comes about, and relationships become properties, the relationship being turned into a property is mathematical.	
Extending and restricting	ER	Extending or restricting meaning of concepts, carrying meaning across, weakening constraints.	
Comparing, sorting and organizing	CSO	Identifying what is the same and what is different, sorting according to the method used to solve, making a connected chain among mathematical concepts.	
Classifying and characterizing	CC	Classifying and sorting information, be alert to ambiguities - organizing data or information. Identifying any similarity or analogous questions – identify possible underlying pattern - Develop a sense of why the answers may be correct. (Mathematical themes: Doing and undoing, Invariance during change, Freedom and constraint).	
Generalizing	GN1	Checking the calculations in general to make sure the generalization is true.	
	GN2	Checking the argument to ensure that the computations are appropriate.	
	GN3	Looking for patterns and relationships.	
	GN4	Seeing the general through the particular.	
	GN5	Extending the result to a wider context by generalizing.	
	GN6	Extending by seeking a new path to the resolution.	
Conjecturing	CJ1	Viewing the side, articulating the generalizing	Viewing the side, articulating the generalizing; Predicting relationships and results.
	CJ2		Checking the consequences of conclusion to see if they are reasonable.
	CJ3		Reflecting on implications of conjectures and arguments.
	CJ4		Checking assumption including implicit ones.
	CJ5		Reflecting on key ideas and moments.
	CJ6		Reflecting on your resolution: can it be made clearer?
Convincing	CV1	Convincing yourself – verification.	
	CV2	Convincing a friend – explanation (Finding and communicating the reasons why something is true).	
	CV3	Convincing a skeptic – justification.	

APPENDIX C

The Pattern of the Use of Mathematical Thinking Powers to Make Sense of Mathematical Structures



(Source: Authors' own elaboration)