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MODESTUM

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Decimal multiplication and division in mathematics textbooks for prospective elementary teachers

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ABSTRACT

This study examined 11 U.S. textbooks written for prospective teachers to investigate how standard decimal multiplication and division algorithms are presented, especially the rationale of both algorithms. Analytical frameworks of various methods used in different textbooks were developed. The findings suggest that half of the textbooks do not fully present the standard algorithms. The results indicate variations in textbooks' explanations for the procedures involved in the standard algorithms. Some procedures are explained more frequently than others. The reason why moving the decimal point in both dividend and divisor keeps the quotient unchanged is provided in most of the textbooks. Half of the textbooks explain why adding the decimal places in the multiplication algorithm. However, only two books explain the rationale for placing the decimal point in the quotient. The findings suggest that textbook writers pay more attention to explaining the rationales of the standard algorithms for decimal multiplication and division.

Keywords: textbook analysis, teacher education, decimal multiplication, decimal division

INTRODUCTION

Decimals and their operations are essential mathematical topics in K-12 mathematics curricula. Decimals are foundations for later mathematics achievement. A deep understanding of decimals is critical to acquiring skills in many non-mathematics areas (e.g., finance and engineering) and is crucial for success in many professions. Even daily, students constantly encounter decimals in activities such as cooking, purchasing, and measuring. However, as pointed out by various research studies, students and even adults demonstrated an inadequate understanding of decimals and their operations (Graeber & Tirosh, 1990; Lai & Murray, 2014; Lortie-Forgues et al., 2015; Siegler & Lortie-Forgues, 2017).

Research has suggested that prospective teachers shared similar struggles with students and demonstrated weakness in understanding decimals and operations (Lai & Murray, 2014; Muir & Livy, 2012; Tirosh et al., 1998). For instance, correctly placing the decimal point in a decimal division problem was difficult for both elementary students (Lai & Murray, 2014) and prospective elementary teachers (Tirosh et al., 1998). Commonly known misconceptions that "multiplication always makes bigger and division always makes smaller" were also found in both prospective teachers (Graeber et al., 1989; Lortie-Forgues & Siegler, 2017) and elementary students (Okazaki & Koyama, 2005; Steffe, 1994). Both populations performed better on adding, subtracting, and multiplying decimals than on dividing decimals (Lai & Murry, 2014; Tirosh et al., 1998). Prospective teachers will become future teachers. If they continue to struggle with understanding decimals, the same difficulties will be carried into their teaching of elementary students and have a negative impact on their students. When elementary students grow up and choose to become teachers, they re-carry these difficulties into their teaching. This process forms an endless and vicious cycle. Teacher education programs seem to be an important place to break this cycle by clarifying misconceptions, correcting mistakes, and deepening teachers' conceptual understanding through effective instructional strategies in university classrooms. Mathematics content courses for prospective teachers are designed to achieve this goal by refreshing elementary mathematics from a higher perspective. They are the few courses that can improve prospective teachers' mathematics knowledge for teaching and therefore influence future elementary students (Laursen et al., 2016; Matthews et al., 2010).

As McCrory (2006) indicated, one way to investigate what is taught in the mathematics content courses is to examine the textbooks written for mathematics content courses for prospective elementary teachers. Therefore, this study aims to investigate how the textbooks for prospective teachers present standard decimal multiplication and division algorithms because the inadequate explanations of the rationales of the algorithms might contribute to prospective teachers' weak understanding of the standard algorithms, as suggested by the research studies (Joung et al., 2021; Lai & Murray, 2014; Tirosh et al., 1998). Mathematics educators have distinguished different meanings of "curriculum". In this paper, "textbooks" or "curriculum materials" refer to "printed, often published resources designed for use by teachers and students during instruction" (Remillard, 2005, p. 213). "Enacted curriculum" or "experienced curriculum" is "what actually takes place in the classroom" (Remillard, 2005, p. 213). Though curriculum materials may not reflect the enacted curricula, the analysis of textbooks provides insight into the opportunities designed for prospective teachers to develop knowledge about decimals and operations because "the books define a substantial element of what students have an *opportunity* to learn" (McCrory, 2006, p. 20) and work as a "channel of influence" (National Research Council [NRC], 2002). Research at the K-12 level suggests that the distributions of practice problems of decimal arithmetic in mathematics textbooks for elementary students could impact what children do and do not learn (Tian et al., 2021). The types of problems children can and cannot solve were consistent with the types of problems presented in the textbooks. Though no similar direct connection is established at the college level, a study (McCrory et al., 2008) that surveys college instructors of mathematics content courses for prospective elementary teachers suggests that the participating instructors tended to use a textbook as their primary resource and depended on the textbook. The findings from this research support the rationale of this study to explore the curriculum materials as a valuable resource to identify the potential opportunities that could be experienced by prospective elementary teachers.

Research has suggested that some pre-service teachers believe that mathematics is a set of meaningless ideas and skills determined by experts (Berk & Cai, 2019; Steffe, 1990; Tirosh, 1990). For these teachers, the study of mathematics seems to become strictly following these made-up rules through memorization instead of through understanding. Helping teachers understand the rationale of the algorithms will change teachers' beliefs to view mathematics as a well-connected and meaningful subject and will eventually impact students' learning of mathematics (Philipp, 2007). Considering the role textbooks play in teaching, the role standard algorithm plays in mathematics education and in influencing learners' beliefs, as well as the recommendation from the professional organizations that prospective teachers should be competent in "explaining the rationales for decimal computation methods" (Conference Board of the Mathematical Sciences, 2012, p. 27), this study set out to investigate the presentation of decimal multiplication and division algorithm in textbooks designed for prospective mathematics teachers. The purpose of the textbook analysis is not to evaluate individual textbooks but to describe their similarity and differences to inform mathematics educators and instructors who will be using these textbooks of the worthy-to-be-attended differences which could potentially impact prospective teachers' learning. Specifically, the research questions are the following:

RQ1. How do textbooks for prospective elementary teachers present the standard algorithm for decimal multiplication?

a. How is the standard algorithm for decimal multiplication stated in the textbooks?

b. How is the rationale of the decimal multiplication algorithm explained in the textbooks? Specifically, in what way do the textbooks explain why adding the numbers of decimal places in factors determines the number of decimal places in the product?

RQ2. How do textbooks for prospective elementary teachers present the standard algorithm for decimal division?

a. How is the standar algorithm for decimal division stated in the textbooks?

b. How is the rationale of the decimal division algorithm explained in the textbooks?

i. In what way do the textbooks explain why moving the same number of decimal places in dividend and divisor preserves the value of the quotient?

ii. In what ways do textbooks explain why the divisor needs to be a whole number, but the dividend does not need to be when executing the standard algorithm?

iii. In what ways do the textbooks explain why the decimal point in the quotient needs to be lined up with the decimal point in the dividend?

LITERATURE REVIEW

Research on Analysis of Textbooks Written for Prospective Elementary Teachers

Research on textbooks written for prospective elementary teachers has revealed interesting findings that have informed both research and practices. McCrory (2006) analyzed 20 textbooks written for prospective elementary teachers by both research mathematicians and mathematics educators. McCrory (2006) noticed that the four books written by research mathematicians differed from those written by mathematics educators in various aspects. The purpose and design of the textbooks written by mathematicians tended to be narrative and focused on a few selected mathematical topics, whereas the books written by mathematics educators were encyclopedic, long, well indexed, and comprehensive. McCrory (2006) also noticed the mathematicians written textbooks were coherent and rigorous. The National Council on Teacher Quality (Greenberg & Walsh, 2008) analyzed 257 course syllabi and 18 required textbooks in 77 institutions across the U.S. to investigate the quality of elementary teacher preparation programs. The research team examined 12 essential topics in four critical areas (numbers and operations, algebra, geometry and measurement, and data analysis and probability) to evaluate if the topics in each were clearly presented. The results have indicated significant variability in the quality of the textbooks and that two-thirds of the courses used no textbook or a textbook that did not adequately cover these topics. McCrory and Stylianides (2014) examined 16 mathematics textbooks written for mathematics content courses for prospective elementary teachers to investigate the opportunities provided in these books for prospective teachers to learn about reasoning and proving. The researchers used the table of contents and index to locate content relevant to reasoning and proving and found that most of the reasoning and proving relevant topics were not explicitly discussed in the textbooks. Maher and He (2023) investigated definitions of length, area, and volume in textbooks written for prospective elementary teachers and noticed the definitions of these three concepts vary from procedural to conceptual to formal. Not a single textbook included a complete definition and most textbooks provided only one definition type. The findings from these studies of textbook analysis suggest the differences in textbook presentations and the inadequate coverage of certain textbook topics.

Difficulties to Learn Decimal Multiplication and Division

Decimal operations are natural extensions of whole number operations and seem easy to be grasped. However, comprehending rational numbers and their representation as decimals with the appearance of the decimal point presents new challenges for learning. Ashlock (2010) summarized common student errors in computing decimals. Instead of counting from right to left, students counted from left to right when placing the decimal point in the last step of a decimal multiplication problem. Ashlock (2010) suggested students use estimation to support locating the decimal point or show more examples to help students discover and identify the right rule. Though both suggestions may improve students' performance on computing decimal multiplication, none of them tackles the root of students' mistakes, namely, students' inadequate understanding of the decimal multiplication algorithm that in the last step, moving the decimal point from right to left one decimal place is equivalent to dividing the result by 10. Ashlock (2010) also noticed when taken as far as digits given in the dividend, if the division does not come out even (when more 0s need to be appended to the dividend to complete the calculation), one common student mistake was to write the remainder as an extension of the quotient. Students seemed to mix the procedure of writing the remainder (e.g., R3) to the right of the quotient in a whole number division problem with the procedures in a decimal division problem. Ashlock (2010) suggested emphasizing the meaning of division, using either partitive (fair share) or repeated subtraction (measurement) model, to help students understand the meaning of remainders and how to properly handle the remainders in decimal division problems.

Research has also revealed that inconsistent ways of locating the decimal point for four operations may create additional confusion for learners (Hiebert & Wearne, 1985). For instance, why do we align decimal points when adding and subtracting decimals but performing differently for multiplication algorithm? Muir and Livy (2012) reported prospective elementary teachers' confusion about the four algorithms. When asked to multiply two 2-decimal place decimals, prospective teachers tended to have two decimal places in the product instead of four decimal places. Researchers suspected that this error was related to teachers' understanding of place value and overgeneralization of the rule from adding and subtracting decimals to multiplying decimals. Similarly, Lai and Murry (2014) noticed that students aligned the decimal points when multiplying two decimals. Even though this action is not entirely wrong and can still achieve the correct answer, it is unnecessary to do so. This action may come from students' overgeneralization of decimal addition and subtraction algorithms. Tirosh et al. (1998) also reported prospective teachers' misapplication of steps in the decimal multiplication algorithm to the decimal division algorithm. When dividing decimals, prospective teachers tended to move the decimal point several places to the left in the last step of the algorithm, where the proper procedure is to align the decimal point in the quotient with the decimal point in the dividend. Prospective teachers' actions may be associated with their consideration to compensate for the previous step when the decimal point was moved several places to the right to make the divisor a whole number. Prospective teachers failed to understand the rationale of two algorithms which determines that the compensation is required in the multiplication algorithm but not in the division algorithm. All reviewed studies have suggested the need for prospective teachers to fully understand the meaning of decimal multiplication and division as well as the rationale of the algorithms so teachers will not generalize one algorithm to another based on superficial features. Mathematics is a subject where similar structures are found, connections are established, and generalizations are made. However, how can a learner identify which generalization is legitimate and which is not? The key lies in if the learners truly understand the rationale of the procedures, so they can compare and contrast their mathematical structures, to determine if a generalization could be made from one procedure to the other.

Conceptual Understanding and Procedural Fluency

Nowadays with computers and calculators widely available in classrooms, one may argue that studying the standard algorithm is obsolete and unnecessary. However, the standard algorithm is not limited to its computing purpose but is an essential component of mathematics as a discipline. Understanding *why* the standard algorithms of decimal operations work supports students' conceptual understanding of closely related mathematical concepts such as the meaning of decimal point, place value, decimal fractions, and the properties of real numbers (e.g., distributive property). The community of mathematics educators has been advocating for promoting students' conceptual understanding and procedural fluency for a few decades (Kilpatrick et al., 2001; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Star, 2005). Procedural understanding involves knowledge about skills, procedures, and rules, which can be carried out flexibly, accurately, efficiently, and appropriately as steps to complete a mathematical task (Hiebert & Wearne, 1986; Skemp, 1976). Conceptual understanding involves comprehending mathematical concepts, operations, and relations between related concepts (Wearne & Hiebert, 1988). Students with solid conceptual understanding have their knowledge well connected and organized into a coherent whole. They learn new ideas by connecting them to what they've already known. Students tend not to forget the facts and methods; even if they are forgotten, they can be easily reconstructed. Western educators seem to believe that students should acquire conceptual understanding first before employing the rules (Li, 2006). However, Chinese scholars have different views about the intertwined relationship between conceptual understanding and procedure fluency and stress that the fluency in the procedures may support one's development of conceptual knowledge (Rittle-Johnson et al., 2001). My focus on unpacking the rationales of the standard algorithm supports both conceptual understanding and procedural fluency.

METHODOLOGY

Data Sources

Eleven textbooks commonly used in the U.S. to teach mathematics content courses for prospective elementary teachers were analyzed. The selection of the textbooks is guided by previous studies (e.g., Jones et al., 2017; McCrory, 2006; McCrory & Stylianides, 2014; Greenberg & Walsh, 2008) on analysis of mathematics textbooks written for prospective elementary teachers. Some textbooks used in the previous studies are hardly found for sale, which suggests a narrower market share, and are therefore excluded from the analysis. These selected textbooks include a wide variety of textbooks written by mathematics educators and mathematicians to represent both groups' approaches to writing textbooks. To be consistent across the textbooks, only the student editions of the textbooks were examined. The narrative sections of the textbooks were examined but not the practice problems. The narrative sections include an introduction, description, and justification of the algorithms as well as the worked examples that demonstrate how the description and justification of the algorithms are applied to specific mathematical problems.

Procedures

A similar method described by McCrory and Stylianides (2014) was used to locate the content. This study took the perspective of the prospective elementary teachers and course instructors as users of the textbooks and imagined where a student or instructor would look if they wanted to locate a specific mathematical topic. Not every page of the books was examined. Instead, tables of content and indexes were used to locate the sections of decimal multiplication and division in the textbooks. Unlike reasoning and proving, a mathematical practice that intertwines with all mathematical topics analyzed in McCrory and Stylianides (2014), decimal multiplication and division are usually presented as a separate section that is easy to locate. Though decimal multiplication and division could be presented elsewhere as an application of other mathematical topics, my focus on standard algorithms makes it less likely that the standard algorithms would only appear in chapters for their applications without appearing first in a section that introduces the algorithms.

Once the content was located, two researchers (the author and another researcher) independently examined the content using the analytical frameworks presented in the section below. Each researcher individually started with open coding to document various methods used in different textbooks to present the standard algorithms of multiplication and division. They then met, discussed, and compiled a list of codes of methods and applied them to code five textbooks. They met to refine the codes by resolving the coding discrepancy. They then independently coded the rest of the six textbooks using the new codes and met to address the discrepancy of their coding until consensus were reached for all coding.

Data Analysis

The data were first examined by looking at how the textbooks present the standard algorithms (RQ1a and RQ2a). Then the textbooks' rationales to justify the standard algorithms (RQ1b and RQ2b) were explored. When examining the rationales, four aspects were looked at, as stated in RQ1b and RQ2b. For multiplication, the textbooks explain why adding decimal places in each factor to obtain the decimal places in the product was examined. For division, three aspects were examined:

(a) the reason why the quotient stays unchanged when the decimal points are moved the same number of decimal places in dividend and divisor,

(b) the reason why the divisor needs to be a whole number but the dividend does not need to be, and

(c) the reason for lining up the decimal points vertically in the quotient and dividend.

Textbooks' approaches to addressing all four aspects will be illustrated in the analytical perspectives section.

Analytical Perspectives

Standard algorithm of decimal multiplication

Steps of the multiplication algorithm: To make it easier to refer to the components of the standard algorithm for decimal multiplication, the following three steps are used to describe it. Some books may present the algorithm in a different order or combine two steps. The form of the presentation does not affect the analysis. The focus is whether the books show all the steps. The three steps serve as the analytical framework to answer RQ1a. MS1 to MS3 were used to designate each step where MS stands for multiplication step:

MS1. Ignore the decimal point and multiply as with whole numbers to obtain the product a.

MS2. Add the number of decimal places in each factor to obtain the sum s.

MS3. Place the decimal point in the product a so that there are s digits to the right of the decimal point.

Methods to explain the rationale of the multiplication algorithm: When answering RQ1b and examining the rationales, the focus is on how textbooks explain why adding the number of decimal places in MS2. MS1 and MS3 are relatively easy to explain and are based on the connection between decimals and fractions. For instance, if a decimal has two decimal places, it can be written as a decimal fraction (a fraction with a power of 10 as its denominator) with 100 or 10^2 as its denominator. In total, three methods were used by different textbooks. To ease the readers, the same multiplication problem 2.37×4.5 was used to illustrate all three methods.

Method 1. Decimal fraction

$$2.37 \times 4.5 = \frac{237}{100} \times \frac{45}{10} = \frac{237 \times 45}{100 \times 10} = \frac{10665}{1000} = 10.665. \tag{1}$$

In this method, the two decimals are first written as two decimal fractions. As explained previously, the number of decimal places in each factor corresponds to the number of zeros in the power of 10 in the denominator. 2.37 has two decimal places; thus the decimal fraction representing it ($\frac{237}{100}$) includes two zeros in the denominator. Another critical observation to understand this method is to notice when multiplying the power of 10, the resulting power of 10 has the number of zeros obtained by adding the number of zeros in each of the individual power of 10. In this example, two zeros in 100 and one zero in 10 make three zeros in 1,000 in the equation $100 \times 10 = 1000$.

A slight variation of this method, as demonstrated in some books and the following, is to expand 100 into 10×10, to help students see how the numbers of 10 multiplied connect to the number of decimal places in each factor.

$$2.37 \times 4.5 = \frac{237}{100} \times \frac{45}{10} = \frac{237}{10 \times 10} \times \frac{45}{10} = \frac{10665}{(10 \times 10) \times 10} = 10.665. \tag{2}$$

Method 2. Exponent

$$2.37 \times 4.5 = \frac{237}{100} \times \frac{45}{10} = \frac{237 \times 45}{100 \times 10} = \frac{237 \times 45}{10^2 \times 10^1} = \frac{237 \times 45}{10^{2+1}} = \frac{10665}{10^3} = 10.665. \tag{3}$$

Similar to method 1, method 2 first connects the exponents to the numbers of decimal places in each factor. In this example, the two decimal places in 2.37 correspond to the exponent of 2 in $\frac{237}{10^2}$. Then the law of exponents $a^m \times a^n = a^{m+n}$ (adding exponents when multiplying) is applied to show why adding the number of decimal places in MS2.

A variation of method 2, as shown below, is to use negative exponents instead of positive exponents.

$$2.37 \times 4.5 = (237 \times 10^{-2}) \times (45 \times 10^{-1}) = (237 \times 45) \times (10^{-2} \times 10^{-1}) = (237 \times 45) \times 10^{(-2)+(-1)} = 10665 \times 10^{-3} = 10.665. \tag{4}$$

Both method 1 and method 2 first turn a decimal multiplication algorithm into a whole number multiplication algorithm (MS1) and eventually turn the resulting whole number product back to a decimal product (MS3) by first rewriting the decimal, either using a fraction notation (method 1) or exponential notation (method 2). The key in both methods is to explain *why* adding the number of decimal places in MS2. Intuitive but incorrect thinking would be to multiply the number of decimal places (e.g., 2×1 instead of $2 + 1$) because the context is to multiply two decimals, not to add two decimals. Both methods, through applying the place value concept, the rule to multiply powers of 10, or the use of the law of exponents, show that $100 \times 10 = 1000$ or $10^2 \times 10^1 = 10^{2+1} = 10^3$ rather than $100 \times 10 = 100$ or $10^2 \times 10^1 = 10^{2\times1} = 10^2$, thus explain why the decimal places should be added in MS2.

Method 3. Enlarging decimals and shrinking product: This method was illustrated by Beckmann (2017). The method first connects the decimals to their corresponding whole numbers by multiplying the appropriate power of 10; then completes the whole number multiplication using the standard algorithm (MS1). At last, the product is adjusted by dividing the powers of 10 multiplied before (MS3). The key is to notice that the numbers of 10s multiplied in MS1 are the same as the total decimal places in both factors, which is also the number of 10 divided in MS3 (**Figure 1**).

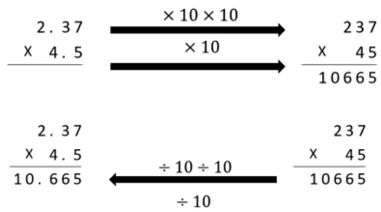

Figure 1. A diagram to illustrate method 3: Enlarging decimals and shrinking product (Beckmann, 2017)

Standard algorithm for decimal division

Steps of the standard algorithm for decimal division: The standard algorithm for decimal division is described in three steps, as shown in the following, where DS1 stands for division step 1:

DS1. Shift the decimal point of the divisor and dividend the same number of places to make the divisor a whole number.

DS2. Calculate the quotient by long division.

DS3. Insert the decimal point in the quotient such that the decimal point is aligned with the decimal point of the dividend.

Similar to the standard multiplication algorithm, some books may present the steps in different orders and state them differently. For instance, instead of using the language of moving the decimal point, some textbooks may use the language of multiplying by the power of 10. The form of the presentation does not impact the analysis, the focus is on whether the books present all the steps. Similar to the multiplication algorithm, a uniform meaning of the steps was created to ease the readers. The above three steps serve as the analytical framework to answer RQ2a. Wu (2011), a research mathematician, wrote a section addressing decimal division using a different method involving profound mathematics. Considering the purpose of this paper as to illustrate ways to explain algorithms accessible to elementary teachers, Wu's (2011) approach will be briefly described in the results section.

Methods to explain the rationale of the standard algorithm for decimal division: RQ2b was answered by addressing its three subquestions: RQ2b1, RQ2b2, and RQ2b3.

Collectively, the textbooks use five different methods to explain why moving the decimal points to the same number of places in both divisor and dividend would not change the value of the quotient (RQ2b1). The same decimal problem 19.38 ÷ 5.7 was used to illustrate all the five methods found in the textbooks.

Method 1. Decimal fraction

$$19.38 \div 5.7 = \frac{1938}{100} \div \frac{57}{10} = \frac{1938}{100} \div \frac{570}{100} = \frac{1938}{570} = 3.4. \quad (5)$$

In this method, the two decimals are first written as two decimal fractions. Then both denominators are made the same using the equivalent fractions. Last, the fraction division can be completed by dividing two numerators ($\frac{a}{c} \div \frac{b}{c} = \frac{a}{b}$).

Method 2. Equivalent fraction

$$19.38 \div 5.7 = \frac{19.38}{5.7} = \frac{19.38}{5.7} \times 1 = \frac{19.38}{5.7} \times \frac{10}{10} = \frac{19.38\times 10}{5.7\times 10} = \frac{193.8}{57} = 193.8 \div 57 = 3.4. \quad (6)$$

Although both method 1 and method 2 use fractions as intermediate steps, method 1 turns the decimals into decimal fractions. Instead, method 2 first rewrites a division problem into a fraction problem using the connections between the division and fraction $a \div b = \frac{a}{b}$. Then the fraction is multiplied by a power of 10 such that the denominator of the fraction becomes a whole number. At last, the new fraction is rewritten into a division problem with a whole number divisor.

Method 3. Definition of division: The definition of division method uses the definition of division and the algebraic properties of the equation to show $19.38 \div 5.7 = 193.8 \div 57$, but it does not involve fractions. To prove $19.38 \div 5.7 = 193.8 \div 57$, the quotient 19.38 ÷ 5.7 is first named as c, namely, $19.38 \div 5.7 = c$. Then, because the division is the inverse operation of the multiplication, $19.38 = 5.7 \times c$. Multiplying both sides of the equation by 10 leads to $19.38 \times 10 = (5.7 \times c) \times 10$. Rearranging the above equation makes $19.38 \times 10 = (5.7 \times c) \times 10 = (5.7 \times 10) \times c$. Rewriting the above equation using the inverse relationship between multiplication and division leads to $(19.38 \times 10) \div (5.7 \times 10) = c$. Namely, $193.8 \div 57 = c$. Because $19.38 \div 5.7 = c$ we have $19.38 \div 5.7 = 193.8 \div 57$.

Method 4. Property of division: The property of division method uses algebraic properties of multiplication, division, and parenthesis to prove $19.38 \div 5.7 = 193.8 \div 57$, as shown in the following line of reasoning:

$$19.38 \div 5.7 = (19.38 \times 10 \div 10) \div 5.7 = 19.38 \times 10 \div 10 \div 5.7 = 19.38 \times 10 \div 5.7 \div 10 = 19.38 \times 10 \div (5.7 \times 10) = 193.8 \div 57. \quad (7)$$

Method 5. Reinterpreting unit: Before explaining this method, two division models need to be reviewed: the partitive/fair share model and the repeated subtraction/measurement model (Fischbein et al., 1985). These two models are introduced when teaching whole number division but can be extended to decimal division. The partitive division problem provides the total number of objects and the number of groups; the question asks how many units are in one group. The repeated subtraction division problem gives the total number of objects and the group size; the question asks how many groups can be formed. Taking $6 \div 2 = 3$ as an example, a partitive word problem looks like this: Dave has six apples and wants to give them to two friends evenly. How many apples does he give to each friend? A repeated subtraction word problem looks like this: Dave has six apples and wants to give each friend two apples. How many friends can he give the apple to?

The reinterpreting unit method examines decimals using smaller units specified by the place value. For instance, considering 19.38 ÷ 5.7, instead of thinking of 19.38 as 19 whole and 38 hundredths, we can consider 19.38 either as 193.8 tenths or 1938 hundredths. Similarly, we can think of 5.7 as 57 tenths or 570 hundredths. Using the repeated subtraction model, we interpret 19.38 ÷ 5.7 as the solution to the word problem "how many 5.7s are in 19.38"? However, if switching to the smaller units (tenths or hundredths), the same word problem can be reinterpreted as "how many 57 tenths are in 193.8 tenths" or "how many 570 hundredths are in 1938 hundredths"? The solutions to the last two word problems would be $193.8 \div 57$ and $1938 \div 570$, respectively. Because all three word problems are identical, other than using different units, the three solutions should also have the same value, suggesting that $19.38 \div 5.7 = 193.8 \div 57 = 1938 \div 570$.

To answer RQ2b2, whether textbooks explain why the divisor needs to be a whole number, but the dividend does not need to be a whole number was examined. Unfortunately, no textbook provides any explanations.

To answer RQ2b3, how textbooks explain why the decimal points need to be lined up in quotient and dividend was examined. Only one method was noticed below.

Method 1. Partitive model for division: This method was illustrated by Van De Walle et al. (2018). The method uses the partitive model for division and unpacks the long division algorithm by considering the place value. The following example, along with its explanation, shows how this method explains the location of the decimal point in DS3 (**Figure 2**).

$$
\begin{array}{r}
3.4 \\
57\,\overline{)\,193.8} \\
\underline{171} \\
22.8 \\
\underline{228} \\
0
\end{array}
$$

Figure 2. An example to illustrate method 1: Partitive model for the division (Van De Walle et al., 2018)

Explanation of the algorithm: Put 193 ones in 57 groups. Each group gets 3 ones (put 3 above the ones place), or 171 ones in all. That leaves 22 ones. Trade 22 ones for 220 tenths, making 228 tenths. Put 228 tenths in 57 groups. Each group gets 4 tenths (put 4 above the tenth place), or 228 tenths in all. That leaves no remainder. Thus, the quotient has 3 ones and 4 tenths, which makes the decimal 3.4, and the decimal point in the quotient aligns with the decimal point in the dividend.

This method is a generalization of the explanation of the whole number long division algorithm using the partitive model. It generalizes the place value from whole number places to include decimal places. Because all place values in the quotient line up well with the dividend during the process of dividing, the decimal points also line up. The key observation of this method is to notice that the nature of lining up the decimal points is to line up the place values.

RESULTS

Textbooks' Presentation of the Standard Decimal Multiplication Algorithm

Table 1 provides information on the extent to which each textbook addresses the standard algorithm for decimal multiplication. Out of the 11 textbooks (**Appendix A**), nine books (Beckmann, 2017; Bennett, 2015; Billstein et al., 2019; Long et al., 2014; Musser et al., 2019; Parker & Baldridge, 2008; Sonnabend, 2010; Van De Walle et al., 2018; Wu, 2011) stated all three steps of the standard decimal multiplication algorithm. The other two books (Bassarear, 2019; Sowder et al., 2016) provide answers to decimal multiplication problems through other means, such as using nonstandard algorithms or alternative methods (e.g., calculating partial products and drawing a rectangular array), but do not solve the problem using the standard algorithm. Among the nine books that mention the standard algorithm, four books demonstrate the algorithm using specific examples. The other five books state the standard algorithm in general language, similar to those described in MS1 to MS3. Some books use letters and algebraic language to facilitate the description. No textbook presents partial steps.

Table 1. Textbooks' presentation of the standard algorithm for decimal multiplication

Reference	Not mentioned	Illustrated 3 steps using specific examples	Stated 3 steps using general language
Bassarear and Moss (2019)	X		
Beckmann (2017)		X	
Bennett (2015)		X	
Billstein et al. (2019)			X
Long et al. (2014)			X
Musser et al. (2019)		X	
Parker and Baldridge (2008)			X
Sonnabend (2010)			X
Sowder et al. (2016)	X		
Van De Walle et al. (2018)		X	
Wu (2011)			X

A careful examination of the language used by books reveals differences that might lead to students' misconceptions and are worthy of being attended by mathematics educators. In seven out of the nine textbooks (Beckmann, 2017; Billstein et al., 2019; Long et al., 2014; Musser et al., 2019; Parker & Baldridge, 2008; Van De Walle et al., 2018; Wu, 2011), the textbooks writers emphasize the process and use the action language (insert/place/put the decimal point) to describe MS3 and connect MS3 to the product of the two whole numbers in MS1. However, the other two books (Bennett, 2015; Sonnabend, 2010) do not describe the action but describe the number of decimal places in the product as a static result from multiplying two decimals. For instance, Parker and Baldridge (2008) wrote: multiplying the two numbers, ignoring the decimal points. The number of decimal places in the product *is* [emphasis added] the sum of the number of decimal places in the two factors (p. 206). At first glance, the second sentence in this quote seems to be a combination of MS2 and MS3, but they are slightly different when a special case appears. That is when the product obtained from multiplying the two whole numbers in MS1 has zero(s) at the end. For instance, when calculating 4.62×3.5, 462 and 35 are first multiplied to obtain 16170 (MS1). By applying MS3 of the standard multiplication algorithm, the decimal point is inserted to be right to digit 6 to make it 16.170, so that there are three digits to the right of the decimal point in the product. But if Parker and Baldridge's (2008) description were applied, the focus would be less on the process and more on the final calculation result. Three (= 2 + 1) decimal places will be expected in the final result of the calculation. However, writing 16.170 as 16.17 by deleting the end 0 brings an issue of a mismatch between the number (2) of decimal places in the product (16.17) and the sum of the number of decimal places in the two factors (2 + 1 = 3). Therefore, Parker and Baldridge's (2008) language may need to be modified slightly to avoid the confusion caused by the end zeros.

Textbooks' Presentation of the Rationale of the Decimal Multiplication Algorithm

The findings suggest that the textbooks vary significantly in how they explain why adding decimal places to the factors (MS2). Three books (Bassarear, 2019; Sonnabend, 2010; Sowder et al., 2016) do not provide the explanations. Among them, two textbooks

(Bassarear, 2019; Sowder et al., 2016) do not mention the standard algorithm and therefore provide no rationale. Sonnabend (2010) presented an inductive reasoning activity by giving students three decimal multiplication problems, asking them to discover the number of decimal places in the product in relation to the number of decimal places in both factors; students are asked to summarize the rule they notice. This activity provides excellent opportunities for prospective teachers to explore mathematics and discover mathematical patterns independently, but the activity ends only with finding the pattern without further exploring *why* the pattern always works on any two decimals. The rest of the eight books use three methods described in the analytical perspectives section: decimal fraction, exponent, and enlarging decimals and shrinking product. **Table 2** provides information on the distribution of the textbooks regarding the methods the books use when explaining MS2. While most books use one method, two books use two methods. Five books use the decimal fraction method. Four books use the exponent method. Only one book uses the enlarging decimals and shrinking product method.

Table 2. Textbooks' presentation of the rationale of the standard algorithm for decimal multiplication related to MS2

Reference	Decimal fraction	Exponent	Enlarging decimals and shrinking product
Bassarear and Moss (2019)			
Beckmann (2017)	X		X
Bennett (2015)	X		
Billstein et al. (2019)		X	
Long et al. (2014)		X	
Musser et al. (2019)	X	X	
Parker and Baldridge (2008)	X		
Sonnabend (2010)			
Sowder et al. (2016)			
Van De Walle et al. (2018)	X		
Wu (2011)		X	

In addition to classifying the methods used, the clarity of the textbook presentation was also examined. Three books (Bennett, 2015; Parker & Baldridge, 2008; Van De Walle et al., 2018) present one of the three methods, but the authors leave the understanding of the explanations of the rationale completely to the audience. For instance, **Figure 3** displays Parker and Baldridge's (2008) explanation using the decimal fraction approach. Instead of drawing attention to the middle two mathematical expressions, which explain why adding the number of decimal places, they drew attention to comparing the number of decimal places in 7.378 (the end of the equations) to the number of decimal places in the two factors 2.17 and 3.4 (the beginning of the equations). His comments that "That observation leads to a general procedure for decimal multiplication" supports discovering MS2 rather than explaining MS2.

$$2.17 \times 3.4 = \frac{\square}{100} \times \frac{34}{10} = \frac{217 \times 34}{1000} = \frac{7378}{1000} = 7.378.$$

In the numerators of this calculation we see the whole number multiplication $217 \times 34 = 7378$. The denominators show that the number of decimal places in the answer (namely 3) is the sum of the number of decimal places in 1.02 and 2.3 (2 in the first, 1 in the second). That observation leads to a general procedure for decimal multiplication.

Figure 3. Explaination of the decimal multiplication algorithm in Parker and Baldridge (2008, p. 206)

The rest of the five books (Beckmann, 2017; Billstein et al., 2019; Long et al., 2014; Musser et al., 2019; Wu, 2011) clearly explain why adding the number of decimal places. As shown in **Table 2**, among the five books, four use the Exponent method, which easily demonstrate that adding up the exponents corresponds to adding up the number of decimal places. Taking Wu's (2011) explanation as an example (see **Figure 4**), he wrote clearly that the purpose of his explanation is to "… see that the basic reason that the number of decimal digits in the product equals the sum of the number of decimal digits in each factor is equation (1.4)" (Wu, 2011, p. 269). The Eq. (1.4) referred here is $10^{m+n} = 10^m \times 10^n$. He also attempts to make his explanation generalizable by stating that "noting at the same time that the reasoning in the general case is the same" (Wu, 2011, p. 269).

We now justify the algorithm using the example of 1.25×0.0067, noting at the same time that the reasoning in the general case is the same.

$$\begin{aligned} 1.25 \times 0.0067 &= \frac{125}{10^2} \times \frac{67}{10^4} \\ &= \frac{125 \times 67}{10^2 \times 10^4} \qquad \text{(product formula).} \end{aligned}$$

Now we multiply 125×67 in the numerator instead of 1.25×0.0067; this corresponds to (i). Therefore,

$$1.25 \times 0.0067 = \frac{8375}{10^2 \times 10^4}.$$

By equation (1.4) on page 30, $10^2 \times 10^4 = 10^{2+4}$, so that

$$1.25 \times 0.0067 = \frac{8375}{10^{2+4}}.$$

But by the definition of a decimal, the right side is 0.008375, and this corresponds exactly to (ii). In particular, we see that the basic reason that the number of decimal digits in the product equals the sum of the number of decimal digits in each factor is equation (1.4).

Figure 4. Explanation of the decimal multiplication algorithm in Wu (2011, p. 269)

Textbooks' Presentation of the Standard Decimal Division Algorithm

In this section, an analysis of textbooks' presentation of the standard decimal division algorithm in 10 textbooks, excluding the textbook written by Wu (2011), is reported. Wu's (2011) book uses a completely different approach involving advanced mathematics. The readers cannot easily understand the approach without intensively studying one chapter of his textbook. His method will be briefly explained later in this section. **Table 3** suggests that six textbooks state all three steps of the standard decimal division algorithm using general language. Four textbooks (Bassarear, 2019; Long et al., 2014; Sowder et al., 2016; Van De Walle et al., 2018) do not present a general description of all three steps of the standard division algorithm. The authors of these four books hint at the standard algorithm through a specific example and show partial steps. In addition, Sowder et al. (2016) design a series of division problems with the same digits but different places for the decimal point ($56 \div 7, 56 \div 0.7, 5.6 \div 0.7$). They ask the readers to find the quotients first and discover a pattern regarding the relationship between the quotients and the locations of the decimal points in the dividend and divisor. Van De Walle et al. (2018) uses estimation as the primary tool to divide decimals. Because none of the four books summarizes the complete algorithm using general language, it is unlikely that a new learner would be able to discover and summarize the algorithm by examining one specific example, not to mention that some examples selected are presented incorrectly (e.g., misalignment of the place value) or do not involve a decimal quotient, which makes the discovery of DS3 impossible.

Table 3. Textbooks' presentation of the standard algorithm for decimal division using the general language

Reference	DS1	DS2	DS3
Bassarear and Moss (2019)	X		X
Beckmann (2017)	X	X	X
Bennett (2015)	X	X	X
Billstein et al. (2019)	X	X	X
Long et al. (2014)	X		
Musser et al. (2019)	X	X	X
Parker and Baldridge (2008)	X	X	X
Sonnabend (2010)	X	X	X
Sowder et al. (2016)	X		
Van De Walle et al. (2018)		X	

Wu (2011) introduces an entirely different approach using advanced mathematical knowledge not typically required by pre-service elementary teachers. He presents the fractions and decimals algorithms together in one lengthy chapter. A complete understanding of his approach requires an in-depth read of the entire chapter. Therefore, his approach is only briefly described here. Interested readers can take an in-depth read of his book. Wu (2011) first proves that a division of decimals can always be reduced to the division of two whole numbers, which then can be rewritten as a fraction. For instance, $\frac{24.3}{1.57} = \frac{2430}{157}$. Next, he proves that any fraction can be written into a whole number plus a proper fraction. Using the same example, $\frac{2430}{157} = 15\frac{75}{157}$. Last, he describes steps to convert the proper fraction to a decimal. He distinguished two cases when implementing the above algorithm:

(a) when the proper fraction can be written as a terminating decimal and

(b) when the proper fraction cannot be written as a terminating decimal.

The purpose of this paper is not to fully unpack Wu's (2011) method, so only case (a) is briefly discussed here using the formula:

$$\frac{m}{n} = \left(\frac{m \cdot 10^k}{n}\right) \times \frac{1}{10^k} = q \times \frac{1}{10^k} = \frac{q}{10^k}, \tag{8}$$

where k is the minimum whole number making $\frac{m \cdot 10^k}{n}$ a whole number. The existence of k is guaranteed when the proper fraction $\frac{m}{n}$ can be written as a terminating decimal. $\frac{m \cdot 10^k}{n}$ is renamed as q. The formula can be demonstrated using an example: $\frac{5}{8} = \left(\frac{5 \cdot 10^3}{8}\right) \times \frac{1}{10^3} = \frac{625}{10^3} = 0.625$. In summary, Wu's (2011) method first turns a decimal division problem into a fraction. He then uses the property of fraction, whole number division, and definition of decimal to obtain the quotient to the original decimal division problem. Using $0.15 \div 0.24$ as an example, Wu's (2011) approach is demonstrated, as follows:

$$0.15 \div 0.24 = 15 \div 24 = \frac{5}{8} = \left(\frac{5 \cdot 10^3}{8}\right) \times \frac{1}{10^3} = \frac{625}{10^3} = 0.625. \tag{9}$$

Wu's (2011) method does not clearly state how to position the decimal point as it is stated in DS3. It describes the process as "obtain a decimal by the judicious placement of a decimal point in the quotient" (p. 297). Wu's (2011) approach is mathematically rigourous and he proves every mathematical conclusion (e.g., theorems) needed for his method. However, his approach is not easily accessible to pre-service elementary teachers and students.

Textbooks' Presentation of the Rationale of the Standard Decimal Division Algorithm

This section presents how textbooks explain the rationale of the standard division algorithm from three aspects:

(a) the reason why the quotient stays unchanged when the decimal points are moved the same number of decimal places in dividend and divisor,

(b) the reason why the divisor needs to be a whole number but the dividend does not need to be, and

(c) the reason for lining up the decimal points vertically in the quotient and dividend.

Each aspects corresponds to one research question RQ2b1, RQ2b2, RQ2b3.

Moving the decimal points preserves the value of the quotient

Table 4 summarizes textbooks' presentation of why moving the decimal points in both dividend and divisor the same number of decimal places preserves the value of the quotient. Two textbooks (Sowder et al., 2016; Van De Walle et al., 2018) do not provide an explanation. For the eight books (Bassarear, 2019; Beckmann, 2017; Bennett, 2015; Billstein et al., 2019; Long et al., 2014; Musser et al., 2019; Parker & Baldridge, 2008; Sonnabend, 2010) that present the rationales, the mostly widely used method is the equivalent fraction method. For the other four methods, including the decimal fraction method, definition of division method, property of division method, and the reinterpreting unit method, only one textbook uses each method, respectively. The findings also suggest that the books written by Musser et al. (2019) and Sonnabend's (2010) both generalize their explanation from a generic example to a general algebraic proof involving a, b, c. For instance, Musser et al. (2019) illustrate the definition of divisioni method and present an algebraic argument to prove $(a \times 10^n) \div (b \times 10^n) = a \div b$. Notice that in a decimal, moving a decimal point n times to the right is equaivalent to multiplying this number by 10^n. Thus, the equation $(a \times 10^n) \div (b \times 10^n) = a \div b$ explains why moving the decimal places n times in both dividend and divisor preserves the value of the quotient. Sonnabend (2010) proves $a \div b = ac \div bc$ by showing $a \div b = \frac{a}{b} = \frac{a}{b} \cdot \frac{c}{c} = \frac{ac}{bc} = ac \div bc$ (p. 327).

Table 4. Textbooks' presentation of why moving the decimal points preserves the value of the quotient

Reference	Decimal fraciton	Equivalent fraction	Definition of division	Property of division	Reintepreting the unit
Bassarear and Moss (2019)		X			
Beckmann (2017)		X		X	X
Bennett (2015)		X			
Billstein et al. (2019)		X			
Long et al. (2014)		X			
Musser et al. (2019)	X		X		
Parker and Baldridge (2008)		X			
Sonnabend (2010)		X			
Sowder et al. (2016)					
Van De Walle et al. (2018)					

As shown in **Table 4**, two textbooks presented more than one method. Musser et al. (2019) presents the decimal fraction method and the definition of the division method. Beckmann (2017) presents three methods: the equivalent fraction method, the property of division method, and the reinterpreting unit method. When explaining the reinterpreting unit method, Beckmann (2017) first uses changing the unit of money as an introductory example. In her example, $4.5 \div 1.27$ is the solution to the word problem "how many pounds of plums can we buy for \$4.50 if plums cost \$1.27 per pound" (Beckmann, 2017, p. 272)? Similarly, $450 \div 127$ is the solution to the word problem "how many pounds of plums can we buy for 450 cents if plums cost 127 cents per pound" (Beckmann, 2017, p. 272)? Because both word problems are the same, other than using different units (dollar vs. cent), the two solutions should be the same too. Therefore, $4.5 \div 1.27 = 450 \div 127$. Beckmann (2017) then generalizes the reinterpreting unit method from money problem to base ten blocks. By looking at the same blocks but with different interpretations of the meaning of units, different decimal divisions can be produced. These decimal division problems have the same digits but different places for decimal points. Because all division problems are derived from the same blocks, their solutions are equal; therefore the different decimal division problems have the same quotient.

Making the divisor a whole number

Eight books (Beckmann, 2017; Bennett, 2015; Billstein et al., 2019; Long et al., 2014; Musser et al., 2019; Parker & Baldridge, 2008; Sonnabend, 2010; Sowder et al., 2016) emphasize that the divisor needs to be a whole number when describing the standard algorithm for decimal division while two textbooks (Bassarear, 2019; Van De Walle et al., 2018) do not. Bassarear (2019) only provides one illustrative example of decimal division where the divisor and dividend have the same number of decimal places. The author describes moving a certain number of decimal places without explaining the purpose of the action. It is unclear to the readers if the purpose is to make the divisor, dividend, or both to be integers. Van De Walle et al. (2018) does not fully describe the standard algorithm. Unfortunately, no textbooks explain *why* the divisor needs to be a whole number, but the dividend does not need to be. No textbooks emphasize the importance of the whole number divisor and its effect on implementing DS3. Mathematically speaking, if the divisor is not a whole number, the decimal points in the dividend and divisor would not align up as described in DS3.

Lining up the decimal point in the quotient with the decimal point in the dividend

Eight textbooks (Bassarear, 2019; Beckmann, 2017; Bennett, 2015; Billstein et al., 2019; Long et al., 2014; Musser et al., 2019; Parker & Baldridge, 2008; Sonnabend, 2010; Sowder et al., 2016; Van De Walle et al., 2018; Wu, 2011) do not explain why the decimal point in the quotient needs to be lined up with the decimal point in the dividend. Musser et al. (2019) attempts to address this issue by briefly mentioning that "this can be justified using division of fractions" without showing the justification process. Two books (Sonnabend, 2010; Van De Walle et al., 2018) explain using the reinterpreting the unit method. The explanations from both textbooks are not as clear as the example demonstrated in the methodology section, but both explanations illustrate the gist of reinterpreting the unit method. Van De Walle et al. (2018) provides an example $23.5 \div 8$. The division was presented using the standard division algorithm, with explanations provided for why the algorithm works. Van De Walle et al. (2018) stated the following:

> Trade 2 tens for 20 ones, making 23 ones. Put 2 ones in each group, or 16 in all. That leaves 7 ones. Trade 7 ones for 70 tenths, making 75 tenths. Put 9 tenths in each group, or 72 tenths in all. That leaves 3 tenths. Trade the 3 tenths for 30 hundredths. (Van De Walle et al., 2018, p. 427)

Though shown as a specific example, the example has the potential to explain why the decimal points in the dividend and quotient need to be aligned. For instance, 23 ones divided into eight groups makes two ones in one group; thus two ones should be aligned with the digit “3” (the ones place value). A similar reason applies to the position of digit “9” in 9 tenths. In order for all the place values to be properly aligned, decimal points also need to be aligned. However, the textbook does not clearly point out this connection and leaves it to the readers to interpret.

Sonnabend (2010) uses a more straightforward example 0.8÷4, and stated:

> Each of the 8 columns represents 1 tenth, so the 2-column result represents 2 tenths. This exercise illustrates how 0.8 ÷ 4 is computed by dividing 8 by 4 and then placing the decimal point in the quotient above the decimal point in the dividend. (Sonnabend, 2010, p. 326)

In addition to the explanation, two long division examples are illustrated: 8 tenths divided by 4 to yield 2 tenths, and 0.8 divided by 4 to yield 0.2. The two long division examples are displayed side by side, with a clear purpose for comparison. By showing how 2 tenths and 8 tenths are aligned, Sonnabend (2010) hinted why decimal points need to be aligned so that 2 and 8 can both be at the same place value of tenth. However, Sonnabend’s (2010) example is oversimplified and only involves one decimal place. The presented example and its conclusion would be more generalizable if a multidigit problem were presented.

DISCUSSIONS AND CONCLUSIONS

Studies have reported pre-service teachers’ inadequate understanding of the standard algorithms for decimal multiplication and division (Lai & Murray, 2014; Muir & Livy, 2012; Tirosh et al., 1998). Due to the potential impact of textbooks on students’ learning, this study explores the possible reasons for pre-service teachers’ weak understanding from the perspective of the textbooks used for mathematics content courses. Eleven U.S. textbooks’ presentations of the standard decimal multiplication and division algorithms were examined to investigate the extent of the support provided by the textbooks to facilitate pre-service elementary teachers’ learning of the standard algorithms. The purpose of this study is not to evaluate individual textbooks but to describe their similarities and essential differences to inform mathematics educators and instructors who plan to use these textbooks. The findings indicate differences in textbooks’ emphasis on standard algorithms. Half of the textbooks analyzed do not fully present the standard algorithms but only show a couple of examples to demonstrate the process without summarizing the algorithms. The other half of the textbooks present the algorithms in all steps.

Then textbooks’ explanations for the procedures involved in the standard algorithms were explored. In total, four aspects were examined. For multiplication, whether the textbooks explain why adding the number of decimal places in factors to determine the place of the decimal point in the product was examined. For division, whether the textbooks explain why moving the decimal points in both divisor and dividend keeps the quotient unchanged, why only the divisor needs to be a whole number, but the same requirement does not apply to dividend, and why lining up the decimal points in the quotient and dividend were examined. The results indicate variations in textbooks’ treatment of the above four aspects. Looking across the textbooks, most of the textbooks attempt to explain why moving the decimal points keeps the quotient unchanged for the division algorithm; about half of the textbooks explain why adding the decimal places in the multiplication algorithm. Only two textbooks explain why lining up the decimal points in quotient and dividend. No textbooks explain why the divisor needs to be a whole number, but the dividend does not need to be. Looking at each textbook, no textbook addresses three or four aspects. Five textbooks address two of the four aspects, five address one aspect, and one textbook provides no explanation of any aspect. The average number of aspects explained by textbooks is 1.36 aspects per textbook. A possible reason for missing justifications for a particular step (e.g., DS3) may be due to textbook writers’ overlook that the justification on one or two steps represents the justification for all steps. Dividing the complete algorithm into three steps provides an analytical perspective for textbook writers to examine if their books address the rationale for all steps. Research has reported prospective teachers’ misapplication of steps in one standard algorithm to another algorithm (Lai & Murray, 2014; Muir & Livy, 2012; Tirosh et al., 1998). For instance, Tirosh et al. (1998) reported that in DS3, instead of simply lining up the decimal points in dividend and quotient, prospective teachers tended to incorrectly move the decimal point several places to the left, to compensate for the action they performed in DS1, where the decimal point was moved the same number of decimal places to the right. Prospective teachers’ incorrect actions could be related to their confusion between the two algorithms. They were not aware that in the standard algorithm for decimal multplication, when an expression 3.2×1.7 was changed to 32×17 in MS1, the two expressions 3.2×1.7 and 32×17 are not mathematically equivalent; therefore, an adjustment of decimal places needs to happen in MS3. However, in the standard algorithm for decimal division, an expression $3.2 \div 1.7$ is mathematically equivalent to $32 \div 17$. Thus, no adjustment of decimal places is required in DS3; lining up the decimal point in the dividend and divisor is sufficient. The results from this study of textbook analysis suggest that textbook writers need to pay more attention to how they unpack the rationales for the standard algorithms. A clear presentation of the rationales would deepen students’ understanding of the nature of the algorithms so that students can fully understand why performing one action in one algorithm is valid but performing the same action in another algorithm would be illegitimate.

The findings suggest that the textbook authors seem to be aware of the significance of explaining the rationale of the standard algorithms and all books collectively present multiple methods. But surprisingly, textbook authors are consistent in attending to the same aspects (e.g., explain why moving the decimal point keeps the quotient unchanged) but overlooking the other aspects (e.g., how to place the decimal point in the quotient). There is no clear reason to explain this high consistency. It may seem to be too straightforward to justify some of the procedures so textbooks writers decided not to do so, as Musser et al. (2019) wrote in their book, “this can be justified using division of fractions” without providing further explanation. However, students and teachers’ common mistakes reported in the research studies (Graeber & Tirosh, 1990; Lai & Murray, 2014; Lortie-Forgues et al., 2015) suggest the opposite. That is, learners have difficulties in understanding procedures such as placing the decimal point in the right place. Though there is no solid evidence that learners will certainly avoid making the mistakes if the textbooks provide appropriate explanations, the attempt made in the textbooks provide the needed information if learners actively search for explanations to support their learning. Textbooks’ presentation of the explanations will also remind novice mathematics instructors of the significance of unpacking these procedures. Considering college mathematics instructors’ tendency to use a textbook as their primary resource (McCrory et al., 2008), the support from a clearly written textbook could potentially impact novice mathematics instructors’ selection of the content to present in the classroom. In addition, providing explanations to mathematical procedures support the belief that mathematics is not a set of procedures or rules set by mathematicians, but a subject with connections and sense making, the type of belief reported lacking in pre-service elementary teachers (Steffe, 1990; Tirosh, 1990).

Even in the era of technology, professional organizations such as Association of Mathematics Teacher Educators (2017) still emphasize in the standards for the preparing teachers of mathematics that prospective teachers should be “taught by linking these procedures to important mathematical structures and properties” because “even though the standard algorithms often continue to be emphasized in elementary schools, candidates may not know [emphasis added] that the familiar algorithms that they learned in school” (p. 75). As indicated by the standards for the preparing teachers of mathematics, the purpose of the standard algorithms is to provide not only a means for calculation but also a place for intertwining and connecting essential mathematical concepts. The methods used to justify the rationales consist of applying essential mathematical concepts such as place value, property of exponents, models of divisions, and property of fractions. Among these mathematical concepts, the two models of divisions, partitive and repeated subtraction models, are essential to explain the hard-to-explain procedures, such as why moving the decimal point in DS1 preserves the value of the quotient and how to locate the decimal point in the quotient properly in DS3. As suggested by Ashlock (2010), emphasizing the meaning of division, using either a partitive (fair share) or repeated subtraction (measurement) model may help mathematics learners avoid making mistakes in computation.

The findings also suggest that though some textbooks set out to be a good start to providing a valid rationale, their explanations do not support learning as they intend to. Some textbooks use vague languages that could lead to misunderstanding and misconceptions. The Results section discuss how using an action language versus a static language may result in learners’ mistakes in incorrectly positioning the decimal point in decimal multiplication. The action language stresses the process, while the static language emphasizes the result. Research has reported students’ mistake of misplacing the decimal point in the product due to the confusion discussed above. For instance, Muir and Livy (2012) gave the pre-service teacher the following item: The decimal point on the calculator is not working. It shows the product of 12.68 × 1.55 is 19,654. Show what the correct answer should be. Muir and Livy (2012) found that the most common mistake is 1.9654. Students focused on having four decimal places (result) but forgot that the decimal point should be inserted (process) to 196,540, the product of 1,268 and 155, not 19,654. Concentrating on the result rather than the process seems to cause mistakes. Textbook writers need to be aware of how this nuance in language could potentially impact prospective teachers’ understanding. Clear language involving action needs to be used when describing the standard algorithm for decimal multiplication.

The results of the study also indicate that some textbooks focus on the discovery of the procedures and offer opportunities for pre-service teachers to observe how the procedures work. Other textbooks provide justifications to explain why the procedures work. Sometimes the same textbook adopts both approaches when presenting different content. For instance, Sonnabend (2010) presents an example that aims to have students discover the pattern to position the decimal point in the product when presenting the standard algorithm for decimal multiplication. When presenting the standard algorithm for decimal division, he presents an example that has the potential to explain why DS3 works using the reinterpreting the unit method. Aricha-Metzer and Zaslavsky (2019) investigated students’ productive and non-productive example-use for proving and distinguished the uses of empirical and generic examples. An empirical example concentrates on the specifics but fails to be seen in a general way. A generic example is a specific example that can be seen as representing a class of cases and supports the development of complete proof. An empirical example is typically used to show how a mathematical idea works but not why it works. The first example of multiplication presented by Sonnabend (2010) is an empirical example, while the second example involving division is a generic example that has the potential to be developed into a mathematical proof and illustrates why. If the learners only acquire the skills to discover but not to justify, they stay at the stage of inductive reasoning rather than moving to the higher stage of deductive reasoning. Research has reported pre-service teachers’ inadequate understanding of mathematical proofs (Knuth, 2002; Martin & Harel, 1989; Stylianides et al., 2007). Pre-service teachers tended to incorrectly believe that discovering a pattern and checking it on a couple of examples proves the validity of the noticed pattern (Knuth, 2002; Martin & Harel, 1989; Stylianides et al., 2007); the missing appropriate justifications might reinforce the incorrect belief that checking examples is sufficient as a mathematical proof. Textbook writers need to be aware of the difference between empirical and generic examples and present the justifications that promote the learning of deductive reasoning and the explanation of why the algorithm works in addition to how it works.

As pointed out by many tests and research studies (e.g., Liu et al., 2014; OECD, 2014), understanding of decimal multiplication and division is weak even in top performing countries on international comparisons of mathematical achievement. Research also reported that teachers in multiple countries (e.g., Israel, Hong Kong, Australia, and United States) struggled with understanding decimal multiplication and division (Lai & Murray, 2014; Joung et al., 2021; Tirosh et al., 1998). The findings of this study indicate that the lack of explanations in textbooks for teachers may contribute to the teachers’ weak understanding of decimal multiplication and division in the U.S. Though the study does not investigate the textbooks in other countries, international scholars could revise and apply the analytical frameworks developed in this study to examine textbooks in their countries. International mathematics educators can also adopt some methods when teaching the rationale of the standard algorithms for decimal multiplication and division.

This study has several limitations. Only 11 textbooks were examined. Even though the sample was carefully selected to be representative, there might be unanalyzed textbooks that provide new ways of explaining the standard algorithms that are missing from the analysis. In addition, the differences between the written and enacted curricula was acknowledged. It is possible that mathematics instructors who deem the standard algorithms as important explain the algorithms in ways not described in textbooks during their teaching practices. Future research can investigate how mathematics instructors teach algorithms to provide a complete picture of the actual learning happening in the classrooms. Studies can also be conducted if the ways of teaching algorithms affect pre-service teachers' understanding of the standard algorithms. Researchers can also explore how pre-service teachers read and interpret textbooks to examine the impact of the curriculum materials on their learning.

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APPENDIX A: LIST OF TEXTBOOKS

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