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**ELEMENTARY SCHOOL STUDENTS' INTUITIVE CONCEPTIONS OF RANDOM  
DISTRIBUTION**

**Sibel Kazak**

**Jere Confrey**

**ABSTRACT.** This research focuses on fourth-grade (9-year-old) students' informal and intuitive conceptions of probability and distribution revealed as they worked through a sequence of tasks. These tasks were designed to study students' spontaneous reasoning about distributions in different settings and their understanding of probability of various binomial random events that they explored with a set of physical chance mechanisms. The data were gathered from a pilot study with four students. We analyzed the interplay of reasoning about distribution and understanding of probability. The findings suggest that students' qualitative descriptions of distributions could be developed into the quantification of probabilities through reasoning about data in chance situations.

**KEYWORDS.** Statistics Education Research, Probability, Random Distribution, Elementary School, Student Intuitive Conceptions.

**INTRODUCTION**

Stochastic ideas and intuitions are widely used in almost every field of our lives, from sciences to sports and from games to legal cases, as we make decisions under uncertainty. Data and chance are two related topics that deal with uncertainty, and statistics and probability are the mathematical ways of dealing with these two ideas, respectively (Moore, 1990). Hence, knowledge of probability and data analysis becomes of critical importance for ordinary citizens to make judgments in chance situations as well as to make decisions on the basis of numerical information. When the National Council of Teachers of Mathematics (NCTM) publicly strengthened their emphasis on these topics in the school mathematics curriculum at all grade bands (e.g., NCTM, 1989, 2000), probability and data analysis began to be introduced from Pre-kindergarten through Grade 12 in the United States.

Unfortunately, there is an artificial separation between chance and data analysis topics in instruction, which some researchers (Steinbring, 1991; Shaughnessy, 2003) have called attention to. More specifically, there is no conceptual connection in the curriculum between

theoretical random distribution (i.e., probability distributions of random variables) and empirical data distribution (i.e., statistical distributions of data). In the traditional approach, for example, probability is taught primarily based on the classical definition (i.e., when all outcomes in an event are equally likely, the probability of the event occurring is the ratio of the number of outcomes in the event to the number of outcomes in the sample space) and maybe with the frequentist approach in which the empirical probability of an event is the relative frequency of occurrences in a large number of trials (i.e., as the ratio of favorable outcomes to the total number of trials). In teaching data analysis, the focus is mainly on the ideas of center, spread, and shape of data, which are not necessarily probabilistic. In this instructional or curricular approach, the conceptual link between probability and data analysis is not treated until the discussion of statistical inference (in advanced levels) when the idea of probability is needed to provide the basis for quantifying statistical uncertainty.

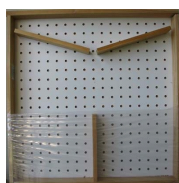
To begin to address this artificial division between the treatment of probability and data topics in instruction, this study aimed to focus on the role of the notion of distribution as a link between data and chance. In this paper, we report on results from a pilot study whose preliminary findings were presented at the ICOTS-7 (Kazak & Confrey, 2006). The purpose of this study was to help develop conjectures about possible *conceptual trajectories* [i.e., possible pathways that students can navigate during any particular set of instructional episodes (Confrey, 2006)] along which students' ideas about distribution and probability in chance situations might develop. These conjectures informed a larger study (Kazak, 2006)

### THEORETICAL FRAMEWORK

Extensive research has been done on probabilistic reasoning across a wide range of age groups. Researchers have documented persistent erroneous conceptions (or misconceptions) and strategies that adults use in judging the likelihood of uncertain events (e.g., Kahneman & Tversky, 1972; Tversky & Kahneman, 1973) and how students interpret and make statements about probability (e.g., Konold, 1991). Shaughnessy (2003) claims that students' strategies related to the questions about the likelihood of outcomes and the results of repeated trials in various uncertain or chance situations have roots in both probability and statistics. Accordingly, Shaughnessy points out a close link between the notion of sample space in probability and the nature of variation in statistics. Furthermore, he suggests that students should be introduced to probability through looking at data from simulations. This approach assumes that statistics should motivate probability questions in the context of real data.

From the cognitive perspective, some researchers studied how children develop the ideas of chance and probability (e.g., Fischbein, 1975; Piaget & Inhelder, 1975). Piaget's work, for example, focused on the formation of the physical aspects of the notion of chance, the basis of the quantification of probabilities, and the development of the combinatoric operations in (4-to-

12-year-old) children's ideas (Piaget & Inhelder, 1975). Piaget and Inhelder anticipated that children could build upon their intuition of random mixture to reason about the fortuitous distributions. They looked at children's understandings of different forms of distributions of balls in inclined boxes (see Figure 1 as an example) with a funnel-like opening in the middle of the upper part of the box and equal-sized (2, 3, 4, or more) slots partitioned by a divider at the bottom. Children were asked to experiment with the boxes dropping balls from the funnel and then to explain the arrangements of the balls in the slots. The researchers found that young children (four to six years old) lacked an understanding of a distribution of the whole as they failed to predict or generalize the symmetrical dispersion of the balls in the slots. Although 7-to-10-year-old children began to understand the dispersion as a whole with more or less precise symmetry, they failed to recognize the role of large numbers of balls (i.e., the relative fortuitous difference of the number of balls in the slots would diminish as more balls were dropped repeatedly). The children of ages 11 to 12 began to quantify the distributions looking at the number of balls in the slots, such as "just about equal-eight more" (p. 47). They also came to understand the role of large numbers in the regularity of distributions.



**Figure 1.** The split-box for marble drops.

In a more recent study, Horvath and Lehrer (1998) conducted classroom research in which second graders rolled one or two dice and used bar graphs to record their results and to represent the number of different ways of getting each outcome in the sample space. The researchers considered several aspects of normative stochastic reasoning in documenting young students' intuitive probabilistic reasoning when rolling two dice. Those included randomness, the distinction between certainty and uncertainty, the nature of the experimental trials, sample space and probability distribution, the relationship between individual outcomes and patterns of outcomes, the treatment of difference between the expected and actual outcomes, and the validity of evidence. Many children did not think the experiment of rolling dice was completely random prior to collecting data by rolling dice. Rather, they believed that they could predict the next outcome in rolling dice based on, for example, which numbers were "lucky." During the initial experimentation, the students noted the different ways students tossed the dice. Half of the class believed that tossing the dice different ways could affect the results of the experiment. After a class discussion, the students agreed on rolling dice out of a cup to eliminate any possible bias on the outcomes. As they had more experience with the tasks, most students began to expect that global patterns of events (the shape of distribution in the long run) were more predictable than a single outcome. The graphical representation of the number of ways of getting each sum in the

sample space helped students reason about the general shape of the distribution and the significance of results as it highlighted the structure of the sample space. Moreover, the class discussion on the number of ways to get each outcome (e.g., two ways to get an outcome of 3 are “2 and 1” and “1 and 2”) engaged the second graders in thinking about the link between the sample space and the actual outcomes. However, a few students came to understand residuals (difference between predicted and actual results) in terms of the relationship between outcomes and sample spaces.

In addition to research on novice reasoning, the historical development of probability provides insights into its different interpretations. Hacking (1975) noted the duality of the concept of probability during the transformation of the old concept of sign in the medieval periods (i.e., evidence of testimony by the authority found in natural signs) to the inductive evidence (i.e., evidence of things in recognition of the connection between natural signs and frequency of their correctness). This dual nature of probability, which still exists, includes (1) an epistemic notion of probability, referring to degree of belief supported by evidence and (2) a statistical notion of probability, concerned with stable frequencies of occurrences of certain outcomes (Hacking, 1975). Hence, the concept of probability was historically used both to describe the degrees of belief relative to our background knowledge and to refer to the tendency of certain random events. This duality in effect recognizes both formal and informal uses of probability that can be encountered in children’s reasoning about uncertain events.

According to Steinbring (1991), “beginning with very personal judgments about the given random situation, comparing the empirical situation with their intended models will hopefully lead to generalizations, more precise characterizations, and deeper insights” (p.146). In other words, Steinbring suggests that subjective probabilities based on our knowledge, but not simply a matter of opinion, can be checked by experiment. Kazak and Confrey (2005) pointed out that when talking about probabilities, one draws upon a variety of evidence, such as personal knowledge or belief, empirical results, and theoretical knowledge. We think young students’ understandings of probabilities, in particular, are based on their personal and experiential knowledge about the world. Therefore, it is critical to involve students simultaneously in conducting experiments and thinking about the sample space so that the empirical and theoretical interpretations of probability develop together.

## METHOD

### Participants

Research participants were four fourth-grade students (9-year-olds), whom we refer to as Brad, Jim, Kate, and Tana (pseudonyms). The participating students were recruited through their classroom teacher at a local elementary school in St. Louis, Missouri (U.S.A.). The four students met with the first author for about an hour over four sessions in the spring of 2005.

## Data Sources and Analysis

The major sources of data included student-produced artifacts, videotapes of student responses in individual interviews, and videotapes of whole-group discussions as they participated in the tasks. A qualitative analysis of these data was conducted by identifying emergent categories based on students' responses and actions that reflected their understandings and strategies in reasoning about distributions and probability.

## Tasks

To emphasize the connection between data and chance starting from the early grades, the focus of the tasks designed for this study was on the ideas of probability and distribution. Our notion of distribution comprises two ideas: (1) a statistical interpretation in which data are viewed as an aggregate in reasoning about distributions and comparing distributions in data analysis (e.g., Cobb, 1999; Lehrer & Schauble, 2000); and (2) a probabilistic sense in which we consider the possible outcomes (or sample space) of a chance experiment (e.g., Horvath & Lehrer, 1998; Vahey, 1997). The important distinction between these two is that the former deals with distribution of real data, described by center and spread statistics (e.g., mean or variance) while the latter refers to theoretical models of variables resulting from a random experiment and described by the probabilities assigned to all their possible outcomes (e.g., probability distribution). In this study, while the first two tasks were designed to serve as a starting point of reasoning about statistical distributions, the last task about the binomial distribution of rabbit hops entailed an informal understanding of a probability distribution. Next, we describe each task.

### *Task 1: Non-graphical Distributions*

To examine informal, non-graphical understandings of distributions, students were shown on two occasions the digital pictures of real objects distributed in space. These included a buffalo herd, a sheep herd, wild flowers in a plateau, leaves under a tree in the fall. They were asked: (1) to describe what they noticed in the pictures; (2) whether they could see any pattern; and (3) to explain where they saw more (and less) of the objects and why they thought there were more or less there.

### *Task 2: Dropping Chips Experiment*

Students' understandings of distributions with center clump were examined as they conducted experiments in which they were asked to predict, generate, and interpret distributions of dropped chips. The experiment involved dropping 20 chips all at once through a tube

positioned 15 inches above the floor and centered over a white sheet of paper. Students were first asked to predict approximately where the majority of the chips would end up. Each pair then conducted the experiment after which we discussed the results as a group. Students then conducted another trial. Next, they were asked to predict results of dropping 20 chips when the tube was 30 inches above the floor. Students then compared the results of the 15-inch and 30-inch drops. After these experiments, each pair was asked to create a game in which they gave different points for chips based both on the distance they landed from the target and the height from which they were dropped. Each pair played their own game as well as each other's game.

### *Task 3: Split-box*

The split-box task (see Figure 1) was used to investigate students' understanding of distributions of results from a device in which the chance mechanism was evident. This split-box [adapted from Piaget and Inhelder (1975)] had a centered funnel-like opening on the upper part from which marbles could be dropped. Beneath a partition were two slots of the same size into which the marbles could go. It was constructed with the hope of inviting the observation that a marble was equally likely to go right or left, and thus of producing a uniform distribution.

Students were first asked to predict whether a single marble would go to the right or left slot. They then dropped the marble and commented on the actual results in comparison with their expectations in 10 trials. Next, students dropped 10, 50, and 100 marbles. Before each trial, they were asked to make predictions about whether they would get more marbles on one side or the other and by how much, or the same number of marbles in each slot. After conducting an experiment, they then were asked to discuss the resulting distribution of marbles and compare it to their expectations.

### *Task 4: Flipping a Coin*

The purpose of this task was to examine students' conceptions and reasoning about the results of repeatedly flipping a coin 5 or 10 times prior to the Hopping Rabbits task in which students were asked to simulate the rabbit hops by flipping a coin. When they tossed a coin 5 and 10 times, students first made predictions about the outcomes and then discussed the results. After this experiment, they were asked to make predictions about the number of Heads/Tails for 5 coin-tosses and 10 coin-tosses.

### *Task 5: Hopping Rabbits*

In this task, students were given a problem that could be modeled by a binomial distribution. The *Hopping Rabbits* activity was adapted from Wilensky (1997) in which one of

the research participants created this model in an attempt to make sense of normal distributions. The aim in this task was to use simulation to help students link the observed frequency of outcomes to the probability of outcomes as determined through analysis of the sample space.

Suppose there are a number of rabbits on a land where each rabbit can choose to hop only right or left. For each hop, rabbits are as just likely to hop right as left. We want to know where a rabbit is likely to be after five hops.

Students began by making predictions and then were asked to simulate where the rabbits would be after 5 hops by tossing a coin 5 times. While one student was flipping a coin, the other kept track of each hop and marked where the rabbits ended up on the number line after 5 hops. After the simulations, there was a whole-group discussion about the aggregate distribution of final position of the rabbits. As part of this discussion, students offered their conclusions about the most likely/the least likely outcomes (i.e., the final position after 5 hops), the set of all possible outcomes, and the different ways to get those outcomes (see Table 1).

**Table 1.** All possible ways to get to the final position after 5 hops with the coin simulation.

		Combinations						
		5T	4T1H	3T2H	3H2T	4H1T	5H	
Permutations (All possible ways)				TTHH	HHHT			
				HTHT	THHH			
				THTH	HTHT			
				TTHT	HHTH			
				HTHT	THHT			
			TTTT	THTH	HTHT	HHHT		
			TTHT	HTTH	THHT	HHHT		
			TTHT	HHTT	TTHH	HHTH		
			THTT	THHT	HTTH	HHTH		
		TTTT	HTTT	TTHH	HHTH	THHH	HHHH	
		-5	-3	-1	1	3	5	
			Final Position = # of H - # of T					

## RESULTS

In this section, we present our analysis of how the students made sense of various random distributions in each activity. The findings presented below illustrate students' intuitive conceptions about distribution and probability, methods they used to understand and explain the random events, and their reasoning about random distributions.

### Task 1: Non-Graphical Distributions

When students were asked in the interviews to describe what they noticed in the pictures of a buffalo herd, sheep herd, and leaves under a tree, they focused primarily on various patterns and the notion of density. They also attempted to explain the patterns in terms of specific causes. Common to the students' initial response was the amount of things in the pictures, i.e., "lots of buffalos" and "too many leaves." When asked to explain where they saw more and less of them, students talked about different patterns, such as aggregates and spread-out-ness, in their own informal language:

**Kate:** "Most sheep *gathered up* together".

**Jim:** "Over here they [buffalos] are kind of *spacing out*, but over here they look like *jamming up* a little bit".

**Tana:** "They [flowers] are *scattered*".

Furthermore, students tended to explain the patterns they observed by providing causal interpretations:

**Kate:** "They are all gathered right here" [pointing to the right upper corner of the picture]

**Sibel:** "What do you mean by "gathered"? Can you explain it?"

**Kate:** "They are probably on a trail together".

**Jim:** "There are just little [leaves] that made this far. Probably because the wind would have to be blowing long enough in the right direction for those get there. But that would happen to fewer leaves because mostly they would fall down by the trunk".

To follow up students' initial interpretations of the natural distributions during the one-on-one interviews, the whole group discussion of selected pictures (a buffalo herd, a sheep herd, wild flowers in a plateau, leaves under a tree in the fall) focused on how students reasoned about the density while estimating the number of things in the pictures. For instance, those who talked about the density used "bunch, pile, or crowd" to refer to high-density areas and "separate, left alone, or spaced out" to refer to low-density regions:

**Brad:** "There are more [buffalos] in the [right] corner".

**Sibel:** "How do you know there are more?"

**Brad:** "Because there is like a *big bunch* together".

**Sibel:** "How about the others [in the middle]?"

**Brad:** "They are just *separate*".

When students paid attention to the number of buffalos in the picture, the discussion about estimating the amount of animals led them to develop a method:

**Brad:** "There is a lot of them".

**Kate:** "They are bunched up".

**Sibel:** "Can you count them?"

**Kate:** "Yeah, you can estimate. I'd say 270".

**Brad:** "I'd put them in a cage".

**Jim:** "If I could have something to mark, I would separate this into four pictures to cover every buffalo. And then count one group, write that down. Count the next group and write that down".



In this exchange, each student had a different idea about how to estimate the number of buffalos in the picture. For example, Brad explained that he would put the buffalos in a cage in groups of hundreds for counting. However, Jim divided the picture of buffalos into (approximately) equal quarters to show his method of estimation.

**Sibel:** “Do you think there are equal numbers of buffalos in each quarter?”

**Jim:** “I have doubt”.

**Sibel:** “Why?”

**Jim:** “Because over at this corner [left, upper] there is not much. Then on this side [right, upper] over here it looks like there is more. This area [right, lower] doesn’t hold too much. And on this side [left, lower] there is fair amount”.

Jim’s use of equal partitions to estimate the number of buffalos gives a sense of relative density. When he talked about the number of buffalos in each quarter, he contrasted the areas that held “more” or “fair amount” of buffalos with those that contained “not much.” Jim’s method of clustering buffalos in (about) equal-sized sections seemed to entail the concept of density (i.e., buffalo per common unit), while Brad’s method did not.

## **Task 2: Dropping Chips Experiment**

In pairs, students first dropped 20 chips through a tube when the tube was 15 inches above the ground. While describing the distribution of the chips, students initially focused on where most of the chips were on the floor by partitioning the chips into two groups, one centered close to the tube and the others further away. For example, Jim said: “Because I was holding this [the tube] right about here. So, they kind of stacked up right here [showing a smaller area in the middle] but they are about around here [showing a larger area around that middle one].” Note that his explanation suggests a notion of density indicated by the small area right under the tube (i.e., the middle clump) and an expected spread shown by the larger area around that.

When students were asked to conduct the same experiment while holding the tube 30 inches above the ground, their predicted plot of the distribution of the chips would still have that middle clump which was a bit bigger, showing the density under the tube. On the plot, students also expected more area outside of the middle one to accommodate the expected spread of the chips. After the experiment, Tana looked at the distribution of the chips on the sheet and pointed to the region where a few chips landed all together under the tube: “this is like a pile and the rest is separate.” In Tana’s explanation, the “pile” indicated the middle chunk with higher density as opposed to the low density and more spread outside of that middle.

While comparing the results for the chips dropped from the two different heights, Kate commented that the chips dropped from the higher position were “more separate and scattered,” while the ones dropped from the lower height were “gathered around” in the middle. Then she explained that the height at which the chips were dropped might affect the outcomes. Next, in

the discussion about the effect of the height variable, students were able to offer reasons and conjectures for why the chips distributed that way. For example:

**Jim:** “When they landed, I noticed that a lot of them started rolling around. That might have affected it”.

**Brad:** “If it goes up higher, then they will spread like *almost everywhere*”.

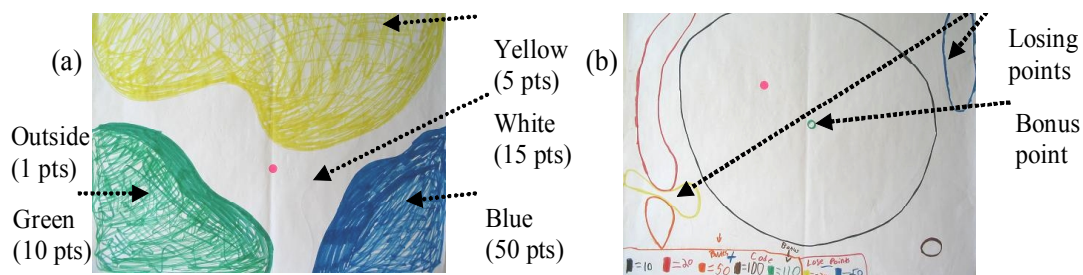
**Sibel:** “Almost everywhere. Hmm. That’s interesting”.

**Tana:** “Maybe a little bit around here [region under the tube] and the rest of them are *all over*” [beyond the region below the tube].

**Sibel:** “So, do you expect more “all over the place” when you do it higher?”

**Kate:** “Yeah. If it was like a foot-long, it would probably be close to each other”.

To foster students’ reasoning about the effect of height on the distribution of chips, we asked them to design a game using the chips. Each pair chose a higher position than 15 inches to drop the chips in their games considering its effect on the spread. Jim for example said “we did it at higher level so they’d roll around more” but he was not sure if that would give a “higher or lower chance” to win the game before playing. In their game, Kate and Tana divided the sheet into four regions of different sizes as seen in Figure 2a. They assigned the highest point to the blue region at the lower right corner as Tana explained, “sometimes the smallest part is hard to get on.” In Jim and Brad’s game (see Figure 2b), the bigger circle in the middle had the lowest score while a very small area close to the point where the chips were dropped got the highest point in the game, which they called “the bonus point”. They also had an idea of “losing points” that are assigned to the two regions outside of the bigger circle for the chips expected to roll around randomly. By including “bonus point” and “losing points,” Jim and Brad began to express different chances that might exist in a typical middle region and the outside of that region.



**Figure 2.** Student-generated games: (a) Kate and Tana’s game (b) Jim and Brad’s game.

### Task 3: Split-box

Students began with making predictions about whether, when released, the marble would go to the left (L) or right (R) compartment in the split-box. Kate conjectured that the marble might go to right without hitting the middle divider, but if it “bumps up” against the divider, then it might go to left. In her reasoning, the outcome of the experiment depended on

which path the marble would take when it was released from the top. In predicting the outcomes of successive trials, some students tended to choose the opposite of what occurred. Based on the previous result, Brad, for example, made a conjecture about a pattern of results, such as L-R-L-R. It seemed that Brad expected a pattern of alternating outcomes, which could be attributed to a conception of randomness in sequences (e.g., Cohen, 1960; Falk, 1981).

In the subsequent experiments, students were given 10, 50, and 100 marbles. Students inclined to carry out various investigations by letting the marbles fall down through the funnel. They attempted to understand the mechanism of the physical apparatus by watching how the marbles bounced off the middle divider and repeated experiments possibly to find out an algorithm to predict outcomes. For instance, they dropped different numbers of marbles from each side of the funnel, such as five marbles from each side, or six marbles on the left side and four on the right side, or all on one side. After several experiments, Jim conjectured “I think that if we put more on this side [left], it has a bigger chance to go on this side [right] because they are opposites and it might go something like that and in this something like that [showing possible paths from the left-top to the right-bottom and vice versa].”

When asked to predict the number of marbles in each side of the split-box, students tended to make their predictions unequal, but “close to even,” such as 6 to the left and 4 to the right, or 27 to the left and 23 to the right, or 49 to the left and 51 to the right. Although students used the notion of “50-50” to refer to the equal distribution of marbles in each slot when they dropped 100 of them, their predictions for the results were mostly “close to equal” (i.e. “48-52”) for 100 marbles. Similarly in Piaget and Inhelder (1975), children (7 to 11-year olds) expected about equal number of balls between the right and left slots, but with no recognition of any equalization as the number of balls increases.

#### **Task 4: Flipping a Coin**

Similar to the findings in the previous task, predicting “even” or “close-to-even” results (i.e., 5 heads and 5 tails or 6 heads and 4 tails) was common in flipping a coin 10 times. However, Kate and Tana seemed to show no expectation that one outcome may be more likely than the others. For instance, Tana made her prediction as “1 heads and 9 tails” and Kate was inclined to give two predictions, such as “5 heads and 5 tails” and “3 heads and 7 tails.” After flipping a coin 10 times, Brian and Jim got 5 heads and 5 tails while Tana and Kate got 3 heads and 7 tails. Kate interpreted these different results as “anything can happen” which suggests reasoning based on the outcome approach (Konold et al., 1993). Using this approach, one is focused on whether you can make correct predictions, not on quantifying the probabilities of the various outcomes. Consistent with this outcome-oriented view, she pointed out “When you flip a coin, there is no right answer. So there is an estimate. It could be heads ten and tails zero.”

### Task 5: Hopping Rabbits

When students were asked to make predictions about where a rabbit would likely be after five hops, they initially responded with a deterministic mindset:

**Jim:** “If I were a rabbit, I’d know where I’d land.”

**Sibel:** “You would know?”

**Jim:** “Yeah, because I get to do it...Or, I could just tell the rabbit what happens next”.

However, introducing the idea of simulation of the rabbit hops with coin tosses (i.e., students assigned Tails to the right and Heads to the left) helped students consider different possibilities due to the chance associated with the coin flip. Based on the number of hops, students first noted the range of possible outcomes (from -5 to 5 on the number line given that they start at 0). They gave a range of responses to the question “where do you think they are most likely to be after 5 hops?”

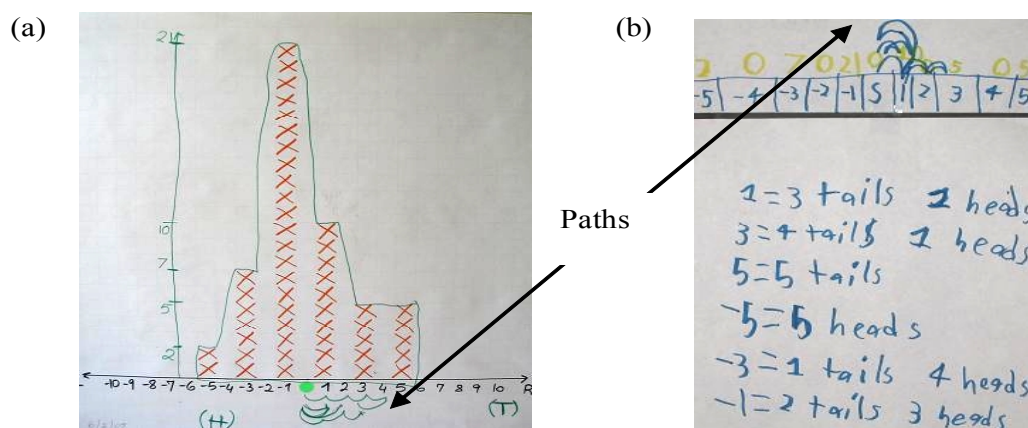
**Kate:** “I think most of them on this side “[right].

**Jim:** “One. I think it is going to be this”.

**Brad:** “Three”.

**Kate:** “More here” [on four].

Kate’s last prediction “4” for the most likely outcome after 5 hops led to a new discussion about whether it would be possible to land on an even number on the number line after 5 hops. Jim’s strategy was to try different combinations of five hops (see the paths in Figure 3) to convince others that it was impossible to land on even numbers on the number line after an odd number of hops.



**Figure 3.** Student-generated inscriptions for the rabbit hops.

After each group conducted their simulations and plotted their outcomes on the graph paper (see combined results in Figure 3a), students were asked to interpret them. They began by comparing the individual points (i.e., “-1 has the most”, “1 is the second”) as well as aggregates of data (i.e., “There is a majority in the negative side than the positive side”). Moreover, Tana

made a conjecture that since there were more rabbits on the negative side, the coin must have landed on Heads more than Tails (they assigned Heads to left initially). So, she was able to make a connection back to the coin flipping based on the compound results. Students also noted that the outcomes were “spaced out” on the graph. They acknowledged the different probabilities of various outcomes by referring to them as “easy to get” and “hard (or rare) to get.” Jim described different ways to get to the places on the number line arguing that, “usually to get to negative one, you want THTHH and it’s only three Heads and two Tails or sometimes it went HHTTH.” When they attempted to quantify the likelihoods of outcomes by figuring out the possible ways to get an outcome, students used a range of inscriptions (Latour, 1990), including lists, paths, and stacked plots (Figures 3a and 3b). Note that Jim’s list of combinations to obtain each final outcome in Figure 3b is a critical step towards quantification of probability which involves recognizing the respective ordered arrangements of those combinations in a sequence. This example might add a new level of understanding about constructing an idea of chance and probability which, according to Piaget and Inhelder (1975), essentially depends on the ability to use combinatoric operations in random mixture cases, such as in urn problems. That is, constructing conceptions of sample space, combinations, and permutations in the context of five coin-tosses to determine the final locations of the rabbits enables students to develop the quantification of probability viewed as the relationship between the part and the whole with regard to the outcomes of a random event.

## DISCUSSION AND CONCLUSION

The goal of this study was to develop conjectures about various ways in which fourth-grade students’ reasoning about distributions and understanding of probability might develop. The findings of this study suggest that students’ spontaneous understanding of distributions and probability in various chance settings can possibly evolve into the notion of a probability distribution, which involves a quantitative perception of probability.

The existing literature documented that students expressed the qualitative characteristics of data distributions as clumps, clusters, bumps, hills, gaps, holes, “spread-out-ness,” and “bunched-up-ness” (Cobb, 1999), on graphical representations (i.e., the frequency plots). This pilot study data revealed similar findings. Those included informal language to talk about different aspects of distributions (i.e., “gathered up together”, “spaced out”, “jamming up”, “in a pack”, “ groups”, “scattered”, “bunched up”) as well as causal explanations (i.e., “the wind blows the leaves”, “they [sheep in a group] are probably friends”, “there is better grass there”) to explain the patterns. Also, students tended to care about patterns when they paid attention to the arrangement (the way things are distributed) of and quantity (“more” or “less”) of things.

Through the experiment with dropping chips, student strategies involved showing and drawing hypothetical borders around where most of the chips were expected to land. When asked

to drop the chips from a higher position, students tended to draw a bigger middle region and some small ones around that to accommodate the middle clump and a bigger spread or more extreme data points (i.e., “this is like a pile and the rest is separate”). In designing their own games with dropping chips, students tended to assign a bigger score to the locations where the chips were not very likely to land, such as the smallest area which was not in the middle and the region which was close enough to the middle (where the tube was supposed to be held) but very small. So, they might have some intuitive ideas about chance in games, such as a dart game, which involves both skill and chance.

The split-box was used to investigate equiprobability (a uniform distribution of marbles). However, there is a much more complex situation than that. For example, the marbles may or may not hit the middle divider because of the way they roll down depending on the box surface (smooth and flat, or not), the marbles (uniform and perfectly round, or not), and the force applied to the marbles (even, or not). Then, as stated by Piaget and Inhelder (1975), there is physics involved in this task. The findings from this study confirm that argument. Students developed conjectures about how the marbles would roll based on the ways of releasing them from the top (i.e., dropping five marbles from each side or one on the left side and nine on the right side, or dropping them in a lined-up position or pouring all the marbles into the funnel). When students believed in causality rather than pure chance in the physical apparatus, they tended to inquire about any systematic pattern and algorithm to explain these patterns. It could be possible that they start believing that chance takes over in the absence of a determining cause in the apparatus. When predicting the outcomes in the split-box (i.e., the number of marbles in each slot) and flipping a coin many times (when the chances are equally likely), students often gave responses “close to equal.” This could imply that they believed the outcome would be as close as possible to but not always perfect (equiprobability). In this sense, perhaps they were trying to communicate both expectation and variability. Also, students might be more likely to use the outcome approach (Konold, 1991) in equiprobable events if they perceive the task as making accurate predictions of single trials.

In the *Hopping Rabbits* task, students’ initial responses indicated a distinction between the role of flipping a coin to simulate random rabbit hops and that of deterministic decision-making. For example, “I could just tell the rabbits what happens next” encompasses a deterministic view whereas the coin simulation is considered more chance-based, such as “Then it’s 50-50.” Also, the question was initially about a single rabbit. Once they simulated the random rabbit-hops by flipping a coin and recorded their results on a stacked-dot plot, students noted the possible/impossible outcomes of the event and began to describe the likelihood of possible outcomes qualitatively (i.e., “easy to get” or “hard to get”) by using inscriptions, such as paths. These inscriptions as initial steps in a modeling process suggest that students have intuitive ideas about permutations and combinations, which can serve as a strong basis for developing quantitative conceptions of probability and distribution. Also, Jim’s way of representing different

combinations to obtain each outcome and the beginning of showing different ways for each combination with paths indicate the importance of permutations and combinations in understanding of probability, particularly in transition from noting all possible ways to get each outcome to quantifying the probabilities of those outcomes.

In general, the findings from this study suggest that the students' informal conceptions of chance and data could be developed into more powerful ways of thinking in probability and distribution through a sequence of tasks. Moreover, the use of simulations to model a chance event by a binomial distribution where students can build upon their previous experiences could be a great potential to link the discussions of probability and distribution. In this modeling approach, it is important to note the role of student-generated inscriptions, such as paths for rabbit hops, in supporting their arguments in intuitive understanding of distributions in chance events and of compound probabilities. Furthermore, this study adds to our knowledge relating to children's understanding of the distribution as an overarching idea in the recent studies in statistics while focusing on explaining how children began to develop the important concepts related to probability and distribution in chance events.

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Author : **Sibel Kazak**  
E-mail : [skazak@srri.umass.edu](mailto:skazak@srri.umass.edu)  
Address : Scientific Reasoning Research Institute  
Lederle Graduate Research Tower #420  
University of Massachusetts  
710 N. Pleasant St., Amherst, MA 01003  
Phone : (1-413) 545-5865  
Fax : (1-413) 545-4884

Author : **Jere Confrey**  
E-mail : [jconfrey@wustl.edu](mailto:jconfrey@wustl.edu)  
Address : Department of Education  
One Brookings Drive  
Box 1183 Washington University, St. Louis, MO 63130  
Phone : (1-314) 935-6324  
Fax : (1-314) 935-5979