**International Electronic Journal of Mathematics Education**
2026, 21(1), em0864
e-ISSN: 1306-3030
https://www.iejme.com

MODESTUM

**Research Article** **OPEN ACCESS**

# Constructing prime number and prime factorization with realistic mathematics education: The case of 6th grade students

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**Citation:** Keyik, N., & Yilmaz, R. (2026). Constructing prime number and prime factorization with realistic mathematics education: The case of 6th grade students. *International Electronic Journal of Mathematics Education, 21*(1), em0864. https://doi.org/10.29333/iejme/17798

**ARTICLE INFO**

Received: 26 Nov. 2024

Accepted: 24 Dec. 2025

**ABSTRACT**

This research investigates the construction of prime number and prime factorization concepts through realistic mathematics education (RME). The research was conducted in a classroom setting with 6th grade students and was designed qualitatively as a case study. A readiness test was prepared to assess the students' preliminary knowledge of these concepts. The teaching process was implemented using activities based on RME principles. Considering the test results, researcher observations, and teacher views, three participants with different knowledge levels were selected, and two interviews were conducted with each of them. The data obtained from the teaching process and interviews were analyzed according to the APOS (action-process-object-schema) theoretical framework. The results demonstrate that the participants conceptualize the prime number and prime factorization as objects. It has been determined that the coordination involving divisor, multiple, odd-even numbers, factor tree, divisibility rules, and algorithm processes plays a fundamental role in conceptualizing prime numbers and prime factorization.

**Keywords:** concept formation, prime number, prime factorization, realistic mathematics education, APOS theory

## INTRODUCTION

The central and significant portion of mathematics curricula encompasses numbers, their properties, and relationships. Starting with natural numbers in primary school, students progress to integers and rational numbers in middle school. The curriculum concludes with introducing irrational, real, and complex numbers in high school, where students compare their characteristics (Australian Curriculum and Assessment Reporting Authority [ACARA], 2022; Ministry of National Education [MNE], 2018, 2024; National Council of Teachers of Mathematics [NCTM], 2000). Throughout this progress, natural numbers are significant in understanding and relating to the entire number system. Engaging with numerical concepts is often highlighted as a means to demonstrate their practical utility in problem-solving, foster a deeper understanding of mathematical principles, unveil the elegance inherent in mathematics, and shed light on the human aspects embedded in the historical evolution of numerical systems.

On the other hand, prime numbers can be described as the building blocks of natural numbers (Burkhart, 2009; Van de Walle, 1998), and understanding the multiplicative relationships among numbers necessitates a comprehensive understanding of prime numbers (Zazkis & Liljedahl, 2004). Despite recognizing the abstract nature of prime properties surpassing the comprehension level of schoolchildren, there has been a longstanding practice of globally incorporating basic number theory, including the fundamental theorem of arithmetic (FTA), into educational curricula. This trend persists due to the belief that the intellectual depth associated with prime properties extends beyond the immediate grasp of students. It is noteworthy that, as observed by Maaß and Doorman (2013), although inquiry-based learning has gained traction recently, the teaching of number theory appears to have endured and remained largely unchanged in many educational systems worldwide. While it is commonly asserted that exploring prime numbers and conducting research on them predominantly falls within mathematical content, Freudenthal (2006) raises the perspective that such endeavours may go beyond mere mathematical pursuits. Research has shown that prime numbers are incorrectly defined and that even if they are defined correctly, there are difficulties in their applications (e.g., Zazkis, 2005; Zazkis & Liljedahl, 2004) and that there are difficulties in prime factorization when performing operations and misdistinguishing between prime factors and factors (e.g., Lenstra, 2000; Sutarto et al., 2021; Triyani et al., 2012; Yilmaz & Dündar, 2021; Zazkis, 1999),

This article was compiled from Nehir Keyik's master's thesis under the supervision of Rezan Yilmaz, which was submitted to the Institute of Undergraduate Education, Ondokuz Mayis University.

there are various misconceptions such as the fact that whether the numbers especially 0, 1, and 2 are prime, prime numbers are small numbers, classifying prime numbers as composite, having limited understanding of the definition of prime numbers, often forming incorrect rules without reference to the underlying arithmetic concepts. (e.g., Özdeş, 2013; Sutarto et al., 2021; Zazkis & Campbell, 1996a, 1996b; Zazkis & Gadowsky, 2001). Also, Bu (2008) characterized middle grades' understanding of prime numbers as

> "an isolated definitional knowledge of prime numbers, pervasive procedural and rule-based approach to problem-solving, an inclination to seek empirical evidence in spite of other forms of argumentation, a narrow view of primality due to curricular limitations such as primes being only small numbers, and a lack of sensitivity to the representational features of numbers" (p. 18).

Although the necessity and importance of understanding prime numbers are emphasized by the studies conducted, it is seen that the research on the subject is still quite insufficient. When the related studies are examined, the understanding of prime numbers is generally conducted, and these studies are generally carried out on prospective teachers and undergraduate students. Also, studies examining how these concepts are abstracted are almost non-existent.

Given that mathematical concepts emerge from the genuine need for knowledge and relationships in real life, there is a prevailing belief that teaching these concepts should adopt a more informal and real-life contextual approach (Campbell & Zazkis, 2002; Gravemeijer, 1999). The idea is that when students connect their learning to their own experiences and engage in the process of mathematization, mathematics becomes more meaningful to them (Carpenter & Lehrer, 1999; Gravemeijer & Terwel, 2000). However, integrating informal meanings and familiar contexts into mathematics education should not be isolated from the simultaneous development of conceptual foundations (Campbell & Zazkis, 2002). In this context, realistic problems within environments where students actively reinvent mathematical structures using their methods and the models they create can facilitate this process (Gravemeijer, 1999). The sequence of steps, starting with developing operations-relations in real-life contexts, then transferring the same structure to other contexts, and finally formulating a standard structure through symbolization (Treffers, 1991), constitutes the path toward generalization and abstraction (Mitchelmore, 2002). Considering that a concept is a cognitive structure formed through abstraction (Von Glaserfeld, 1991; Yilmaz & Argun, 2018), the environment in which abstraction occurs becomes crucial in concept formation. Hence, APOS (action-process-object-schema) Theory which depends on Piaget's reflective abstraction and serves a model for mental constructions of understanding with the genetic decomposition (Arnon et al., 2014) may be a strong framework to suggest the conceptualization whenever the instruction is designed in accordance with the theoretical principles of realistic mathematics education (RME) which is rooted in Freudenthal's (1973) view of mathematics as a human activity and highlights the re-invention of formal knowledge as learners transform informal, real-life understandings through contextual problems (Treffers, 1991; De Lange, 1996).

Given that the understanding of prime numbers and prime factorization is initially acquired during middle school (ACARA, 2022; MNE, 2024; NCTM, 2000), there is a perceived significance and necessity in investigating the development of these concepts at this educational level, particularly due to the limited research in this domain. It is believed that analysing the evolution of these concepts by facilitating a transition from informal to formal situations through real-life contextual problems can provide valuable insights into both the learning and teaching processes. This research focuses on constructing the prime number and prime factorization concepts among 6$^{th}$ grade learners, employing APOS theory to define its genetic decomposition. Hence, the research concentrates on constructing these concepts by offering a fine-grained account of abstraction stages and proposes a task sequence by designing the instruction per RME theory. Moreover, the research builds upon the limited research conducted at the middle school level and aligns with current national and international curriculum standards. Therefore, the central research question is framed as follows:

*How do the construction processes of the prime number and prime factorization concepts unfold for 6$^{th}$ grade students in an RME-based teaching environment within the APOS theory framework?*

## THEORETICAL FRAMEWORK

### Prime Number and Prime Factorization

Prime numbers are introduced by their properties in the 6$^{th}$ grade, and prime factors of natural numbers are taught using the factor tree and division algorithm in this grade. Relatively prime numbers are taught in the 8$^{th}$ grade (ACARA, 2022; MNE, 2018; 2024; NCTM, 2000).

Prime numbers are generally defined in school textbooks as "integers greater than 1 whose divisors are only 1 and themselves" or "natural numbers greater than 1 that have no natural number divisors other than 1 and itself." Also, it can be defined as "a prime number is a natural number that has exactly two factors." The last definition may help students to exclude the number 1 from the list of primes (Zazkis, 2005). In the definitions, it does not use orders, and the exclusion of the unit is more integrated into the definition. Also, they are simplified variations of the irreducible property in rings (Kiss, 2020), and researchers (e.g., Varga, 1969; Zazkis et al., 2009) report teaching instructions leading to the definition based on the irreducible property to help students discover primes. Exploring prime numbers in this way is thought to be a helpful form of teaching that is accessible to students of this age. The FTA states that any natural number greater than 1 can be represented by a unique product of prime numbers (Argün et al., 2014; Curtis & Tularam, 2011). Hence, when teaching is planned based on the definition of "if a prime p divides a product ab of two integers a and b, then p must divide at least one of those integers a and b," which is the key in the proof of the FTA (prime property or Euclid's lemma), it is almost impossible that students must search for all products ab of the integers a and b to decide whether the numbers are prime or not and check whether p divides this product and if so, whether it divides at least a or b (Kiss, 2020). Therefore, planning the teaching as mentioned and completing the gap as much as possible would be meaningful.

Prime factorization, also known as prime number decomposition, is the expression of a number as the product of prime factors. Prime factorization helps to decompose a number into its smallest prime parts. Students can make connections between the prime structure of a number and the number of its factors (NCTM, 2000), so it also provides a way to relate to several mathematical topics, such as divisibility, simplifying, square roots, and fractions (Kurz & Garcia, 2010). According to the FTA, every positive integer greater than 1 can be represented uniquely as a product of prime numbers, up to the order of the factors. Several properties of arithmetic depend on this uniqueness, so it is an essential property of the whole number (Mason, 2006).

Factor tree and division algorithm techniques are taught in education for prime factorization. Students generally start by dividing the number by the first prime number, 2, and continue dividing by 2 until it can no longer be divided. Then they divide by 3, 5, 7, etc., until the only numbers left are prime numbers. However, this process is generally directed to carry out procedures, likely leading to rote learning instead of reasoning and problem solving (Jäder et al., 2020).

It is beneficial to teach prime numbers and prime factorization in environments where students can explore these concepts through real-life applications rather than relying on rote, procedural methods. Prime numbers are crucial not only for understanding mathematical relationships but also in various real-world disciplines. Prime numbers are used to keep important information safe, and one of the foremost applications of prime numbers is encryption. Also, physicists research how prime numbers govern our universe, and some musicians use prime numbers to break free from traditional musical rules. In nature, certain insects' life cycles are governed by prime numbers, impacting their survival (Curtis & Tularam, 2011). Since this research was conducted with $6^{th}$ grade students, contextual problems appropriate to their level were employed: the property of prime numbers–being divisible only by 1 and themselves–was represented through rectangular board areas, while prime factorization was through simple encryption tasks. Hence, the instruction was supported by making these concepts' critical role in daily life at this grade level visible and enabling the students' modeling.

### Realistic Problems and RME

RME, based on Freudenthal's (1973) idea that mathematics is a human activity, emphasizes learners' re-invention of formal mathematical knowledge by transforming their informal, real-life knowledge through contextual problems (De Lange, 1996; Treffers, 1991). This process mirrors how real mathematicians work, as learners abstract their informal knowledge into formal mathematics and then relate it back to real-life situations (Gravemeijer, 1999; Gravemeijer & Terwel, 2000). Realistic problems grounded in meaningful contexts facilitate this transformation, offering opportunities for modeling and mathematization (van den Heuvel-Panhuizen, 2005). RME offers a distinctive approach to context problems which, compared to their broader use, highlights the potential benefits of these problems in enhancing mathematical understanding (Beswick, 2011). In addition to serving concept formation, it also enables students to develop their modeling skills.

In RME, three key elements guide instructional design: guided reinvention through progressive mathematization, didactical phenomenology, and self-developed models (Gravemeijer, 1994, 2001). Guided reinvention, a cornerstone of the teaching approach, involves the systematic configuration and organization of problems to uncover rules and reveal the mathematical elements within, and this strong intuitional component prompts the students to reinvent the mathematical concept (De Lange, 1987; Fauzan, 2002; Yilmaz, 2020). Progressive mathematization unfolds as a two-stage process: horizontal and vertical mathematization. In the horizontal stage, informal strategies are employed for schematization, transforming real-life problems into mathematical ones. Vertical mathematization then involves abstracting the conception into the realm of symbols and solving the problem using different models or finding relevant algorithms with the aid of mathematical language and informal strategies (Gravemeijer & Terwel, 2000; van den Heuvel-Panhuizen & Drijvers, 2014). Didactical phenomenology necessitates working with phenomena meaningful to students during the learning process, allowing them to organize and engage with concepts in ways that serve four essential functions: concept formation, model formation, applicability, and practice (Gravemeijer, 1994; 2001; Treffers & Goffree, 1985). Self-developed or emergent models act as bridges, connecting students' informal knowledge with formal knowledge while problem-solving. Initially, students develop a model that evolves into a dynamic and holistic representation aligned with their mathematical thinking through processes of generalization and formalization (Gravemeijer, 2001; Treffers, 1991). These models (model for) that are dependent on the context transform into the models (model of) independent from the problem situation (Zandieh & Rasmussen, 2010).

The development of instructional design based on the RME principles involves several vital elements. These include initiating learning with a solid foundation of concrete, content-rich problems that support the mathematical organization and encourage students to follow informal solution processes within a context (construction and concretization). Additionally, it entails utilizing models and schemes emerging during the problem-solving process to bridge gaps in students' abstraction levels and facilitate cognitive transitions (levels and models). Special assignments are given to students to elicit their independent ideas and reflections (reflection and special assignment). Group work is emphasized to create a social context and promote interaction, allowing students to learn the concept in a spiral relationship with their prior knowledge (structuring and interviewing) (Gravemeijer & Terwel, 2000; Treffers, 1991; van den Heuvel-Panhuizen & Drijvers, 2014).

### Concept Formation and APOS Theory

Our comprehension of a mathematical concept evolves through introspective contemplation, taking shape within our cognition. Consequently, this personal understanding emerges from internal meanings and diverges from the collectively acknowledged formal definitions or formulas associated with the concept. Given the inherent challenge of directly constructing mathematical concepts, the necessity arises to construct mental frameworks for interpreting them. APOS theory serves as a conceptual model elucidating the mental construction of understandings. Propounded by Dubinsky in 1984 and 1986 and further refined in collaboration with colleagues such as Asiala et al. (1997), APOS theory draws inspiration from Piaget's concept of reflective abstraction. Rooted in the constructivist philosophy, it unfolds through the interplay of action, process, object, and schema. The theory focuses on the cognitive processes occurring during the acquisition of a mathematical concept. The expected mental structures and mechanisms for learning a concept are called genetic decomposition, informing the structure of educational environments, teaching materials, and assessments of student achievements and challenges in problem-solving (Agacdiken & Yilmaz, 2023; Arnon et al., 2014).

According to the APOS theory, the first step in structuring a concept is action. At this stage, the learner transforms the objects previously constructed in his mind (Asiala et al., 1997). While transformations are carried out with the help of external stimuli, visualization cannot yet be performed. The individual is given algorithms to follow and is expected to follow them. In the action stage, concepts are static and cannot be acted upon (Arnon et al., 2014). When the learner concentrates on the action and repeats it constantly, he/she begins to perform the same transformations without any external stimulus. This structure, formed by the internalization of actions, is called process (Asiala et al., 1997). Processes are not only formed by the internalization of actions but a new process can also be formed by the coordination of two processes or the reversal of a process (Dubinsky, 1991; Dubinsky & Moses, 2011). These transformations are dynamic, unlike those in the action stage. If the process of internalizing the action and transforming it into a process can be perceived as a whole rather than being a process performed by the learner, and if it is realized that other actions and processes can be applied, the process can be encapsulated as an object (Agacdiken & Yilmaz, 2023). A schema is a collection of actions, processes, objects, and other schemes used to solve a problem related to any mathematical concept in the individual's mind. Hence, we can express our schema related to a concept as the framework of the set of relationships we have regarding that concept.

In the APOS theory, genetic decomposition is a hypothetical model that describes the mental structures and mechanisms that can be constructed to learn mathematical concepts. Genetic decomposition begins as a hypothesis based on researchers' experiences with learning and teaching the concept, their mathematical knowledge, previously published studies on it, and its historical development. It remains a hypothesis until empirically tested and is considered a preparatory stage, referred to as pre-genetic decomposition.

Genetic decomposition is inherently theoretical and constitutes an analytic hypothesis that describes the cognitive structures students may construct as they develop mathematical concepts. Hypothetical learning trajectories in RME function as pedagogical hypotheses that anticipate possible developments in students' strategies and representations during instruction. If RME aligns with the constructivist approach to learning, genetic decomposition also will be consistent with RME's goals and dimensions. Within this aim, the hypothetical learning trajectory informed the design and sequencing of instructional tasks within the RME framework. At the same time, the genetic decomposition was used as a complementary analytic lens to interpret students' conceptual constructions. The research includes a genetic decomposition modeling the construction of the concepts of prime numbers and prime factorization consistent with RME's hypothetical learning trajectory.

## METHOD

### Design of the Research

This qualitative investigation explored the construction processes of prime numbers and prime factorization among 6$^{th}$ grade students using realistic problems. Conducted as a case study (Yin, 2009), the teaching process followed the RME approach, emphasizing knowledge construction through realistic problems. Students' mental structures were analysed within the APOS theory framework. The research process was carried out as follows:

Administering the readiness test to the classroom of 15 students; identifying three participants with three different levels; distributing the students into three heterogeneous groups of five, each with a different participant; conducting lesson 1 and lesson 2, including context problem 1; conducting the first interviews with each participant; conducting lesson 3 and lesson 4, including context problem 2; conducting the second interviews with each participant; analyzing the data; reporting.

### Participants

This research involved three participants from a state middle school in the Black Sea Region, where one of the researchers was a mathematics teacher. Participation was voluntary, and students were selected through purposive sampling. As the instructional environment was designed for heterogeneous groups, a readiness test specified in data collection tools, covering topics related to prime numbers and prime factorization, was administered to the entire class. Based on the test results, classroom observations, and input from other mathematics teachers regarding students' academic performance and personal characteristics, 15 students were grouped heterogeneously into groups of three.

From these groups, three participants (pseudonyms: Murat, Ali, and Selim) were selected for individual interviews using maximum variation sampling. The aim was to include students who could express their ideas clearly, while representing different readiness levels (advanced, upper-intermediate, and intermediate). All three participants correctly determined the side lengths of the rectangles. Murat answered all 20 readiness test questions correctly except for an oversight. Ali made mistakes in adding and subtracting fractions with different denominators, incorrectly adding numerators and denominators without equalizing them. He also erred in comparing decimal and percentage values due to a lack of understanding of their relationships. Furthermore, Ali misinterpreted a question asking for decimal representations, writing the pronunciations of fractions instead. His other errors included miscalculating the rectangle's perimeter and making a mistake in applying the divisibility rule by 3. Selim miscalculated decimal representations and their pronunciations, made one mistake in finding divisors, and only partially applied the divisibility rule by 3. Additionally, Selim made an incorrect random choice regarding side lengths. Considering the total test results, Murat answered 19 questions out of 20 correctly, Selim answered 16, and Ali answered 13 correctly. In addition to test scores, students' responses to open-ended questions were considered, particularly those that reflected different or notable statements. Readiness levels were classified as advanced when students demonstrated near-complete accuracy with correct explanations, upper-intermediate when partial conceptual errors were present alongside relatively correct explanations, and intermediate when multiple conceptual and procedural errors were observed. The participants, their readiness levels, and group compositions are presented in **Table 1**.

**Table 1.** Participants in groups and their levels (students in groups were abbreviated with S and coded with numbers)

| Groups | Participants and students in groups | Participant levels |
|---|---|---|
| Group 1 | Murat, S1, S2, S3, S4 | Advanced |
| Group 2 | Ali, S5, S6, S7, S8 | Intermediate |
| Group 3 | Selim, S9, S10, S11, S12 | Upper-intermediate |

### Teaching Process and Data Collection Tools

The instructional approach adhered to the principles of RME, fostering a learning environment where students could engage in group discussions and freely explore mathematical concepts. The contextual problems provided a structured introduction to concepts, allowing students to connect the given scenarios with their existing knowledge. Furthermore, the learning process encouraged the emergence of informal solution strategies, didactic phenomena, and their modeling processes, contributing to the abstraction of mathematical concepts. Following group problem-solving sessions, interviews were conducted to gather insights. During the instructional process, contextual problems were followed by in-class activities involving related concepts, exercises, and problem solutions. These were conducted to support concept formation and its relationships with other concepts. Similar processes were then reinforced with homework assignments.

Data collection involved various tools, including a readiness test, observations, interviews, and worksheets used during group activities. The test comprised 20 questions, including multiple-choice and open-ended items covering fundamental concepts like factor/divisor, multiples, quotients, divisibility rules, fractions (simplification, expansion, addition, subtraction), and finding the side lengths of a rectangle based on its area. The test was revised according to the opinions of six mathematics educators and one language educator. Then it was applied to different 6$^{th}$ grade students in the same school and put into the final form.

The observations were unstructured, and the students were observed during the instruction, gaining insight into classroom dynamics, academic progress, and personal situations. Video recordings documented group activities and whole-class interactions. Each student's contributions were traced with different coloured pencils on individual and group papers.

Interviews were semi-structured and conducted at the end of each lesson and were designed to capture students' thoughts and problem-solving processes. Both interview forms included questions that asked the students to recall problem 1 and problem 2 solved in class, to explain what they did while solving these problems, what they thought, how they how they reached their solutions, why they thought so, and to explain the concept they modelled. Additional interview questions addressed the construction and application of relevant mathematical concepts. Participants documented their thoughts, solutions, and responses on paper during the interviews. They were encouraged to think aloud in these interviews and during the interviews, as in the classroom environment, materials (such as a tape measure, ruler, and algebra tiles) were also ready. The first interview with Murat, Ali and Selim lasted approximately 18 minutes, 15 minutes, and 16 minutes, and the second interview lasted 17 minutes, 18 minutes, and 15 minutes, respectively.

The first interview focused on participants' reflections on "determining prime numbers and their properties," exploring their problem-solving approaches for the problem which was related to the commemoration day of the National Victory planned to be celebrated on the agenda (in **Figure 1**), and prior knowledge application. The second interview reflected on "determining the prime factorization of natural numbers" after solving the second problem (in **Figure 2**).

The school administration plans to decorate a corridor with visuals and maxims highlighting a national celebration day. Student-prepared content will be printed and displayed on a board along the corridor wall. The teacher will order a board from an affordable online shopping site, which takes orders based on area measurements instead of width and height. To estimate the board's area, the teacher will choose natural number dimensions. To avoid confusion from variations, which specific area measurements should our teacher choose for the board?

**Figure 1.** Board problem (Source: Authors' own elaboration)

To prevent damage to the board ordered for visuals and maxims related to the national celebration day, it will be stored in a room in the school until it is used. Teachers and administrators who use a two-digit password can enter this room with a coded lock on its door. The password can only be deciphered by using the prime factors of the code. If some students have entered this room, how could they know the deciphered password and the prime numbers that decode it? If you chose a strong password, which numbers would you use? What would you suggest for finding the numbers that can decode a password with many digits?

**Figure 2.** Password problem (Source: Authors' own elaboration)

### Data Analysis

The content analysis of transcribed data from teaching and interview process videos involved coding within the framework of the genetic decomposition of the prime number and prime factorization concepts. The analysis was conducted iteratively, comparing and examining the coded data repetitively (Creswell & Poth, 2016). Subsequently, an inductive approach was applied to thematize the coded data. Multiple experts independently repeated the data analysis, and the final themes were established through consensus. This process aimed to conduct an in-depth analysis of the prime number and prime factorization concepts, guided by the theoretical framework of APOS. Care was taken to ensure that the information obtained was highly descriptive, presenting the findings directly to the reader.

Before the teaching process, a genetic decomposition was prepared, offering researchers preliminary insights into how students might conceptualize the prime number and prime factorization. This information was valuable for designing the RME environment and planning the teaching process. This pre-genetic decomposition was refined based on the codes identified during data analysis, providing a foundation for understanding and addressing students' construction of these concepts. These revisions were explicitly indicated in the results section, thereby preserving the coherence of the analysis. The pre-genetic decomposition related to the prime number and prime factorization was structured as follows:

#### *Action*

Realizes that the number considered for the area of the given board in the context must specify a unique rectangle by determining the side lengths of the rectangle that will constitute the area.

#### *Process*

Repeats the action of finding the rectangle's side lengths that give the considered area, determines all areas specifying a unique rectangle, and *interiorizes* the action to the process. Various *coordination* can be performed to find the side lengths that give the considered area: Coordinates with the process of a multiplier/factor is addressed, and the factors of the number giving the area can be found to determine whether this area specifies a unique rectangle or with the process of divisor to ascertain the condition of specifying a unique rectangle or with the process of the rules of divisibility to consider the validity of the area for a unique rectangle. Additionally, the conditions of being odd or even natural numbers can be included in the process.

#### *Object*

By *generalizing* the idea that the divisors of numbers considered as the area of a unique rectangle are only 1 and itself, *encapsulates* the natural numbers greater than 1 that have no natural number divisors other than 1 and itself, then constructs the prime number as object (1).

*De-encapsulates* the object of the prime number into the process and then, by *coordinating* this process with the processes of factor tree or algorithm or long division or multiplier/factor, finds the multiples/factors of the number that was considered as a password in the context and then, determines the prime ones among them. By *generalizing* the prime ones of the factors, *encapsulates* the expression of a natural number (represented uniquely) as the product of its prime factors into the object (2) of prime factorization.

## RESULTS

The research organized its findings on prime numbers and prime factorization using the APOS theoretical framework. The main focus was on differences in how these concepts are constructed. The study aimed to present its results in two sections, which discuss prime numbers and prime factorization separately by synthesizing the constructions developed by participants during group studies and interviews. So, students' construction processes are reported in detail. To enhance readability, representative excerpts illustrating the APOS-based analysis are provided in **Appendix A**.

### Construction of Prime Number

In the classroom, Murat and his friends read about the problem many times and tried to understand it because there was no numerical information for the problem. Murat investigated whether different rectangles provided the area he obtained when he gave 1 unit for the short side and any number to the long side, both by setting with algebra tiles and then representing on the paper. Murat tried to find the divisors of all numbers from 1 to 100, sequentially, by long division. As a result of his efforts, Murat said,

> Teacher, I found it! The smallest area is 2. Because if the area is 1, it will be a square, there is only one divisor … If the number we find can only be divided by 1 and itself, we can use that number. Otherwise, we cannot use it. For example, is 3 … divisible by 3? Yes … Is it divisible by 1? Yes. All numbers are divisible by 1 and itself. But it should not be divided by any other number.

One of the group members, S1, asked Murat if the area could be 6 $m^2$, and Murat modelled a rectangle with an area of 6 $m^2$ using algebra tiles and told S1,

> If it says 6, the sides will be 3 and 2. Also, it will 6 and 1. There are two rectangles here. But there has to be one rectangle.

Then, he stated that the given area should represent a single rectangle. This data clearly demonstrates that Murat was at the action stage. He tested divisors by trial and error and checked rectangles through concrete models. Moreover, his response to S1's question shows that he had not yet generalized the concept at the process or object level; he was only experimenting with concrete models. Therefore, his understanding at this stage was spontaneous but still experimental and pre-procedural. His models with algebra tiles and representations are given in **Figure 3**.

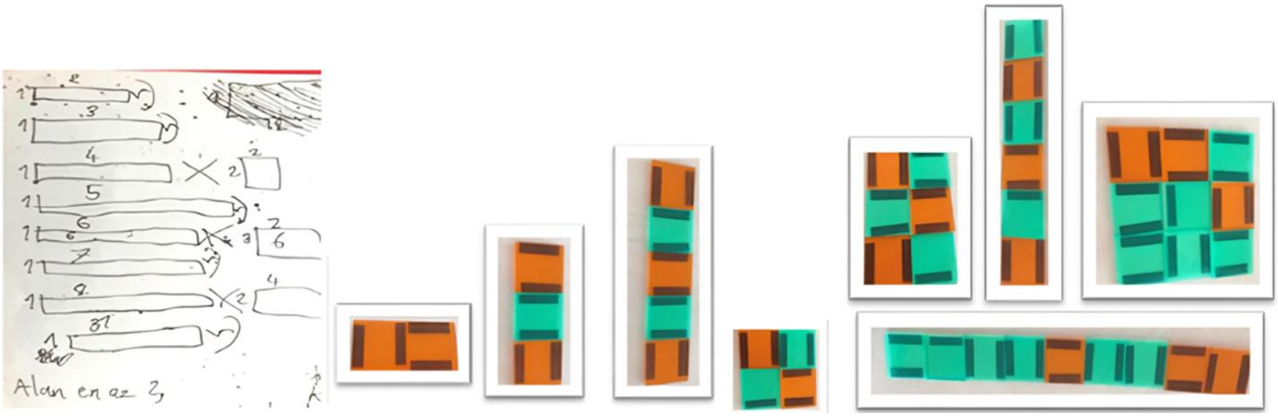


**Figure 3.** Models with algebra tiles and representation of Murat in the group work (Source: Authors' own elaboration)

His group member S2 tried to find a pattern while looking at the relationship between numbers indicating the area of a single rectangle, and expressed his thought as

> Look! Doesn't it go like this? 2 … skipped a finger, 3 … 2 fingers skipped, 5 … 2 fingers skipped 7?

But Murat stated that

> After 7, 4 skips are passed to 11, and after 11, 2 skips are passed to 13, and that this pattern is wrong.

On the other hand, S1 thought that "… *odd numbers are possible, even numbers are not*" and he could not realize 2 as an odd number. While finding the prime numbers from 1 to 100, Murat tried to find a pattern among the numbers by saying,

> 19, 29, 59, … there are too many nines … Let's make a deal with you. You find it back from 100. Let me find the one after 37,

and then they collaboratively wrote all the prime numbers up to 100 on the paper. Murat thought that the numbers he needed to find should be numbers that "have no divisors other than 1 and itself." This data shows signs of students transitioning from the action to the process stage. S2's search for a pattern reveals his attempt to establish connections among operations, though he made incorrect generalizations. Murat's attempts at pattern searching and his collaboration with S1 to complete the list demonstrate progress toward internalizing the concept, indicating movement into the process stage. These developments emerged spontaneously through the students' own discussions, without teacher or researcher guidance.

Ali and his group of friends first went to the school corridor where the board would be made, tried to estimate its area and thought about what size board could fit. They modeled the board by posing rectangles with algebra tiles. They thought that the short side of the board should be 1unit and then tried on the long side. S5 realized that finding the actual board's area is unimportant, and they need to find out which numbers can replace the area. Moreover, Ali supported her by saying,

> We will do the math, we are not builders.

When it was offered to measure the wall with a tape measure, Ali was not keen on estimating. During this process, the researcher guided them in understanding the problem. Then, Ali said,

> Hmm … I understand ... If it is not divisible by anything other than itself and 1, it becomes that number ... It wants only one thing. It can be divided by itself and it can also be divided by 1.

He understood that the natural number he would choose as an object in the action stage should indicate the area of a unique rectangle. So, this episode shows that Ali was at the action level of APOS. At first, he tried to understand the problem contextually by estimating the board's area, but with the researcher's guidance, he shifted focus to its mathematical essence. Ali's statement about doing mathematics indicates a move toward abstraction from context. The progress here was partly prompted by researcher guidance, but Ali's later definition of a prime as a number with no divisors other than 1 and itself demonstrates movement toward the process stage of understanding.

His groupmate S6 offered that 31 and 41 could indicate the area of a single rectangle, and Ali approved her. Ali also said that the numbers 11 and 13 only have two divisors and added them to the list, as shown in **Figure 2**. S7 realized that the common feature of these numbers is their oddness. Then S6 said that all even numbers are divisible by themselves, 1, and 2 because they are even, and that the numbers they need to find must be odd. The researcher asked the group to write these numbers in order, starting from the area of the smallest rectangle. When considering the area as 1 (square unit), Ali said,

> If we make it 1, it will not be a rectangle. It becomes a square,

and added 2 to the list as a prime number. Likewise, they listed other numbers by posing them with algebra tiles, drawing them on paper, and writing the prime numbers, as in **Figure 4**. While finding the factors of numbers, Ali offered to use the factor tree. When S5 asked whether 51 could be possible, Ali said,

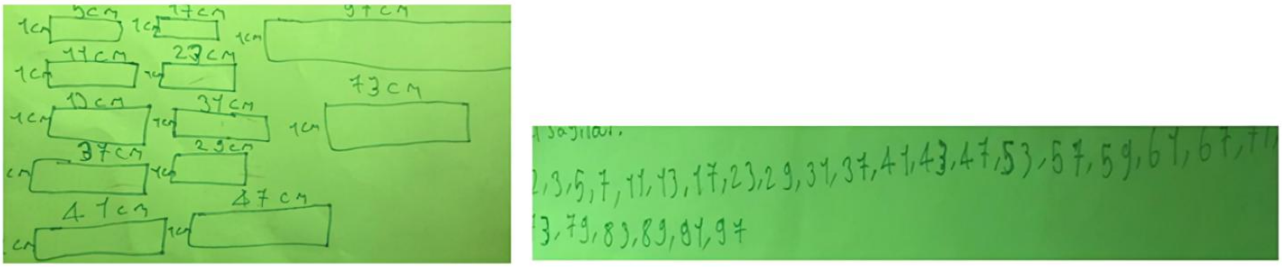


**Figure 4.** Representations of Ali in the group for the board problem (Source: Authors' own elaboration)

> No, it is divisible … It is divisible by 3. Add 5 and 1 to make 6. Since it is a multiple of 3, it is divisible.

They found prime numbers by coordinating natural numbers with multiples, divisors, odd-even numbers, factor trees, and divisibility rules. This data shows that the students had moved into the process stage. They no longer progressed solely by trial-and-error, as in the action stage, but by forming generalizations among numbers. S6 and S7's generalizations on eliminating even numbers and highlighting odd numbers are examples of conceptual coordination. Ali's inclusion of 2 as a special case in the prime list shows a tendency toward a transition from the process to the object stage. The progress here was partly prompted by the researcher's request for sequential listing and partly spontaneous, as students' own discoveries.

Selim tried constructing rectangles with algebra tiles with sides of 3 and 4 units and 4 and 5 units. However, since he could not proceed any further, he asked for help from the researcher. The researcher guided the group with questions to understand the problem and asked them to think. Selim said that if the area is 35 square units, the length of the sides could be 5 and 7 units or 35 and 1 units, and that he understood what it meant to have no confusion on the board, as saying,

> I understand. It only needs to be divided by 1 and itself… Everything is divisible by 1 and itself.

Selim and S9 began to investigate which numbers satisfied this condition. They focused on whether the numbers they thought were divisible by which numbers. S9 noticed that all even numbers are divisible by 2 and eliminated them by saying:

> Then there will be no even numbers. Because they are all divisible by 2. Odd number. It has to be the only one.

While researching the divisors of numbers, Selim stated that 31 is only divisible by 1 and itself and said that the area of the board could be 31 square units. When the researcher asked what other numbers there might be and asked them to rank them from 1 to 100, Selim and S9 said that 2 also indicates the area of a single rectangle, and then they thought in order and included 3 in their list. When they got to 4, they said,

> 4 can't be done … there will be no even numbers.

Selim said,

> If we say 5. Odd number … Oh ... yes, 5 is divisible only by 1 and itself.

Afterwards, they increased the numbers in their lists by proceeding only with odd numbers. Finally, they added the number 97 to their list as shown in **Figure 4**, and stated as

> We calculated numbers that are divisible by only 1 and itself ... Otherwise, the board may be wrong … Even numbers are divisible by 2, so we immediately eliminated those divisible by 2. We chose odd numbers. But ... 2 is even but divisible by only 1 and itself.

This process shows that Selim and S9 were transitioning from the action to the process stage. Initially, Selim engaged in trial-and-error by building models in the action stage, while S9's elimination of all even numbers indicated an attempt in the process stage by generalization. Selim's claim that 31 is prime and his subsequent systematic listing of numbers such as 2, 3, and 5 shows that he was beginning to internalize the procedure. The progress observed here emerged both spontaneously through the students' own discoveries and was prompted by the researcher's request for sequential listing.

When the researcher asked what they thought about the number 101, Selim said,

> 101 has no divisors other than 1 and itself. Is it divisible by 2? Indivisible. Because it is not an even number. Is it divisible by 3? Indivisible. Because if we add 1, 0 and 1, it equals 2. It is not a multiple of 3 and cannot be divided by 3. It is not a factor of 9. It cannot be divided by 9. Since it is not divisible by 2 and 3, it is also not by 6. Since its last digit is not 0 or 5, it cannot be divided by 5. It has to be an even number to be divisible by 4. It cannot be divided by 4. It is not divisible by 7 and 8. So, we can easily order a board with an area of 101 ... We had an advantage of the divisibility rules,

and they constructed their lists for the given numbers as shown in **Figure 5**. Selim's explanation shows that he had reached the process stage. He was no longer relying solely on trial-and-error but was systematically applying divisibility rules to make inferences. Selim's systematic testing of the number 101 using different divisibility rules demonstrates that the concept had been internalized and that he had moved into the level of generalization. The progress here was largely spontaneous; the researcher posed the triggering question, while Selim carried out the reasoning.

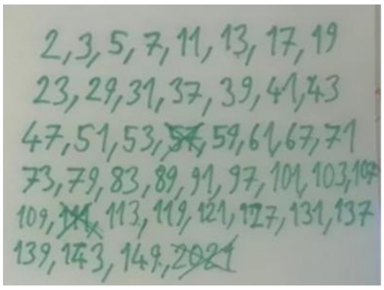


**Figure 5.** Representations of Selim in the group for the board problem (Source: Authors' own elaboration)

In the interviews, the researcher reminded the participants of the contextual problem discussed in class and asked Murat what the area value of the rectangular board meant. Murat replied that it could be numbers indicating the area of a single rectangle. Since Murat investigated how many different rectangles could be formed from the area measurement he had taken, he was considered to be on the action stage. Afterwards, he said that the area measurements he would find should indicate only a single rectangle and be divisible only by 1 and itself. Exploring other numbers that met this criterion can be expressed as starting to internalize the action and progressing to the process stage. By applying the concept of division to the numbers, Murat coordinated with the concept of the divisor. This indicates that he went to encapsulate the process by stating,

> If a number is divisible by any natural number other than 1 and itself, we cannot use that number, if not, we can use it,

for numbers that can measure the area of a single rectangle. The numbers obtained by Murat and the relevant generalization in the interview are given in **Figure 6**.

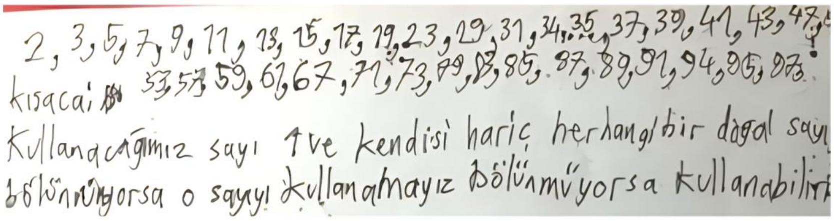


**Figure 6.** Representations of Murat in the first interview (Source: Authors' own elaboration)

In order to understand whether Murat constructed the concept of prime number as an object, the researcher presented a problem about finding the prime numbers which are on the doors of dressing rooms of a swimming pool. Murat successfully identified which of the 42 numbers given were prime numbers. Therefore, it is believed that Murat was passed on to construct the concept of prime numbers as an object. This interview provides a clear example of Murat's progression through the action → process → object stages. Initially, he experimented with creating single rectangles at the action level; later, at the process level, he generalized the principle of "having no divisors other than 1 and itself." His correct identification of primes among 42 numbers in the pool problem indicates that he constructed the concept of prime numbers as an object. The progress here was largely prompted through the researcher's questions, yet the generalization and objectification were achieved through Murat's own mental constructions.

During the interview, the researcher reminded Ali of the contextual problem discussed in class and asked what they thought about its solution. Ali responded,

> I need to find numbers that are only divisible by itself and 1. For example, if I say 26, the lengths of the sides would be 13 units and 2 units or 26 units and 1 unit. So, that wouldn't work.

This response indicates that Ali understood that the number representing the area should correspond to the area of a single rectangle, suggesting that Ali was in the action stage. Afterwards, Ali mentioned the sequence of numbers as 2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 91, 97 noting that all even numbers are divisible by 2, and therefore, apart from 2, no other even number represents the area of a single rectangle and indicated that the numbers found within the class should only be divisible by 1 and themselves. At this point, Ali had moved into the process stage. Instead of proceeding by trial and error, he was generalizing through divisibility rules. Moreover, these ideas were spontaneous; the researcher only posed guiding questions, while Ali generated his systematic inferences independently.

In acquiring these numbers, Ali coordinated concepts such as factors, multiples, divisors, divisibility rules, even numbers, odd numbers, and factor trees. The researcher explained that these numbers have a special name called "prime number." It is thought that Ali was encapsulating numbers that have no divisors other than 1 and themselves into the concept of prime numbers. In response to a pool problem, Ali correctly identified which of the given numbers were prime numbers, indicating progression into the object stage of prime numbers.

During the interview, Selim mentioned,

> To avoid confusion, the area of the board should only be divisible by 1 and itself. All numbers are divisible by these. They shouldn't be divisible by any other number. I systematically thought. Only 2 is an even number. The subsequent even numbers are divisible by 2. I eliminated the evens. Numbers ending in 0 and 5 are divisible by 5; I eliminated those. Numbers whose digits sum up to 3 or multiples of 3 are divisible by 3; I eliminated those. Numbers divisible by both 2 and 3 are divisible by 6; I eliminated those too.

This illustrates that Selim, upon reading the problem in the action stage, understood the need to identify numbers representing a single rectangle's area. Upon internalizing this process and transitioning to the process stage, Selim demonstrated coordination with divisibility rules, even and odd numbers. Selim's explanations provide a strong example of the process stage. His systematic application of divisibility rules in sequence demonstrates the development of his capacity for generalization. This progress was largely spontaneous; the researcher merely reminded him of the problem, while Selim himself constructed the reasoning.

When the researcher asked Selim about which numbers could represent the area of the board and why, Selim responded,

> I wrote down 1 for the shorter side. I then looked at which numbers could be the longer side. They are numbers divisible by 1 and themselves. 2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 91, 97.

Similarly, the researcher explained that these numbers have a special name called "prime numbers," indicating that Selim encapsulated numbers that have no divisors other than 1 and themselves into the concept of prime numbers. Thus, Selim's understanding of numbers that have no divisors other than 1 and themselves as prime numbers was evident. In the pool problem, Selim correctly identified which given numbers were prime numbers (**Figure 7**), indicating progression towards objectifying the concept of prime numbers. This interview reveals Selim's transition from the process to the object stage. His listing of numbers according to the principle of divisibility only by 1 and themselves demonstrates that he coordinated the concept procedurally (process). With the researcher's labelling as prime numbers, Selim integrated numbers with these properties into a single concept as an object. His successful distinction of prime numbers in the pool problem serves as evidence of this objectification. The progress here was partly prompted by the researcher naming the concept and partly spontaneous by Selim's own logical classifications.

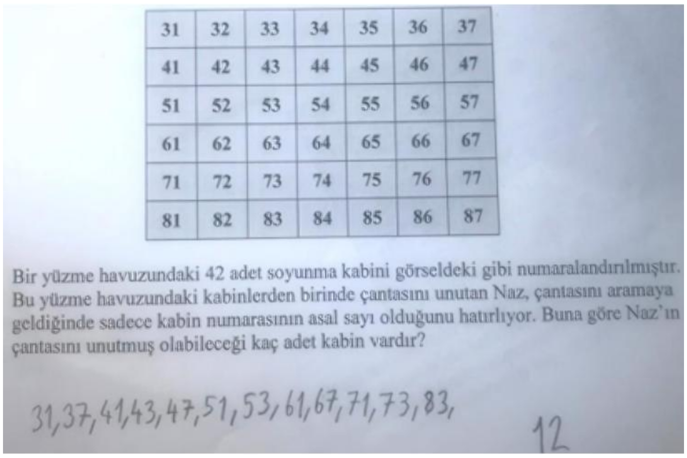


**Figure 7.** Solution of Selim for the pool problem in the first interview (Source: Authors' own elaboration)

## Construction of Prime Factorization

Murat, when initially reading the problem within the group, concluded that

> We will multiply all of these together. For example, you will multiply 2 with all of them. Multiply by 3. Multiply by 5. For instance, when you multiply 23 and 29, you get a result with 3 or 4 digits. So, there's no need to multiply those. Only multiply if the result is two digits. Starting from 2, I will proceed with the multiplications sequentially ...

and concluded that the password consists of a two-digit number resulting from the multiplication of two prime numbers. Murat shared the tasks by saying his group mates,

> You will continue it until the result is 100. For example, if you multiply 3 and 33, you get 99. But if you multiply 3 and 37, it becomes 111. You will multiply until 31,

and decided to multiply all the listed prime numbers sequentially and attempted to find the two-digit multiples of prime numbers as shown in the first part of **Figure 8**. This data shows that Murat was at the action stage. He merely applied mechanical operations at this stage, multiplying primes sequentially to test the results.

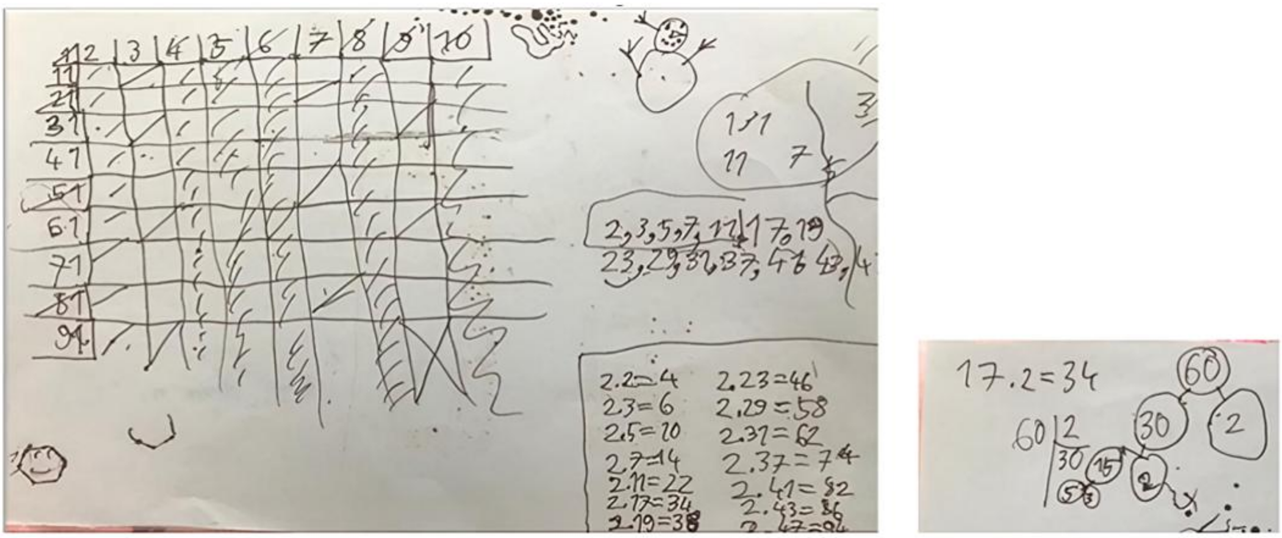


**Figure 8.** Solutions of Murat in the group for the password problem (Source: Authors' own elaboration)

The researcher asked the students whether the problem mentioned that the password consists only of prime numbers. In response, Murat first thought of any number as a password, but instead of finding its prime factors, he preferred to go from the part to the whole by finding two-digit multiples of two-digit primes and said,

> Now, the passwords broken by 11 are 11, 22, 33, 44, 55, 66, 77, 88, 99 ... So, it could be 11.

Following this, the researcher asked the students in the group if the password was 60, what numbers could decode this password. Murat expressed that he understood the problem and explained that the prime factors of 60 are 2, 3, and 5, which were determined using a factor tree and division operation, as shown in **Figure 8**. At first, Murat initially remained at the action stage, as he mechanically listed multiples of prime numbers. However, after the researcher's prompted question about 60, Murat used a factor tree and division to determine the prime factors, indicating a transition to the process stage.

When the researcher asked the group which numbers could crack the password if 90 was the password, the students performed calculations on their papers. Murat found that 2 could crack this password using the factor tree. Finally, when the researcher asked which number would be a strong password to crack, Murat responded,

> 97 ... If we write 1, it is not a prime number, so it cannot crack the password. Also, 97 itself is the password, so it cannot be written. Writing it would reveal the password. 97 would be a strong password.

These explanations are signs that Murat had progressed to the process stage. Rather than merely performing mechanical multiplications, he used the logic of prime factorization as a factor tree and questioned the concept's meaning within a security context. His statement about 97 for a strong password illustrates how he linked the uniqueness of prime factors with the notion of security, transferring the concept into a different problem situation. This progress was partly prompted by the researcher's question and his spontaneous reasoning.

Upon reading the problem, Ali inferred that the password would be a two-digit number that is the product of two prime numbers. Ali instructed all group members to choose and multiply a prime number by another prime number and then write down the two-digit products they found. He began by starting with the smallest prime number, 2, and said,

> Let's write down the products with 2. 2 times 3 does not work. 2 times 5 works. 2 times 7 works. 2 times 11, another 2 times 13 … All prime numbers work. Only the product of 2 and 3 does not work,

and proceeded with the operations in sequence (first part of **Figure 9**). Furthermore, Ali mentioned looking at prime numbers between 2 and 50 to find two-digit results. Other group members performed similar operations. Subsequently, the researcher emphasized to the group for a correct understanding of the problem,

> Does the problem state that the factors of the password consist only of prime numbers?... It is indicated that only factors consisting of prime numbers open the door. For example, if 60 is our password, which numbers solve this password? What are they?

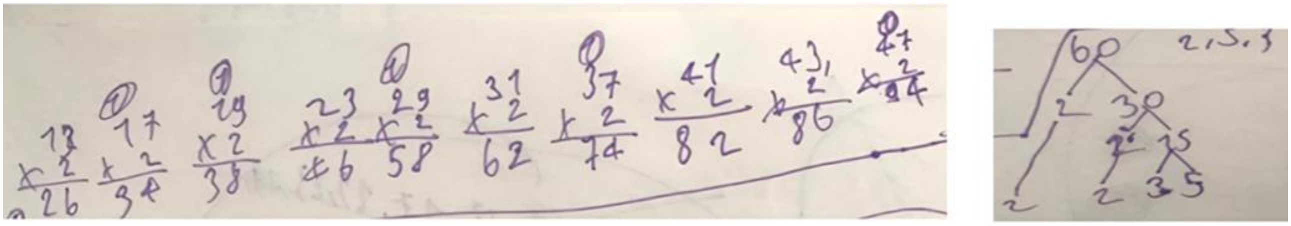

**Figure 9.** Solutions of Ali in the group for the password problem (Source: Authors' own elaboration)

Then Ali responded,

> In that case, we will use the algorithm method. I will perform division ... Okay, okay, I understand … Everyone will find a number. Whether using an algorithm, a factor tree, or finding factors and multiples,

as shown in the second part of **Figure 9**. Ali's initial inductive reasoning through the products of prime numbers represents the action stage. With the researcher's prompted guidance, his shift toward using the prime factorization algorithm and referencing different strategies such as factor tree, divisors and multiples indicates a transition to the process level. Moreover, Ali's organization of group members shows that he structured not only the mathematical procedures but also the problem-solving process itself and demonstrates his movement toward internalizing the concept.

For the given numbers in the problem, Ali provided examples by saying,

> Teacher, I found 23. If 46 is the password, it can be broken by 2 and 23. If the number is 70, we can decode it with 2, 5, and 7. It could be 38. The ones that decode the password are 19 and 2.

For a strong password, Ali suggested numbers like 17, reasoning that numbers like these are strong passwords because they do not have prime factors since they are prime numbers. Later, as a group, they suggested 97 for a strong password.

Like Ali and Murat, Selim interpreted the problem as involving a two-digit natural number composed of the product of two prime numbers and worked with their groupmates to find the solution (**Figure 10**). For example, they mentioned that 14 could be a password because its factors are 2 and 7, but 16 could not because it is the product of 2 and 8, and 17 would not be considered a password since it is already a prime number. When the researcher asked if the password was 60 as a sample, which numbers would solve this password, Selim expressed that he understood that the password's factors did not necessarily have to be only prime numbers and used the factor tree method to find the prime factors of the numbers that came to mind. This episode signs that Ali had moved to the process stage. At first, he was only checking the products of prime numbers as action, but later he realized that primes themselves could directly serve as strong passwords, progressing the concept to the object stage. Selim's comparison of 14 and 16 demonstrates that he recognized non-prime components could not be passwords; his remark that 17 could not be accepted as a password since it was prime indicates the objectification of the concept. After the researcher's guiding question about 60, his use of the factor tree shows his transition to the process stage and his internalization of the prime factorization concept. Ali's progress was mainly spontaneous, while Selim's was partly prompted by the researcher's guidance.

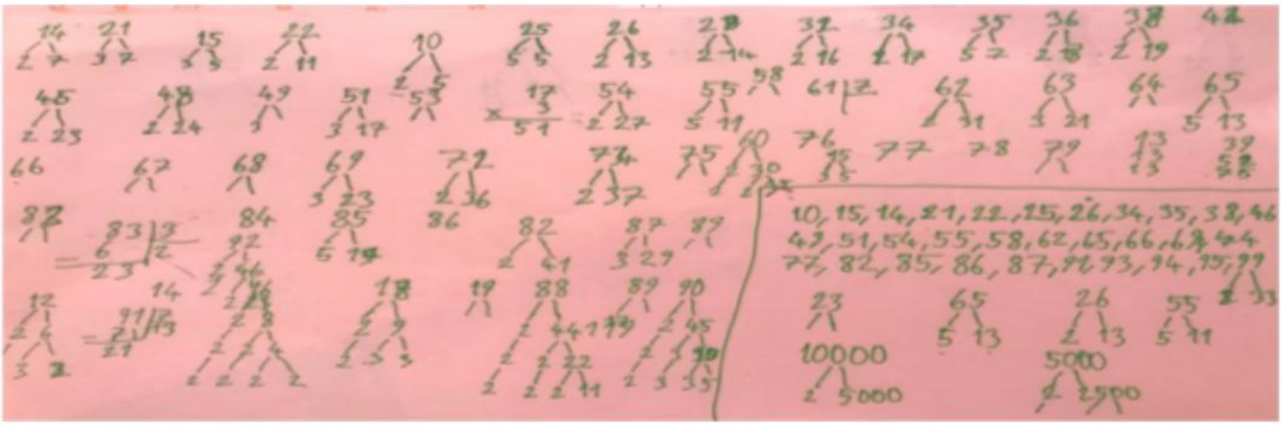

**Figure 10.** Solutions of Selim in the group for the password problem (Source: Authors' own elaboration)

They collectively thought the password should have a few prime factors for a strong password recommendation. Selim explained this as, when the researcher asked how he decrypted a password with many digits, Selim responded,

> If the password is divisible by 2, the last digit must be even. For divisibility by 3, the sum of the digits must be a multiple of 3. Then, for divisibility by 4 ... well, the last two digits must be a multiple of 4. There you go, these are the strategies. For divisibility by 5, the last digit must be 0 or 5. For divisibility by 6, it must be divisible by both 2 and 3. There are no specific strategies for 7 and 8. To be divisible by 9, the sum of the digits must be a multiple of 9. These are the guidelines.

In explaining these divisibility rules, Selim related them to decrypting passwords with multiple digits. Selim was at the process stage. He was no longer limited to finding individual factors but generalized using systematic divisibility rules. Moreover, transferring these strategies into the password-breaking context revealed that he was relating the concept to different problem situations. The progress here was largely spontaneous; the researcher only posed a triggering question.

In the interview, Murat explained,

> I've done this with all prime numbers up to 50. For example, 13 creaks down all these numbers. Eventually, they would all be exhausted. After that, apart from a few passwords, I would do all those numbers. When I entered the numbers, the password would be broken. The remaining numbers would also become new passwords.

Murat transformed the prime number object constructed in the previous problem into a process by de-encapsulating it. Then it was observed that Murat coordinated the concept of prime numbers with the concept of multiplier/factor, assuming that the number that would crack the password was a prime factor. The researcher asked Murat to determine the prime factors of some natural numbers to observe Murat's progression towards encapsulating the concept of prime factor and constructing it as an object. Murat found the prime factors of the numbers using the method of prime factorization tree, as given in the first visual in **Figure 11**. Consequently, it was thought that Murat was progressing towards constructing the object of prime factor. The progress here was partly prompted by the researcher's task to find factors and partly spontaneous, with his reasoning linking prime factors to the password context.

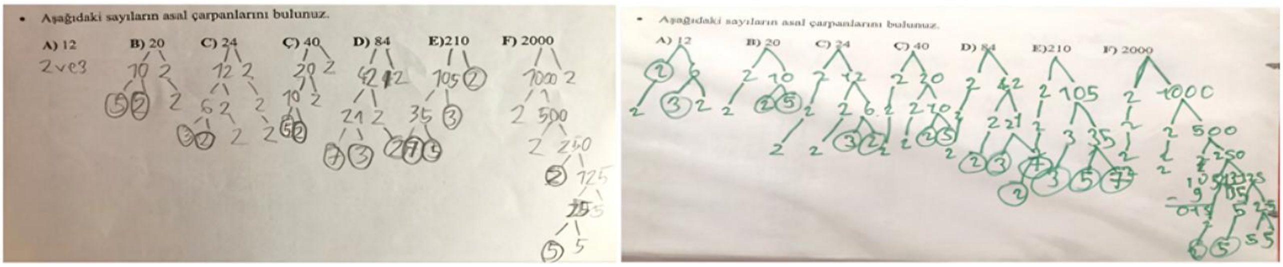


**Figure 11.** Solutions of Murat and Ali in the second interview (Source: Authors' own elaboration)

In the interview, Ali stated that the prime factors of a two-digit natural number, which is a password, can decode the password. Ali mentioned that the prime factors of the numbers opening the door can be determined using factor trees, algorithms, and divisibility rules. The researcher asked Ali to factorize specific numbers into prime factors. Ali used the factor tree method, as shown in the second visual in **Figure 11**, and supported the concept of prime factors with mathematical operations Consequently, it was thought that Ali was progressing towards the object stage, similar to Murat. Initially, Ali used prime factorization only as a procedural operation process. Later, by systematically determining prime factors through the factor tree method and supporting them with mathematical reasoning, he transitioned to constructing the prime factorization concept as an independent object. Here, the progress began as prompted by the researcher's guiding questions but was completed spontaneously through Ali's logical integration of the concept. During the interview, Selim reflected on the group's solution by stating, '

> I gave examples. For instance, I assumed if 98 were the password, the door would be opened by 7 and 2. Similarly, if 15 were the password, the numbers cracking the password would be 3 and 5 … I looked at the prime factors of the password. Every number has at least one prime factor, excluding prime numbers.

Selim emphasized that the numbers decoding the password must be prime and believed they should be the prime factors of the password. Selim identified the prime factors of the password examples he considered by coordinating the processes of prime numbers, factors, factor trees, and algorithms. Selim found the prime factors using the algorithm method for the numbers they were asked about, as shown in **Figure 12**. This indicates that Selim was progressing toward the object stage. He no longer used the prime factorization concept only as a procedural operation process. However, he was constructing it as an independent concept–his examples on 98 and 15 show that he integrated the prime factorization concept into the problem context. The researcher's guiding questions prompted this process, but Selim's conceptual integrations were largely spontaneous.

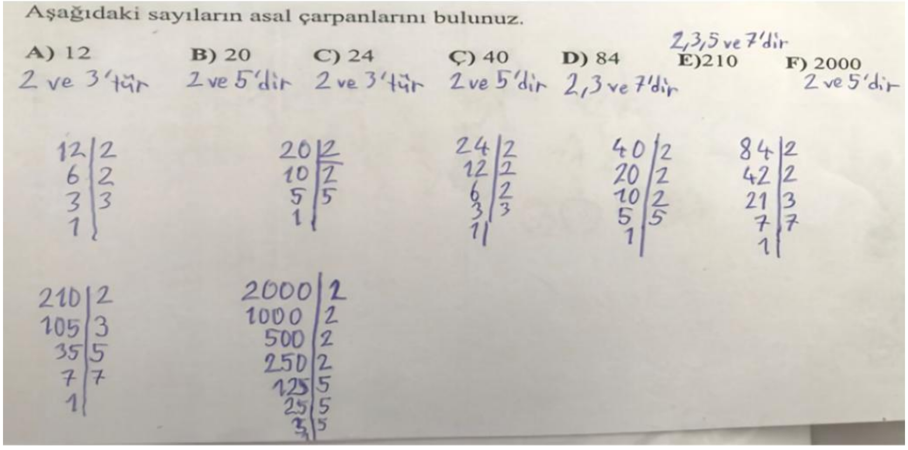


**Figure 12.** Solutions of Selim in the second interview (Source: Authors' own elaboration)

## DISCUSSION

This research aims to examine the construction of the prime number and prime factorization concepts in 6th graders. For this purpose, the instruction was designed according to RME, and students' constructions were examined according to the APOS theoretical framework to support their formation. Each participant's progress reflected a dynamic interaction between contextual problem solving and cognitive development within the APOS framework.

The findings revealed that contextual tasks and representational tools such as algebraic tiles and rectangular board imagery facilitated students' progression from the action to process stages in both conceptual domains. The participants' active manipulation of concrete models helped them visualize divisibility relations and generalize the defining property of prime numbers as being divisible only by 1 and itself. Such transitions align with findings that realistic, manipulatives-based environments foster conceptual shifts from procedural to structural understanding (Gilligan-Lee et al., 2023; Jitendra et al., 2022; Kamii et al., 2001; Rau & Herder, 2021; Uttal et al., 2009).

A key observation was that unique-rectangle imagery–where students recognized that only one possible rectangular configuration could represent a prime area–served as a cognitive bridge that supported students in reorganizing their thinking during the emergence of the concept and helped them overcome potential misconceptions (e.g., considering "1 is prime") without serving as a formal criterion for primality. It should be emphasized that this imagery does not constitute a mathematical justification for excluding 1 from the set of prime numbers; rather, it reflects a context-specific cognitive image that facilitated sense-making within the given problem situation. This finding resonates with Tall's (2004) notion of "conceptual embodiment," suggesting that visual-concrete representations can ground the abstraction of number properties. As Tall (2013) further elaborates, conceptual embodiment refers to the mental construction of meaning from perception and action, where sensory experiences evolve into abstract concepts through reflection and reorganization of imagery. In this research, the concrete manipulation of area and factor pairs not only enabled the internalization of numerical relations but also supported the encapsulation of the prime number concept as an object.

Another important finding demonstrates that students coordinated multiple concepts across readiness levels–including factors, divisors, odd and even numbers, divisibility rules, and factor trees–during the process and object stages. Interestingly, students with lower readiness levels coordinated more concepts simultaneously. This may be interpreted as a tendency reflecting reliance on familiar strategies before encapsulation or a strategy to compensate for less-automated reasoning by engaging multiple familiar representations. This pattern aligns with the idea of emphasizing that strong conceptual coordinations are essential for accurate encapsulation in the APOS cycle (Arnon et al., 2014; Dubinsky et al., 2005).

Real-life contexts in RME were crucial in promoting meaningful learning and enhanced internalization but revealed modeling challenges. The contextual problems–such as determining board dimensions or constructing passwords based on prime factorization–allowed students to link mathematical structures to real-life situations. This process of mathematizing (Freudenthal, 1973) encouraged students to reinvent formal ideas from informal reasoning, as observed in their gradual abstraction of divisibility and factorization patterns. This supports Boaler's (1993) notion that students construct meanings in different contexts beyond a general understanding of the context. Bu (2001) similarly highlights that contextual supports can scaffold learners' reflection and abstraction, demonstrating how carefully designed tasks within RME facilitate conceptual internalization. Therefore, creating contextual problems in line with the RME approach and designing the environment according to the requirements of this approach is crucial for concept formation. However, students' limited prior experience with mathematical modeling led to problem interpretation and representation difficulties. Thus, incorporating realistic situations into mathematics curricula and the design of instructional environments will enable students to see the relationship between mathematics and their daily lives and help them develop problem-solving skills in a meaningful context (Sianturi et al., 2024). Furthermore, this will contribute to the concept formation by serving RME-based environments. All these results will not only contribute to the formation of concepts in a real-life context but will also lay the foundation for advancements in areas such as science, production, communication, environmental change, transportation, and resources as a reflection of mathematical modeling (Geiger et al., 2018).

Overall, the results of this research support studies advocating for the importance of the RME approach in concept formation and studies emphasizing that RME-based activities around contextual problems, collaborative, open to discussion and research, inquisitive, and allowing students to reinvent mathematical knowledge themselves are supportive (e.g., Akgul & Yilmaz, 2023; Bu, 2001; Juandi et al., 2022; Özkaya, 2016; Özdemir & Üzel, 2011; Papadakis et al., 2017).

## CONCLUSION

This research concludes that the RME-based instruction effectively supported students' progression through the APOS stages in constructing prime number and prime factorization concepts. Learning the environment's contextual nature, together with concrete and semi-concrete representations, promoted internalization and encapsulation, leading to the emergence of object-level understanding. The observed consistencies between the pre-genetic decomposition and the students' emergent schemas affirm that contextual, manipulative, and reflective learning conditions foster conceptual growth. Ultimately, integrating RME and APOS frameworks in mathematics education offers a promising approach to fostering deep, connected, and sustainable mathematical understanding.

However, some limitations emerged. Students' limited familiarity with modeling tasks constrained the potential of RME contexts to elicit autonomous reasoning. The curriculum's sequence–introducing set theory after factors and multiples–also restricted students' ability to conceptualize prime numbers as two-element divisor sets. These findings suggest that curriculum alignment and early exposure to modeling activities could enhance concept construction. The fact that the research was conducted with only three students limits the generalizability of the findings. However, the in-depth analysis provided opportunities to closely examine students' conceptual formation processes. Also, similar in-depth case studies have been regarded as valuable in making conceptual formation visible in detail, as emphasized in the literature.

For future research, longitudinal designs exploring how repeated engagement with RME-based modeling influences transitions across APOS stages would be valuable. Moreover, examining digital or dynamic modeling tools alongside concrete manipulative environments may reveal further insights into students' reflective abstraction processes. Also, conducting similar studies at different grade levels may allow comparative analysis of conceptual development processes.

**Author contributions: RY:** conceptualization, data curation, formal analysis, investigation, methodology, project administration, resources, software, supervision, validation, visualization, writing original draft preparation, writing – review & editing: **NK:** conceptualization, data curation, formal analysis, investigation, methodology, project administration, resources, software, validation, visualization, writing – review & editing. Both authors agreed with the results and conclusions.

**Funding:** No funding source is reported for this study.

**Acknowledgments:** The authors would like to thank the participants for voluntarily contributing to this study.

**Ethical statement:** The authors stated tht the study was approved by the Ethics Commission at Ondokuz Mayis University on 09.07.2021 with approval number 2021/599. Written informed consents were obtained from the participants.

**AI statement:** The authors stated that no generative artificial intelligence or AI-based tools were used for the writing, editing, or data analysis of this article. All analyses, interpretations, and wording were entirely performed by the authors.

**Declaration of interest:** No conflict of interest is declared by the authors.

**Data sharing statement:** Data supporting the findings and conclusions are available upon request from the corresponding author.

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## APPENDIX A

**Table A1.** Illustrative alignment of interview and classroom excerpts with APOS stages in the construction of prime numbers

| Data excerpt | APOS stage | Rationale |
|---|---|---|
| *"If it says (the area is) 6, the sides will be 3 and 2. Also, it will 6 and 1. There are two rectangles. But there has to be one rectangle" (Murat).* | Action | The student tests divisors through concrete representations and evaluates cases by trial-and-error without conceptual generalization. |
| *"If it (the area) is not divisible by anything other than itself and 1, it becomes that number" (Ali).* | Process | The student begins to internalize the divisor-checking action and articulates a general criterion, indicating a transition toward process thinking. |
| *"101 has no divisors other than 1 and itself. Is it divisible by 2? Indivisible... by 3? Indivisible … It cannot be divided by 9… it is also not by 6 … it cannot be divided by 5 … It cannot be divided by 4. It is not divisible by 7 and 8 ... We had an advantage of the divisibility rules" (Selim).* | Process | The student applies multiple divisibility rules in a structured and internalized manner. |
| *"Correct identification of prime numbers among 42 numbers in the pool problem" (Murat).* | Object | The student treats prime numbers as a conceptual object and applies the concept flexibly in a new problem context. |

**Table A2.** Illustrative alignment of interview and classroom excerpts with APOS stages in the construction of prime factorization

| Data excerpt | APOS stage | Rationale |
|---|---|---|
| *"We will multiply all of these together … multiply 2 with all of them ... by 3 … by 5 … when you multiply 23 and 29, you get a result with 3 or 4 digits … Only multiply if the result is two digits. Starting from 2, I will proceed with the multiplications sequentially ..." (Murat).* | Action | The student performs sequential operations mechanically, without reference to the conceptual meaning of factorization. |
| *"… Everyone will find a number. Whether using an algorithm, a factor tree, or finding factors and multiples" (Ali).* | Process | The student uses division or factor trees to internalize prime factorization as a systematic procedure and coordinates prime factors with the contextual meaning of decoding a password. |
| *"… I assumed if 98 were the password, the door would be opened by 7 and 2. Similarly, if 15 were the password, the numbers cracking the password would be 3 and 5 … I looked at the prime factors of the password" (Selim).* | Object | The student integrates prime factorization across different problem contexts. |