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EXAMINING "MATHEMATICS FOR TEACHING" THROUGH AN ANALYSIS OF TEACHERS' PERCEPTIONS OF STUDENT "LEARNING PATHS"[1]

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ABSTRACT. How teachers think about student thinking informs the ways in which teachers teach. By examining teachers' anticipation of student thinking we can begin to unpack the assumptions teachers make about teaching and learning. Using a "mathematics for teaching" framework, this research examines and compares the sorts of assumptions teachers make in relation to "student content knowledge" versus actual "learning paths" taken by students. Groups of teachers, who have advanced degrees in mathematics, education, and mathematics education, and tenth grade students engaged in a common mathematical task. Teachers were asked to model, in their completion of the task, possible learning paths students might take. Our findings suggest that teachers, in general, had difficulty anticipating student learning paths. Furthermore, this difficulty might be attributed to their significant "specialized content knowledge" of mathematics. We propose, through this work, that examining student learning paths may be a fruitful locus of inquiry for developing both pre-service and in-service teachers' knowledge about mathematics for teaching.

KEYWORDS. Content Knowledge, Learning Paths, Mathematics for Teaching, Pedagogical Content Knowledge, Students, Teachers.

INTRODUCTION

Presently in mathematics education there is a growing interest concerning the kinds of knowledge mathematics teachers ought to know to teach mathematics effectively; it is known as "mathematics for teaching" (Adler & Davis, 2006; Deborah L Ball, Bass, Sleep, & Thames, 2005; Davis & Simmt, 2006). Adler and Davis (2006) explain that the theoretical foundation of mathematics for teaching is based upon the "epistemological assumption that . . . there is a specificity to the mathematics that teachers need to know and know how to

[1] A synopsis of this research is to be presented at the *Symposium on the Occasion of the 100th Anniversary of the International Commission on Mathematical Instruction* (ICMI), Rome, 2008.

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use" (p. 271). Scholarship in the area of mathematics for teaching is a fairly recent research effort. Theorization about mathematics for teaching is an important endeavor in order to make sense of the many complexities involved in effective mathematics instruction. The resulting outcomes of such theorization can lead to important insights that would ultimately benefit students.

Ball, Bass, Sleep and Thames (2005) propose a framework that describes the knowledge associated with mathematics for teaching. The framework consists of four "distinct domains" (Deborah L Ball et al., 2005, p. 3): (1) *common content knowledge (CCK[2])* — the mathematical knowledge of the school curriculum, (2) *specialized content knowledge (SCK)* — the mathematical knowledge that teachers use in teaching that goes beyond the mathematics of the curriculum itself, (3) *knowledge of students and content (KSC)* - the intersection of knowledge about students and knowledge about mathematics, and (4) *knowledge of teaching and content (KTC)* - intersection of knowledge about teaching and knowledge about mathematics (p. 4).

This research specifically examines the relationship between Ball et al.'s (2005) second domain, specialized content knowledge (SCK), and the third domain, knowledge of students and content (KSC). The locus of this research rests on the third domain, KSC, as the central domain of analysis. The reason being, this particular domain is largely defined by students and student learning, which is the foremost interest of mathematics education. The authors explain that "KSC includes knowledge about common student conceptions and misconceptions, about what mathematics students find interesting or challenging, and about what students are likely to do with specific mathematics tasks" (Deborah L Ball et al., 2005, p. 3).

The third domain has another interesting feature. The other domains, although described as "distinct" by Ball et al. (2005), intersect, if not overlap, most closely within this particular domain. For example, within this domain, knowledge of content intersects with CCK, while the knowledge about students intersects with KTC. Also, domains CCK and KTC intersect within KSC, as does SCK in that teachers may or may not have advanced education in mathematics, which may impact all the other domains, and most particularly the third.

The intersections of the domains, particularly with KSC, are not surprising and are anticipated since the collective of the domains can be viewed as what mathematics teachers do when teaching mathematics. Increased knowledge in *any* of the domains can generally be

[2] The authors of this paper recognize in the reading of the manuscript the challenges associated with multiple acronyms. Although we use all the acronyms provided by Ball et al. (2005) in the early portions of the manuscript, the bulk of the manuscript focuses on only two – KSC (knowledge of students and content), and SKC (specialized content knowledge). The early portion of this manuscript may require some endurance, while the context of this research is established.

viewed as beneficial to the teaching and learning of mathematics, which is one of the central goals of the mathematics for teaching movement. This having been said, in earlier research with pre-service mathematics teachers, Ball (1989) found that teachers with advanced degrees in mathematics (or a related field), or to use Ball et al.'s (2005) domains, high SCK, were not necessarily any better at teaching mathematics. This finding resonated with us and we wondered whether there were instances were SCK might actually interfere with KSC and the development of the other domains.

As Ball (1989) herself demonstrated, teachers *without* sufficient SCK (or other domains) are able to learn both pedagogy and content and become effective teachers of mathematics, hence supporting the mathematics for teaching movement. Our research focus is important in that it deviates from the general concerns over insufficient SCK amongst teachers. Therefore, the research questions guiding this work are: (1) what are the assumptions teachers with sufficient or high SCK make about student thinking (i.e., KSC) or the "learning paths" that students take? And, (2) in analyzing such assumptions, what conceptual and pedagogical insights might be mined to support knowledge development in the other domains defined by Ball et al.? To explore these questions, pairs of students and pairs of teachers were given a common mathematical task. Teachers were asked to model the learning paths students might take.

LITERATURE REVIEW

Three bodies of literature inform this research. The foremost contributor shaping this research is the growing body of scholarship known as "mathematics for teaching," which was introduced earlier, and will briefly elaborate upon. This scholarship suggests that there is a complex, interrelated, and multi-faceted core knowledge required for teaching mathematics that ought to inform how mathematics teacher education is conceived of and how ongoing professional development amongst teachers occurs.

Adler and Davis (2006) suggest that the mathematics for teaching movement is a relatively new way of thinking about mathematics education. However, the term "mathematics for teaching" has been appropriated within scholarship by many and is currently in vogue in terms of research imperatives within this field. However, the underlying epistemology to the various appropriations of mathematics for teaching is fairly consistent and generally agreed upon. Perhaps the most compelling and cohesive aspect of the underlying epistemology, amongst those that theorize about mathematics for teaching, is the view that teachers must know mathematics in such a way to be able to know how to use mathematics to develop student understanding (Adler & Davis, 2006; D. Ball & Bass, 2001; Deborah Loewenberg Ball, 1989; Deborah L Ball et al., 2005; Ball Loewenberg, 2000; Davis & Simmt, 2006).

Although there is arguably a cohesive underlying epistemology to mathematics for teaching, some researchers, in their conceptualization, frame the movement in alternative ways. Davis and Simmt (2006) frame their conceptualization of mathematics for teaching through the "complexity science"[3] lens. This lens views the relationship between teaching and learning as inherently nested, with learning as a collective endeavor. The authors propose alternative domains to those defined earlier in the theoretical framework proposed by Ball et al. (2005). For example, Davis and Simmt describe domains as "nested," and offer different conceptualizations of the domains as follows: (1) mathematical objects, (2) curriculum structures, (3) classroom activity, and (4) subjective understanding. The different emphases between theorists may be attributed to differing loci of attention, or as a related function to the types of questions that particular researchers are exploring. For the purpose of this research, we follow the domains defined by Ball et al. (2005).

Another point of consensus is the underpinning of the movement to the theorizations of Lee S. Shulman (1986; 1987) – the second body of literature informing this research. Shulman was among the first to begin making distinctions between the types of knowledge needed for teaching in his conceptualization of *Pedagogical Content Knowledge* (PCK). Informed by earlier theorizations by Dewey (1969) and Bruner (1966) around teacher's subject knowledge, according to Shulman PCK "goes beyond knowledge of subject matter . . to the dimension of subject matter knowledge *for teaching* [author's emphasis]" (Shulman, 1986, p. 9). He says emphatically that teachers must have "ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9). Yet, as some have argued, knowledge of subject content alone will not necessarily enable an individual to teach that knowledge to another (Deborah Loewenberg Ball, 1989; Sfard, 1997).

Ball et al. (2005), in their conceptualization of mathematics for teaching, attribute their proposed domains of KSC and KTC to Shulman (1986; 1987). Ball et al. state that KSC and KTC "are closest to what is often meant by 'pedagogical content knowledge' — the unique blend of knowledge of mathematics and its pedagogy" (p. 4). The importance of PCK for the education of mathematics teachers has been well document (e.g., Ball Loewenberg, 2000; Langrall, Thornton, Jones, & Malone, 1996). Likewise, the importance of enacting PCK for ongoing teacher practice has also been well documented (e.g., Lampert, 1990; Marks, 1990).

[3] For a more detailed description of "complexity science" see Davis and Simmt (2003).

The final body of literature informing this research considers teachers' beliefs in relation PCK, or in other words teachers' anticipation of student thinking (Feiman-Nemser & Parker, 1990; Murata & Fuson, 2006; Nesbitt Vacc & Bright, 1999). Beliefs and "anticipation," for the purpose of this research are seen as mutually dependent occurrences given that the former shapes the later. Teachers' beliefs often form the very basis of the decisions teachers make.

Some research has shown that largely through professional development, pre-service teacher education, alternative experiences, and so forth, teachers are able to shift early beliefs about teaching, learning, and mathematics that may be negative or situated in a more teacher-directed model (Nesbitt Vacc & Bright, 1999). However, more compelling are the claims that teachers, while seemingly open and motivated to shifting beliefs about teaching and mathematics, continue to be remarkably unchanged in terms of belief systems formed from early experiences (Cohen & Ball, 1990; Gadanidis & Namukasa, 2005; McDiarmid, 1990; Norton, McRobbie, & Cooper, 2000; Stipek, Givvin, Salmon, & MacGyvers, 2001). Despite motivation and openness to change in order to improve teaching and, most importantly, student learning, teachers often resort to what is familiar in terms of *how* and *what* to teach. This is despite, and often because of, curricular, standardized achievement, and pedagogical pressures.

This current research elaborates upon research that examines teacher belief systems in relation to PCK, with a particular focus on beliefs related to student thinking. The importance of teachers being able to anticipate *student* content knowledge based upon the *students'* perspectives rather than teachers' own content knowledge and appraisals of their own classroom teaching has been identified as a key yet under explored issue in the PCK literature (Feiman-Nemser & Parker, 1990; Murata & Fuson, 2006; Nesbitt Vacc & Bright, 1999).

Successful anticipation of student thinking might suggest that classroom practices have adequately met the needs of students and the goals of learning. Alternatively, unsuccessful anticipation of student thinking might reveal to teachers' need to rethink the ways in which learning occurred in the classroom. Equally important is a teacher's ability of to further accommodate student learning when there is a mismatch between the teacher's and the students' perspectives. From a research perspective, examining instances of mismatch may shed significant insight on those areas of mathematics for teaching that likely need increased attention, whether in pre-service teacher education or through professional development for in-service teachers.

Murata and Fuson (2006) propose that successful anticipation of student thinking need *not* be overly complex. In their research examining the thinking of Japanese first graders, they argue that in relation to understanding students' mathematical thinking "there are not 20 to 35 different *learning paths* [authors' emphasis] or strategies for teachers to understand and assist" (p. 424). Rather, student thinking and learning can be isolated to a few specific *and* predictable trajectories, or learning paths. The authors propose that:

> for many mathematics topics, there are a few typical errors that stem from partial but incomplete understandings and some other more random errors from momentary lapses of attention or effort. Likewise, there are usually several solution methods, but these are limited in number and vary in their sophistication, generalizability, and ease of understanding. (p. 424)

Murata and Fuson make clear that these few predictable trajectories are *not* a closed set, and that other trajectories are possible; hence, teachers must be open to these other trajectories.

Not withstanding Murata and Fuson's (2006) clarification of learning paths as possibly open sets, teachers routinely in their pedagogy routinely anticipate the predictable learning paths of students (e.g., in lesson planning, assessment, etc.). We contend an examination of the concurrences *and* contradictions between teachers' anticipation of student learning paths and actual learning paths of students is an important area of research in the mathematics for teaching movement. Accordingly, this research contributes to the existing research.

Examined in this research are teachers' anticipation of students' "learning paths" by interrogating the intersection between Ball et al.'s (2005) conceptualization of SCK and KSC. Through the context of a common mathematical task given to pairs of teachers with high SCK and pairs of students, we examine the assumptions teachers make about student learning paths when teachers are asked to complete a task as they would anticipate a student might complete the task. More simply put, we asked teachers to develop a solution for the task akin to a "solution set" used to evaluate student work. We examine the assumptions teachers make, and consider the tensions between *what* teachers are thinking students ought to be thinking *versus* what actually occurred in response to the common task. We discuss the various assumptions teachers demonstrated, and the implications this evidence might have for the education of mathematics teachers and mathematics for teaching.

METHODOLOGY

Data for this research was collected over the 2005-2006 school years at multiple sites. Students, their parents, and teachers who participated in this research were provided with information about the study, then consents to participate were secured, in accordance with the research policy requirements for the Board of Education in which the research took place.

Participants

In order to examine teachers' anticipation of student learning paths, a common task, as the primary instrument of data analysis was administered to groups of tenth-grade students and a group of mathematics teachers. The students ($n_{total} = 51$) were from two classes ($n_1 = 25$, $n_2 = 25$), each taught by one of the authors of this paper. The composition of students varied according to gender, ethnicity, and socio-economic status. Although, the school where the research was conducted, located in a large urban setting, was known to be of higher socio-economic standing largely based upon location and community demographics.

The course, in which the students were enrolled, was an "advanced" mathematics course, geared toward students who were anticipating post-secondary education. This having being said, the range of abilities within the classes varied. However, the majority of students achieved at least a Level 3- (72%) in this course.

We were the regular mathematics classroom teachers for these students. Aside from the collaboration on select tasks throughout the duration of the course, the actual methods used in each of our classrooms were independently decided upon. The course was provincially set with common curricular goals, assessment guidelines, and so forth. Approved textbooks, as determined by our Board of Education for this course, were largely narrowed to two – with one in particular being the most commonly used textbook in the province.

The mathematics teachers ($n = 27$) who participated in this research were all mathematic department heads from various secondary schools ($n = 30$), some urban and some rural, of one particular board of education. The data was gathered during a monthly organizational meeting where the department heads, in addition to discussing administrative and curricular issues, engaged in a professional development component. We had contacted the system-wide coordinator and requested permission to attend this session and to invite the teachers to participate in our research. In addition to the teachers' "headship" responsibilities (i.e., course assignments, staffing, etc.), the teachers all taught mathematics.

All of the teachers except for one had greater than 10 years of teaching experience, with 19 of the teaches having more than 15 years of teaching experience. All of the teachers, except for one, had advanced teacher training and qualifications in mathematics education. [4]

[4] Each of the teachers, in addition to their university degrees and teaching degrees, also had additional qualifications in mathematics instruction, except for one teacher whose additional qualifications were in biology and chemistry.

The post-secondary educational background for the teachers varied. Most of the teachers had Bachelors of Science degrees (n = 12). Many of the teachers had Bachelors of Mathematics (n = 7). One teacher held a degree in engineering. The remainder of the teachers had undefined Bachelors degrees (n = 7). These undefined degrees likely had a significant mathematical component given that the majority of the teachers also had advanced qualifications in mathematics education that required a minimum amount of post-secondary mathematics education.[5] Consequently, we describe this group of teachers as having high SCK. This sample of teachers is a purposive effort to neutralize concerns over low SCK in relation to mathematics for teaching and to elaborate on contemporary research about teaching mathematics that largely focuses on teachers with low SCK.

The mathematical task that formed the primary artifact of analysis, describe in the next section, involved quadratic relations, which all of the teachers would have taught on multiple occasions. Although the mathematical task was explored in this study with tenth-grade students, quadratics appears in each of the subsequent grades of mathematics instruction in our province. Therefore, the teachers, in this research, likely taught this material on multiple occasions, in multiple courses, during the current school year, as well as during their extensive careers.

The task: What is water pressure?

The teachers and the students in this research completed a common mathematical task – *What is water pressure*? [6] (2007). We developed the task as a final assessment for a quadratics unit. The unit it self spanned approximately four weeks.

The mathematical task involved modeling the water flow from the drinking taps in a school, where the projection of the water from the spout forms a parabolic arch. Pairs of teachers and pairs of students ($n_{teachers}$ = 13, $n_{students}$ = 23) were asked to determine various forms of the quadratic relations (e.g., standard form, factored form, and vertex form) that modeled the current flow of the particular tap they were investigating. Additionally, the pairs were asked to determine the quadratic relation representing an arbitrarily set 'ideal' water flow of 3 cm above the faucet guard at the fountain. The underlining impetus for determining the ideal water flow was based on notions of water conservation; that is, a reduced water flow is potentially is more cost-effective in terms of overall water consumption.

[5] In the province of Ontario, most universities that grant additional qualifications in mathematics instruction require nine full undergraduate courses in mathematics or six full courses plus five full courses in another subject area.

[6] See AUTHOR 1 and AUTHOR 2 (2007) for the complete task and assessment tool that was provided to the students and the teachers.

The task itself was open-ended given that each fountain had a differing flow-rate. The task for the students was a final summative assessment for a unit on quadratic relations. Teachers and students had 70 minutes (one class period) to complete the mathematical task. Teachers and students were randomly paired and assigned a fountain in either their school or the school in which the department heads' meeting was taking place. Each pair was given a chart paper to record their full solution. The group as a whole read through the mathematical task. The assessment rubric was provided with the mathematical task and reviewed prior to beginning the inquiry.

Instructions to the teachers were modified. Teachers were informed of the nature of our inquiry (i.e., anticipation of student thinking). As well as reviewing the task and the assessment rubric, as was done with the students, teachers were given the additional instructions to complete the task as a "level 3"[7] response that might be anticipated from a tenth-grade student. We also relayed to the teachers that students were not permitted to use graphing calculators on the task, but were permitted to use their own scientific calculators. Following the completion of the task, the teachers reconvened for a brief, focus group session to discuss the task, possible teaching and learning dilemmas, and their assumptions.

Data collection and analysis

The primary artefact of analysis was the actual solutions submitted by the teachers and students. However, researcher notes were also made of the focus group discussion. Researcher field notes were taken during the completion of the task for each group of participants as well, documenting other non-mathematical aspects of completing the task (i.e., time at the taps, returning to the taps after initially coming back to the classroom, etc.).

A content analysis, involving the coding and counting of features within the artifacts, was completed for all the solutions (Berg, 2004). A common coding structure was established by initially each independently completing a possible "answer key" for the task. From our individual solutions, we discussed features that might be important or noteworthy if missed or omitted. We recognized that the student and teacher solutions may evoke the development of additional codes or alternative learning paths, as suggested by Murata and Fuson (2006), thus we remained open to this during our coding. One particular pair of teachers' solution did just this, which is outline in our results. Following the establishment of the common coding scheme, each author independently coded all the solutions from all the pairs of teachers and students. We then compared our coding with each other and contemplated differences and

[7] Level 3 represents the acceptable Ministry of Education standards of achievement (i.e., approximately 75% average) in mathematics.

omissions with a goal of reaching consensus. Upon agreement of the codes applied to the artifacts, the collective set of student data was compared against the teacher data by a calculation of means. Finally, we engaged in an analysis of the overall results – including means and artifacts.

We acknowledge that, as mathematics teachers and as teachers of the students in the research, our own beliefs also influenced the codes established in relation to our own anticipations of student learning paths. In our examination of the complete data set, we examine the teachers' anticipation of student thinking, and unavoidably our own anticipation of student thinking as well. The codes used, therefore, emerged from general hypotheses *we* made about potential learning paths of students. In the results section, we detail each of these hypotheses as we report the findings from the coding in an effort to be explicit about the assumptions we made – both about student learning paths, and how teachers might anticipate learning paths in relation to these codes.

In total, there were seven codes used for the analysis. Evidence of an error, either in the solutions or the graphical representations (e.g., incorrect x-intercepts on the graphical representation), was coded as *conceptual error*. Solutions that did not include a diagram, make use of the available physical model, were highly abstract in reasoning (i.e., factoring or using the quadratic formula to find the zeros of the quadratic relation), and/or achieved via graphing calculator (i.e., quadratic regression) were coded as *theoretical reasoning*. The theoretical reasoning code could be viewed as an overly sophisticated response, one of the variances of learning paths proposed by Murata and Fuson (2006). Solutions that were incomplete in one or more of the requirements of the task were coded as *incomplete*, regardless of the extent that the solution was incomplete.

Both researchers, as mathematics teachers emphasized the use of fractions in our courses, as this was consistent with current ministry curricular expectations. Therefore, we thought it was important to see who opted to use decimals over fractions, suggesting, from our classroom experience, perhaps an underdeveloped understanding of operations with fractions. Consequently, solutions that were completed using fractions as opposed to decimals were coded as *fractions.* We had falsely predicted that this code would be used predominantly for the student work.

We anticipated that some solutions might reflect the physical model, as seen, while other solutions might show a transformation of the fountain's water flow to the first quadrant to facilitate more straightforward calculations. Transformation of graphical representations of water fountains that appear to flow into the second quadrant of the Cartesian plane, to the first quadrant were coded as *transformed model*. We also coded instances in which solutions showed the intentional use of *friendly numbers*, indicating that students or teachers understood the numbers used were somewhat arbitrary and could be manipulated slightly to facilitate easier calculations.

Finally, we coded, using the Achievement Chart from the current curriculum documents in mathematics (Ontario Ministry of Education/OME, 2005), the overall level of communication of mathematical findings of the solutions as either a *L1, L2, L3*, or *L4* – with L3 representing current acceptable ministry standards, L4 exceeding standards, and L1 significantly below ministry standards. This code for the overall level of communication is consistent with what Murata and Fuson (2006) describe as "ease of understanding" (p. 424) as a possible variance in student learning paths.

Our results are grouped according to our coding categories, drawing relevant examples from both the teacher and the student solution sets, where necessary to effectively illustrate the result. Overall results from the coding as percentages, for the teachers and students, are also reported in a table format (Table 1). We begin each section of the results stating our assumptions, as hypotheses, of the learning paths that we anticipated might have emerged in relation to each of the codes. A concluding comparison of both sets of solutions is in the subsequent section.

RESULTS

To recall, the task involved determining the quadratic relations that would model the water flow from a school drinking fountain. The task also involved determining the ideal water flow, at an arbitrary height, from the faucet guard of the water fountain. The bowl or the basin of the water fountain, potentially formed the x-axis with the y-axis either being set as the side of the faucet handle or transformed, if the handle was on, say, the right side of a fountain, making the flow of water into the second quadrant of the Cartesian plane. Depending on the flow of each individual fountain, the ideal height, set at 3 cm above the faucet guard, would have resulted in a transformation of the x-intercepts if the fountain were either higher or lower than the ideal. The one intercept from the spout of the fountain would remain constant.

Many of the teachers, much like the students in our respective classes, did not initially anticipate the amount of time that would be required at the fountains in order to gather data. Like some of our students, some teachers returned quickly to the classroom where we had gathered for this meeting. However, they realized rapidly that they did not have the necessary measurements to adequately complete the task, thus returned once again to their fountains.

At the fountains, the teachers appeared to be engaged with the task in much the same ways we had observed with our students. In both groups, we saw much discussion and debate at the fountains while various measurements were taken, as well as later in the classroom where their solutions were being written. During the focus group session, the teachers described the activity as "very interesting" and "highly engaging." There was consensus that this task would be welcome in their classrooms, in their departments, and useful for other grade levels investigating quadratics. We make these points now to suggest that the results we are about to report ought not to be attributed to lack of engagement or interest from either the teachers or the students.

CONCEPTUAL ERRORS

Hypothesis: Approximately 25% of students would perform conceptual errors in their solutions. However, teachers would not make conceptual errors in their anticipation of student learning paths.

Conceptual errors included calculation errors, as well as errors in the graphical representations (e.g., incorrect x-intercepts on the graphical representation) (see Figures 1 and 2). Students, as predicted, did make conceptual errors (<26%), albeit slightly higher than predicted. However, more surprising was the extent to which conceptual errors were made by the teachers (69%). It was remarkable to see the number of conceptual errors *throughout* the teacher solutions. Although an argument could be made that the students may have been more focused on the task given that it was an assessment, our observations, of the teachers engaged in the task, was that there was significant interest in the task and in performing well for each other as colleagues.

The most common and significant error for teachers occurred in relation to the zeros, x-intercepts, or zeros of the graphical representation. One pair of teachers raised this question in their solutions: "Would kids worry about the zeros changing with the maximum?" Unlike the majority of the students, who showed in their graphical representations a transformation of the zeros (i.e., either widening or narrowing) as the flow of the fountain was manually regulated with the handle (i.e., change in the maximum height), many of the teachers did not. This suggests the teachers did not feel that students *would* "worry" about the transformation of the zeros, despite the changes that occurred in the physical model.

Consequently, all the conceptual errors made by the teachers included inaccurate graphical representations showing that, despite increases or decreases in water flow, the points at which the water hit the basin of the water remained constant. Provided that the graphical representation was used for the remainder of the task, the incorrect intercepts made the rest of the solution also incorrect. In *all* cases, for the pairs of teachers and the pairs of students, the error in the intercepts was not the only error. Other errors in basic calculations, distributive properties, factorization, and so forth, plagued the teachers' responses and select student responses.

We suggest here that the significant amount of conceptual errors by the teachers was not intentional. That is, teachers did not construct solutions to the task infused with conceptual errors because that is what they would anticipated student learning paths to have evidenced, even though this was the explicit request of the task. The high number of conceptual errors made by teachers with high SCK is noteworthy, nevertheless.

THEORETICAL REASONING

Hypothesis 2: Few students might use theoretical reasoning, largely in the form of the quadratic formula, in their solutions given the presence of a physical model. Teachers would use the physical model fully given the parameters of the task and therefore would not show evidence of theoretical reasoning.

Theoretical reasoning is overly sophisticated responses (Murata & Fuson, 2006). Solutions that were highly theoretical in their reasoning included: (1) solutions that did not include a diagram, (2) solutions that did not make use of the available physical model, (3) solutions that were highly abstract in reasoning (i.e., factoring or using the quadratic formula to find the zeros of the quadratic relation), or (4) solutions achieved via graphing calculator (i.e., quadratic regression). In the case of the first, second, and fourth points only teacher responses showed evidence of these. The third point was evidenced in both teacher and student solutions. In comparison to the students' responses of 13% reflecting theoretical reasoning, 31% of the teachers' responses showed evidence of theoretical reasoning.

In two instances, teachers used the quadratic formula rather than the physical model to determine the various forms (e.g., standard form, vertex form, and factored form) of the quadratic relation. Even though students did have the quadratic formula at their disposal, they did not resort to the formula given the accessibility to the physical model. Evidence of theoretical reasoning is closely linked to conceptual errors. Teachers in their theoretical (rather than applied) reasoning did not go to the physical model, and therefore missed the transformation of the zeros in their graphical representation and subsequent calculations.

Two pairs of teachers used graphing calculators for their solutions, in spite of having been told explicitly that students did not have graphing calculators available to them during their completion of the task. The use of graphing calculators was evidenced in their solutions. For example, one pair of teachers said that the vertex form of their solution was obtained by "using the curve of best fit." This pair of teachers also reported the r^2 value as an indication of fit in relation to the points on the curve.

The same pair of teachers, who used graphing calculators and reported the r^2 value, also had unique method obtaining measurements at the fountain. These teachers indicated on their solution that the curve representing the water flow was "obtained by holding paper behind water and inserting pen through water to mark paper." Unlike all the other pairs of teachers and students, who used a ruler aligned either horizontally with the basin of the fountain (i.e., x-axis) or vertically with the spout of the fountain (i.e., y-axis), these two teachers put the graphing paper into the water fountain so that the actual water flow hit the paper and made a watermark from which the measurements were then taken.

The use of a graphing calculator, the efforts to achieve maximum accuracy in the measurement of the flow from the fountain, the indication of the r^2 value as a measure of fit, and having put the paper into the water fountain, suggested that this last pair of teachers' reasoning was *highly* theoretical and precision was ultimately an important factor in the successful completion of the task. Overall, teachers were deeply attached to theoretical reasoning approaches and were less practical than the students. Given the goals of the task, this result was unexpected.

INCOMPLETE

Hypothesis 3: Some students might have incomplete solutions. However, few, if any, teachers would have incomplete solutions – despite their anticipation that some students might not complete the task.

Solutions that were incomplete in one or more of the requirements were coded as *incomplete*, regardless of the extent to which the solution was incomplete. The main error by both teachers and students was determining the ideal flow. More students (44%) than teachers (31%) did not complete the task.

We did not anticipate that some pairs of teachers would not be able to complete the task fully. We conclude here that incompletion, on behalf of some of the teachers, may have been as a result of over analysis of the problem. In other words, the teachers that did not complete the task may have been overly distracted with producing a sophisticated result, hence missed some key components of the task (e.g., determining the equation for the ideal flow). We did not perceive the incompletions by the teachers to be intentional.

From our observational notes, teachers seemed highly engaged in the task – in fact, teachers wanted to succeed t the task. During the focus group session, teachers agreed that the task was useful and highly engaging. Furthermore, teachers reported "wanting to do well" given that the results would be under scrutiny by their peers (i.e., the authors of this paper, as colleagues in teaching). Teachers were invested in the task; nevertheless, the results show almost one-third of the teachers had incomplete solutions.

FRACTIONS

Hypothesis 4: Students that struggle with fractions will convert fractions to decimals, in spite of being encouraged throughout their mathematics course to work with fractions rather than convert to decimals. Teachers, however, would predominantly work through the solutions using fractions.

Students who are less comfortable with operations on fractions will convert fractions into decimals in order to facilitate, most particularly, the use of a calculator. In our own teaching of mathematics, we emphasize the importance of being able to work with fractions and discourage the practice of converting fractions to decimals. However, we frequently see those students less comfortable with fractions convert these to decimals. During the focus group session with the teachers following the task, we queried the teachers on their perspectives about fractions – which by consensus were aligned with our own anticipation of student learning paths. Yet, more than half of the teacher pairs (54%) and almost half of the student pairs (48%) used fractions for their calculations during the task. Consequently, it was unanticipated that more than half of the teachers converted their fractions into decimals.

TRANSFORMED MODEL

Hypothesis 5: Students would construct their graphical representations of the physical models as they saw it (i.e., in the second quadrant if the spout was on the right side of the fountain). Teachers would also do the same.

In our initial discussions, about possible learning paths that might be demonstrated, we had agreed that students would likely create graphical models that closely resembled the actual physical model. Surprisingly, we saw a significant amount of students (74%) transform the physical model from the second quadrant to the first quadrant when constructing the graphical representation. This learning path was also unanticipated by the teachers, who predominantly constructed the model as they saw it (38%). The transformation of the model by the students evidenced more sophisticated, problem solving strategies than expected.

A novel finding emerged during the coding of the transformations. Our initial concern was with transformations between quadrants (e.g., flips across the *y*-axis); however, the student responses had one additional transformation that was completely unanticipated by us, and by the teachers. In more than half of the student solutions, students also transformed the x-axis from the basin of the fountain to the point at which the water left the spout (Figure 1). This enabled much more straightforward calculations. We did not see any evidence of this vertical slide in the teachers' responses, suggesting that the teachers also did not anticipate this as a possible student learning path. As Murata and Fuson (2006) state, alternative learning paths are possible; however, given that more than half of the students did this transformation, we do not view this as an *alternative* learning path, but rather one that ought to be anticipated by teachers.

FRIENDLY NUMBERS

Hypothesis 6: Teachers would anticipate that students would select "friendly numbers" (i.e., rounding up or rounding down a measurement to the nearest whole number) when determining measurements for their models of the water fountain, to ease calculations.

Both students and teachers predominantly used friendly numbers. In most instances, each group of pairs rounded to the nearest whole number. For instances in which students or teachers used decimals, the decimals were self-restricted to two decimal points. Therefore, students and teachers rounded were necessary, showing recognition of the somewhat arbitrary aspect of the measurements taken at the fountain.

OVERALL LEVEL OF COMMUNICATION OF MATHEMATICAL FINDINGS

Hypothesis 7: Learning paths of students, with respect to overall level of communication, would be predominantly level 3. The teachers' solutions, given the high SCK, would be predominantly higher.

Using the Achievement Chart for the Province of Ontario (Ontario Ministry of Education/OME, 2005) as our guide, we coded the overall level of communication of mathematical findings for all the samples submitted by students and teachers, with level 3 representing current, acceptable ministry standards of achievement. This code did not assess the overall correctness of the solution (i.e., graphical representation, calculations, etc.), but rather took into account whether there was evidence of a systematic approach, using disciplinary conventions and discursive practices of mathematics, in the solving of the task. Therefore, a solution may have achieved a level 3 or 4 in our coding, but as a result of the incompleteness and/or conceptual and other errors throughout the solution, it may have achieved a lower level when taking these other factors into consideration.

All of the students achieved level 3 or better. Surprisingly, the same was not true of the teacher responses. Although more than 70% of teachers achieved a level 3 or 4, on their communication of mathematical findings, 30% did not. We reiterate here that the teachers appeared to be engaged and motivated to produce good work on the actual task given that their solutions were being examined by peers and within a research context. We also restate that these teachers had both significant academic training in mathematics and mathematics education, plus extensive classroom experience. Despite these important contextual circumstances of the teachers, almost one-third of the teachers were unable to anticipate student learning paths with respect to communication of mathematical findings. Again, we are not suggesting that this was intentional by the teachers. Nevertheless, this result is striking.

IMPLICATIONS FOR MATHEMATICS FOR TEACHING

Our purpose in this research was to examine teachers' anticipation of student learning paths across a common task, and to consider the implications of the results in relations to the mathematics for teaching framework proposed by Ball et al. (2005). This research centered on KSC and SCK, as the central domains of analysis. Earlier research has shown that teachers with advanced degrees in mathematics (or a related field) were not necessarily better at teaching mathematics (Deborah Loewenberg Ball, 1989). We wondered then whether there were instances when SCK may then interfere with KSC, and what insight this may shed on the development of the other domains.

To recall then, the research questions guiding our work were: Therefore, the research questions guiding this work are: (1) what are the assumptions teachers with sufficient or high SCK make about student thinking (i.e., KSC) or the "learning paths" that students take? And, (2) in analyzing such assumptions, what conceptual and pedagogical insights might be mined to support knowledge development in the other domains defined by Ball et al.(Deborah L Ball et al., 2005)?

As our results show, the teachers in this study with high SCK *and* CCK, and significant experience with KCT, were not able to easily anticipate the learning paths of students. In other words, they had difficulty anticipating the KSC. We suggest that this is demonstrated by learning paths evidenced in the teachers' solutions demonstrates just this and can be attributed largely to high levels of SCK.

The conceptual errors and the theoretical reasoning of the teachers, in this research, are intertwined and directly related to SCK. Although many teachers neglected to transform the *x*-intercepts of their graphical representations, we felt that this error was not in relation to the actual mathematics, but rather to the perception that the model was perhaps unnecessary in order for them to complete the task. Consequently, the physical model was not fully incorporated into their solution. The teachers showed more dependency on their theoretical understanding of the quadratic relation (or high SCK). Their results were less precise because of this, although we believe that the teachers' intentions were increased precision. We contend, however, that for both teachers and students, complex calculations do not necessarily imply complex and/or task-related thinking.

Students adapted to the task somewhat more concretely and did better overall because of their use of the physical model. Students' concrete thinking was further evidenced by the creative ways in which their graphical representations were transformed in order to make the calculations more straight forward. This was surprisingly not anticipated by the teachers or us - the researchers. This leads us to question whether student learning paths might potentially be restricted or develop at all, if higher SCK informs the decisions teachers make in relation to KTC.

Also compelling is the evidence that many teachers, with significant SCK, were unable to complete the task. Again, here we assert that over-theorization was the contributing factor. In spite of this assertion, we wonder how teachers come to zoom in on the particulars of CCK and KTC in order to develop an understanding of KSC, given the interference of SCK.

CONCLUDING THOUGHTS

Our research demonstrates that more research is needed to examine, not only those with limited SCK, but also those with significant SCK. Indeed, a different type of mathematics for teaching education may be required for those with high SCK. These teachers, in our opinion, would have been the most likely to anticipate student learning paths, given their personal histories. Yet, the learning paths of these students were not well anticipated, which we contend was not intentional.

However, as concluded by the second author of this paper, who brings more than 30 years of classroom teaching experience to this analysis, despite a teacher's experience and high levels of SCK, student thinking can still be surprising and reveal alternative ways of thinking about mathematics. We surmise, as well, that having high SCK might permit teachers to look at alternative learning paths and see the validity and merit in those learning paths. The ability of those with limited SCK may not have the same sort of elasticity within the domains defined by Ball et al. (2005).

Our findings demonstrate that indeed the loci of mathematics for teaching, within Ball et al.'s (2005) domains, does rest in KSC. Teachers' knowledge of the other domains can be examined, by extension, through an analysis of KSC. Furthermore, we propose that our research suggests that perhaps a fruitful starting point for mathematics teacher education may be through an examination of student learning paths or KSC.

In our research, we did not return to the teachers and present the student results in relation to their own projection of student learning paths. This remains an important area of further inquiry and, as mentioned earlier, could be a productive way of educating both current and in-service teachers about the various domains of mathematics for teaching. Furthermore, we do not explore the factors that do contribute to student learning. Further analysis is needed to see what other factors contribute to student understanding, particularly when the learning paths of students and differ from those projected by teachers.

Our hypotheses of student learning paths, on the whole, coincided with the student samples. We recognize that our role as these students' regular classroom teachers may have influenced this. However, aside from the common task, the individual choices each of us made within our classrooms, as teachers, were not common or shared. We each made independent decisions about our classroom practices. However, like our colleagues, who engaged in this research, our choices are restricted by the current policies in mathematics education (i.e., curriculum, assessment, and authorized texts) so there is a commonality in pedagogy to an extent. This commonality between our fellow teachers and us should have resulted in a more cohesive set of learning paths between the students and the teachers. However, this was not the case.

How teachers think about student thinking potentially correlates to the ways in which teachers teach. By examining teachers' anticipation of student thinking, we can then begin to unpack the assumptions teachers make. Furthermore, we can begin to understand the kinds of additional knowledge that teachers might need to be more effective at teaching mathematics.

Table 1: Overall results from the coding, as percentages, for the teachers and students.

Code	Teachers	Students
Conceptual error	69%	26%
Theoretical reasoning	31%	13%
Incomplete	31%	44%
Fractions	54%	48%
Transformed model	38%	74%
Friendly numbers	92%	96%
Overall level of communication of mathematical findings	L4: 62%	L4: 74%
	L3: 8%	L3: 26%
	L2: 22%	L2: 0
	L1: 8%	L1: 0

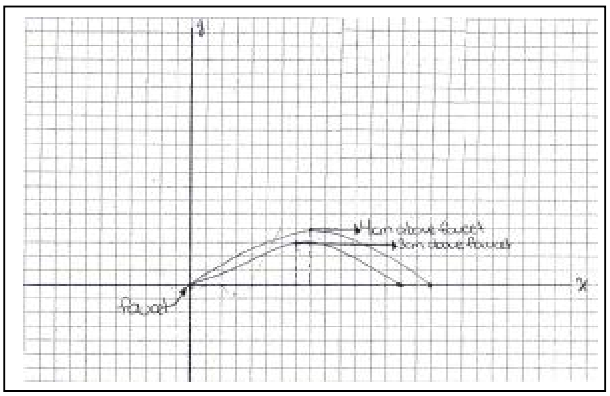

Figure 1: Student sample – x- and y-intercepts transformed.

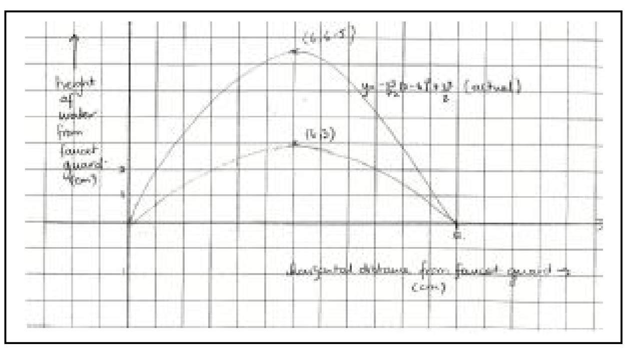

Figure 2: Teacher sample – x-intercepts not transformed.

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