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Connections between Empirical and Structural Reasoning in Technology-Aided Generalization Activities

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ABSTRACT

Mathematical generalization can take on different forms and be built upon different types of reasoning. Having utilized data from a series of task-based interviews, this study examined connections between empirical and structural reasoning as preservice mathematics teachers solved problems designed to engage them in constructing and generalizing mathematical ideas aided by digital tools. The study revealed closer connections between naïve empiricism and result pattern generalization, between naïve empiricism and recognizing a structure in thought, between reasoning by generic example and process pattern generalization, and between reasoning by generic example and reasoning in terms of general structures. Results from this study imply that the ability to generalize based on perception and numerical pattern does not necessarily lead learners to generalize based on mathematical structure.

Keywords: empirical reasoning, structural reasoning, mathematical generalization

INTRODUCTION

Generalizing is a constructive activity that aims to transport a mathematical relation from a given set to a new set for which the original set is a subset, perhaps adjusting the relation to accommodate the larger set. It has been argued that making, representing, justifying, and reasoning with generalizations are crucial components of mathematical thinking and should be at the heart of mathematics activity in school (Blanton, Levi, Crites, & Dougherty, 2011; Mason, Johnston-Wilder, & Graham, 2005). In the past few decades, researchers have identified different forms of mathematical generalizations. For instance, Dörfler (1991) separates between empirical and theoretical generalization. The basic process of empirical generalization is to detect a common quality or property among two or more objects based on perception and then to record it as being common and general. Theoretical generalization is constructed through abstracting the essential invariants of a system of actions taken on/with mathematical objects rather than the perceptual features of the objects themselves. Yerushalmy (1993) distinguishes between generalization from examples and generalization of ideas. Generalization from examples accounts for cases where students establish a generalization by drawing on particular cases or examples in a given set. Generalization of ideas refers to situations where learners construct a more general statement from more specific ideas. Examples are not crucial in generalization of ideas since what matters are the relevant ideas that can be dropped, ignored, relaxed, or combined in order to gain a greater generality. More recently, Mason, Burton, and Stacey (2010) differentiate between empirical and structural generalization. Empirical generalization is the process of forming a conjecture about what might be true from numerous instances. It occurs when a learner looks at several, sometimes many, cases and identifies the sameness among these cases as a general property. Structural generalization arises when a learner recognizes a relationship from one or very few cases by attending to the underlying structure within these cases and perceives this relationship as a general property. These different forms of generalization imply that individual learners can generalize either at an empirical level (empirical reasoning) or based on mathematical structure (structural reasoning). A considerable body of research on pattern generalization has shown that learners of different ages tend to generalize on the basis of spurious numerical pattern rather than the pattern's structure (El Mouhayar, 2018; El Mouhayar & Jurdak, 2016; Küchemann, 2010; Küchemann & Hoyles, 2009), which implies a dominance of empirical reasoning over structural reasoning in generalizing activities as well as the need for students to move from empirical to structural generalization. Although both empirical and structural reasoning occur in the process of constructing, representing, and justifying generalizations, it still remains unclear whether and if so, how the two types of reasoning interact with each other, and how learners' reasoning gradually evolves from empirical to structural.

Meanwhile, with the emergence of interactive software in the early 1990s, a large body of research in mathematics education has considered ways that dynamic technologies such as the Geometer's Sketchpad (GSP) and GeoGebra, which allow their users to create and act on mathematical entities (e.g., constructing and manipulating geometric figures, measuring elements of a geometric figure, executing calculations and algorithms, generating examples, and graphing expressions), might influence mathematical thinking processes among students of all ages. Indeed, researchers have argued eloquently that dynamic environments provide a productive means for learners to search for numerical patterns through measurement and calculation, observe the variants and invariants, formulate and validate conjectures based upon dynamic cases, and extend the perceived relation to a larger set (e.g., Baccaglini-Frank & Mariotti, 2010; Baccaglini-Frank, 2019; Yao, 2020; Yao & Manouchehri, 2019), all of which are essential for generalizing. The inductive nature of dynamic environments triggers researchers to explore the experimental-theoretical interplay during the construction and justification of mathematical knowledge. Although researchers have recognized the existence of an experimental-theoretical gap in dynamic environments and explored ways to bridge the gap through carefully designed tasks and instruction (Christou, Mousoulides, Pittalis, & Pitta-Pantazi, 2004; Leung, 2014; Sinclair & Robutti, 2012), it remains unclear how empirical and theoretical thinking interact with each other in a carefully designed task environment that promotes the habit of looking for and using mathematical structure.

To address this gap in the literature this study aimed to explore the connections between empirical and structural reasoning in the process of forming and extending mathematical ideas within a dynamic geometry environment through a series of carefully designed tasks. It was guided by the following research question: How does empirical and structural reasoning connect in technology-aided generalization activities?

LITERATURE REVIEW AND THEORETICAL BACKGROUND

Empirical and Structural Reasoning in Generalizing Activity

Empirical reasoning is the process of reasoning on the basis of examples to reach a general conclusion. Reid and Knipping (2010) identify the use of empirical reasoning in various mathematical activities, including pattern observing (noticing similarities of given cases), predicting (making a claim about the next case based on past cases), conjecturing (making a general statement from specific cases when the general statement requires additional verification), generalizing (making a general statement that does not require additional verification), and testing (using new cases to test predictions and conjectures). There are different levels of sophistication of empirical reasoning, including naïve empiricism, crucial experiment, and reasoning by generic example (Balacheff, 1988), which indicates that the habit of looking for and making use of structure might or might not present in empirical reasoning. An individual engages in naïve empiricism when (a) the individual gains confidence in the validity of a claim by checking it with specific examples; (b) the examples that the individual chose to consider are not based on conceptual considerations; and (c) the individual obtains no other information from this process apart from verification that the statement holds in these instances (Weber, 2013). The “number-pattern-spotting” approach to solve pattern tasks indicates the dominance of naïve empiricism in students’ generalization activities. A crucial experiment draws on deliberately chosen examples to reach a general conclusion or to make claims about a universal assertion. A generic example in mathematics is an example that illuminates the general rather than the particular properties of the example. When reasoning by generic example, examples are used as a tool to explain abstract thought or the general structure of what is occurring.

Researchers have used different terms to capture learner’s generalization activity on the basis of empirical reasoning, among which include empirical generalization (Dörfler,1991), generalization from examples (Yerushalmy,1993), result pattern generalization (Harel, 2001), and naïve induction and arithmetic generalization (Radford, 2008). These forms of generalization indicate that the simplicity of number patterns may have a stronger appeal than the insight that might be gained from taking a structural approach.

A mathematical structure is a mathematical system with certain specifically recognized properties and theorems that are the logical consequence of these properties. If mathematics is the science of pattern, structure determines the way the pattern is organized. Therefore, there exists a difference between “seeing” a number pattern and “seeing” a structure behind the pattern. A major goal of mathematics education is to nurture the view that conceives of mathematics as a field of intricately related structures rather than a series of computations to be carried out. Researchers have used “structural sense” (Hoch & Dreyfus, 2004), “structural thinking” (Mason, Stephens, & Watson, 2009; Mulligan & Mitchelmore, 2012), and “structural reasoning” (Harel & Soto, 2017; Hawthorne & Druken, 2019) to capture the habit of looking for and making use of mathematical structure.

Many forms of generalization identified in the literature entail some sort of structural reasoning, among which include theoretical generalization (Dörfler, 1991), generalization of ideas (Yerushalmy, 1993), process pattern generalization (Harel, 2001), and generalization through generalizing the reasoning and generalization through unifying specific cases (Ciosek, 2012). Although identified by different criteria, all these forms of generalization require learners to attend to and make use of certain structural aspects of object or process under consideration. Yao and Manouchehri (2019) differentiate among perception-based, property-based, and theory-based generalizations by attending to the learner’s level of attention to mathematical structure, which highlights the role of structure reasoning in shaping the nature of mathematical generalizations.

Like any other way of thinking, structural thinking/reasoning is developmental in nature and evolves gradually with individuals through various social and cultural interventions. Mulligan and Mitchelmore (2009) analyze young children’s responses to mathematical tasks and identify five levels of structural thinking. Learners at the *prestructura*l level attend to salient features that are irrelevant to the underlying mathematical concepts. At the level of *emergent*, learners recognize some relevant features but are unable to organize them appropriately. Learners are said to be at the *partial structural* level when they can recognize most of the relevant features of the structure but their representations are inaccurate or incomplete. Learners reach to the *structural* level

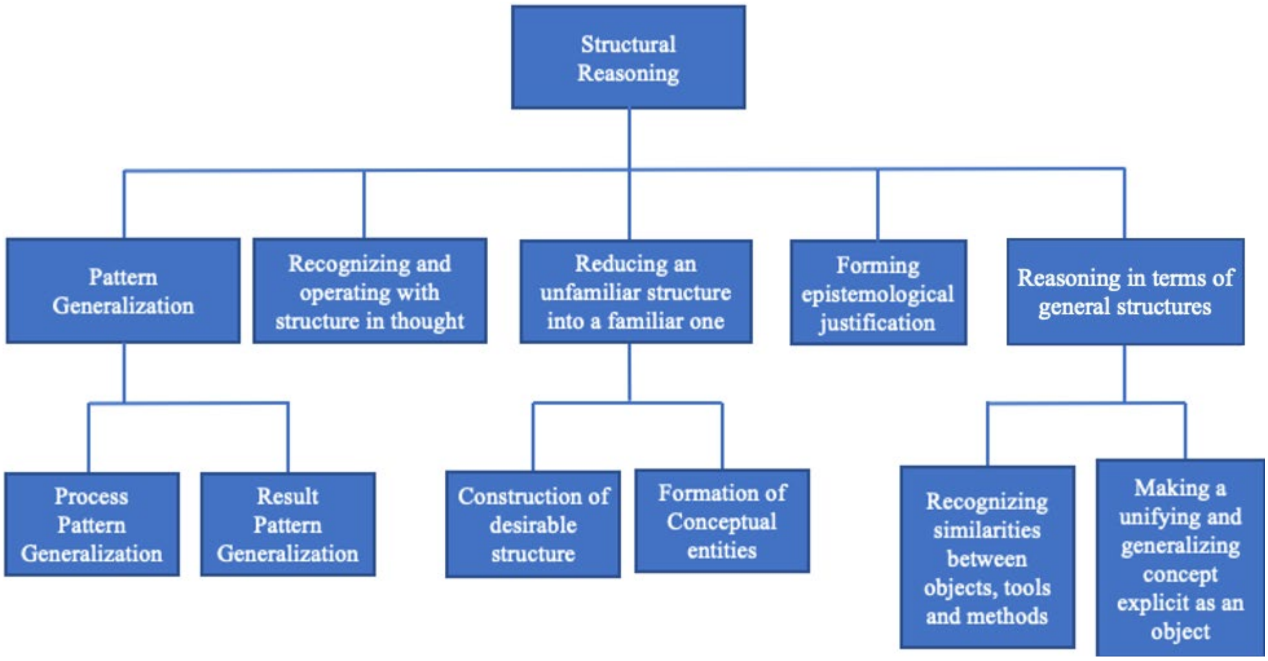

Figure 1. A Typology of structural reasoning (Harel & Soto, 2017)

when they correctly represent a given structure and further move to *advanced* level when they recognize the generality of the structure.

More recently, Harel and Soto (2017) conceptualize structural reasoning as a cluster of abilities to look for and recognize structures, to probe into and act upon structures, to reason in terms of general structures, and to form epistemological justifications. They propose a typology of structural reasoning that instantiates their conceptualization, which includes pattern generalization, reducing an unfamiliar structure into a familiar one, recognizing and operating with structure in thought, forming epistemological justification, and reasoning in terms of general structures (**Figure 1**). Pattern generalization includes process pattern generalization, in which one attends to structure and invariant relationship in the process (e.g., generalizing that the sum of the interior angles of a polygon is $(n-2)\times 180°$ by triangulation), and result pattern generalization, in which one attends solely to regularity in the numerical results (e.g., generalizing that the sum of interior angles of a polygon is $(n-2)\times 180°$ by examining the numerical pattern of the sum of interior angles in the first a few polygons). Reducing an unfamiliar structure into a familiar one includes construction of desirable structure (e.g., drawing an auxiliary line in a geometric figure to decompose it into familiar shapes, transforming an algebraic expression into a previously met structure) and formation of conceptual entities (e.g., working with an algebraic expression for the purpose of investigating or claiming a certain property of it). This might involve decomposing (or chunking) mathematical objects into a variety of familiar sub-structures based on the context and goal at hand. Recognizing and operating with structure in thought involves first taking a step back and looking for properties that are embedded in a mathematical object before selecting a procedure to act on it. It might also involve recognizing equivalent or similar mathematical properties in different forms and multiple representations. When forming epistemological justification, a learner develops reasoning that explains how a piece of knowledge resolves the problematic situation under investigation. Reasoning in terms of general structures includes recognizing similarities between objects, tools, and methods, and making a unifying and generalizing concept explicit as an object. This typology provides a useful framework to examine the embodiment of structural reasoning in mathematical activities. Based on our experience of working with K-12 students and preservice teachers, we believe that pattern generalization, recognizing and operating structure in thought, and reasoning in terms of general structure are categories of structural reasoning closely connected to the process of generalizing.

Mathematical Reasoning with Dynamic Technology

Incorporating dynamic software such as GSP and GeoGebra in mathematics learning has opened a research venue for exploring affordances and limitations of dynamic technologies for fostering mathematical reasoning. When looking at the affordances of dynamic technologies, a common theme that arises is the tools' ability to allow students to access a wide variety of examples. With access to a multitude of examples, students have more cases to refer to when seeking patterns (Kuzle, 2017; Pedemonte & Balacheff, 2016) or find unexpected properties with which they can create new conjectures (Baccaglini-Frank; 2019; Baccaglini-Frank & Mariotti, 2010; Hollebrands et al, 2010). The dragging feature in dynamic geometry environments (DGE) allows its users to quickly generate infinitely many dynamically linked examples. The use of dragging also has the added benefit of maintaining the invariant properties of a set of cases that are the result of the construction process (Richard et al., 2019). Moreover, the measuring facility in DGEs makes it convenient for students to gather data and look for numerical relationships. From this process, students can recognize visual or numerical patterns and abstract the invariant properties of the construction, and then use that information to form generalizations and justifications (Richard et al., 2019; Sinclair & Robutti, 2012).

While these benefits of using dynamic technologies exist, there are concerns about how using these technologies may affect students' mathematical thinking. There are some claims that overemphasis on empirical reasoning that comes from using dynamic technologies may lead to impediments on student's ability to think about abstract justifications (Kuzle, 2017; Olive & Makar, 2010). If this is true, then students would be stuck reasoning empirically and be unable to reason about mathematical structure or other abstract properties. This concern can be seen in the trend found in Hollebrands et al.'s study (2010) where students solely relied on computers to verify claims they were uncertain of.

While these concerns exist and have been raised, there have been responses defending the use of dynamic technologies in mathematics learning. Researchers have argued that dynamic technologies have to be a tool used to develop mathematical thinking rather than simply a tool that executes mathematical operations (Richard et al., 2019). By using dynamic technologies as ways to build knowledge inductively, students can be supported to focus on the more general aspects of the cases they examine and move to thinking more deductively (Baccaglini-Frank, 2019; Komatsu & Jones, 2019; Lachmy & Koichu, 2014).

The discussion around the affordances and limitations of dynamic technologies suggests that empirical and theoretical arguments interplay in students' reasoning. Arzarello et al. (2002) describe ascending and descending as two main cognitive processes, in which students are engaged while investigating mathematical problems in dynamic environments. Ascending processes occur when the student moves from empirical grounds to theoretical considerations in order to freely explore a situation, and look for regularities, invariants, etc. Descending processes occur when the student moves from theoretical considerations to empirical grounds to validate or refute conjectures, and to check properties of cases described by the conjecture. Together, ascending and descending processes capture the back-and-forth movement between empirical and structural reasoning. The study reported in this paper aimed to investigate the connections between empirical and structural reasoning in a dynamic geometry environment through a series of carefully designed tasks.

METHODOLOGY

Method

The data for this study was collected from a series of task-based interviews that were a part of a larger research project aimed to investigate preservice secondary mathematics teachers as learners and teachers of mathematical generalizations in a technology-intensive learning environment. The task-based interview was chosen to obtain knowledge about individual preservice teacher's processes to generalize mathematical ideas and the mathematical knowledge resulting from those processes. Each task in this study often consisted of a sequence of closely related problems that aimed to promote learners to generalize a mathematical idea to a broader domain. Although each task contains different entry points and can be solved in different approaches, all the tasks were designed with the intention to engage learners in searching for and making use of mathematical structures. To this end, each of the tasks often included a sequence of sub-tasks that demanded the participants to recognize a certain structural aspect of a mathematical object under investigation, extend it to solve the subsequent sub-tasks, and reason in terms of the general structure. Moreover, when selecting tasks, we considered how technology could potentially be used by the participants to explore and analyze mathematical relationships, to develop alternative approaches for problem-solving, or to generate new problems that could not otherwise be posed. **Figure 2** presents the tasks used in the interviews. Each task contained one or more GSP or GeoGebra files that allowed the participants to explore the problems. These tasks were chosen to elicit participants' generalizing activities in both geometry (Task 1 and Task 2) and algebra (Task 3 and Task 4) within both familiar (Task 1 and Task 3) and relatively unfamiliar (Task 2 and Task 4) knowledge contexts. These tasks are appropriate for this study in that within each task the participants could generalize either at the empirical level or on the basis of a perceived mathematical structure.

Participants

The participants were 8 junior undergraduate preservice secondary mathematics teachers, of whom 4 are male and 4 are female. The participants were selected based on voluntary participation. The mathematics courses they had taken thus far include the following: Calculus series, discrete mathematics, elementary combinatorics, concepts of real analysis, basic abstract algebra, and linear algebra. The GPA of the 8 participants rang from 3.16 to 3.98 with an average of 3.69 and only the GPA of two participants was below the average. This indicates that most participants in this study performed well academically. During the semester of their participation, all participants were concurrently enrolled in a course that aimed to engage them to learn and teach mathematics with various types of mathematical action technologies (e.g., GeoGebra, Geometer's Sketchpad, TI-Nspire CX CAS, and Fathom Dynamic Data Software). The course took a problem-solving approach and engaged the preservice teachers in the processes of representing, conjecturing, generalizing, and justifying by solving and extending mathematically rich problems in technology-rich learning environments. It also invited preservice teachers to contemplate different ways they could use technology to engage their future students in these mathematical processes. Each participant participated in two task-based interviews, each of which was approximately 1.5~2 hours. During each interview, a participant attempted to solve two mathematical tasks with the technologies they had learned in the course. Participants' interactions with technology were screen-recorded, which allowed the researchers to examine the participant's awareness of mathematical structure and generalizing actions after each session. During each session, the interviewer frequently asked the participant to articulate his/her thinking process and to make a general statement based on his/her exploration. To advance the problem-solving process, the interviewer sometimes also engaged the participant in reflecting on his/her current mathematical activities. Those interactions between the interviewer and the participant were recorded with a camera, focusing on the interviewer's interventions and the participant's reaction. The technology files produced during each interview were collected.

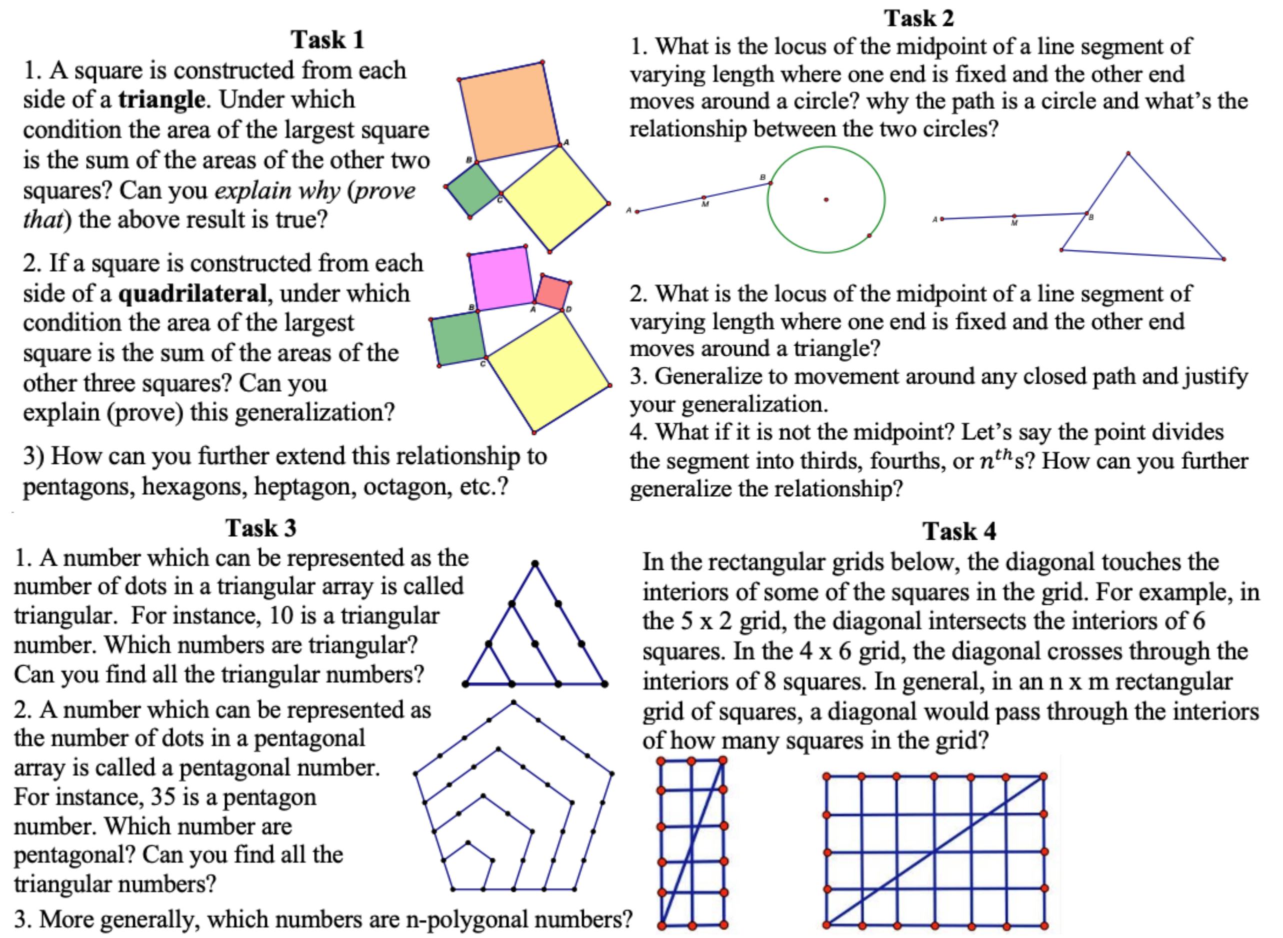

Figure 2. Sample mathematical tasks used in the interviews

Data Analysis

Data analysis in this study consisted of three phases. Firstly, videos of each interview were segmented based on the transition of mathematical tasks and instances of generalizing attempts in each video segment were identified. An activity was identified as a generalizing attempt if a participant engaged in one of the four activities: (1) relating or making a connection between two (or more) situations, problems, ideas, or object; (2) searching for a pattern or a procedure through performing repeated actions; (3) expanding a pattern, a relationship, or a procedure into a more general structure; and (4) deriving a new general statement from an existing generalization. A participant could demonstrate multiple generalizing attempts while completing one mathematical task. Each instance of a generalizing attempt was then transcribed verbatim, including what was said and what was done with technology by the participant. Secondly, each instance of a generalizing attempt was analyzed to determine the types of reasoning that dominated the participant's mathematical actions. Balacheff's (1988) category of empirical reasoning and Harel and Soto's (2017) typology of structural reasoning were used to code the types of reasoning involved in each generalizing attempt. Because of our interest in the types of reasoning that support participants' construction of mathematical generalizations, our analysis of the types of structural reasoning mainly focuses on categories of structural reasoning closely connected to the process of generalizing, including structural reasoning pattern generalization, recognizing and operating structure in thought, and reasoning in terms of general structure. To ensure reliability of coding, the two authors first watched a few sample video excerpts together and coded the types of reasoning involved in each sample video excerpt. This was to ensure that the authors developed a shared understanding of each type of empirical and structural reasoning. The two authors then independently coded all the rest of generalizing attempts in the data. To calculate the percent agreement we took the total number of times in which the two authors agreed and divided that by the total number of classifications made. By this method, the percentage of agreement between the authors was 90.7%, which indicates a high percentage of agreement. The small number of discrepancies in coding were resolved through discussion. The third phase of data analysis involved identifying the instances where a participant shifted from empirical to structural reasoning in his/her process of generalizing a mathematical idea. The number of transitions from one type of empirical reasoning to a particular type of structural reasoning was counted to look for general pattern. Each instance was carefully analyzed to understand mechanism of transition.

RESULTS

In this section we report on the participants' engagement in different types of empirical reasoning, focusing on the connection of a particular type of empirical reasoning to the different types of structural reasoning. Our goal is to show the connections between empirical and structural reasoning that arose from the data.

Connections between Naïve Empiricism and Structural Reasoning

A participant was said to engage in naïve empiricism when the participant formulated and gained confidence in a claim solely based on data from specific examples and the choice of examples was not based on conceptual considerations. **Table 1** shows that naïve empiricism occurred 67 times in the four tasks and in 19 times it did not connect to any type of structural reasoning. This indicates that there was a good chance that a participant engaged in naïve empiricism that did not result in a sense of structure to the problem. The table also shows that the movements from naïve empiricism to result pattern generalization and from naïve empiricism to recognizing and operating with structure in thought were the most prominent types of connection between naïve empiricism and structural reasoning. This indicates that naïve empiricism could often lead to the discovery of a number pattern or recognition of a mathematical relation based on numerical data and visual clues. Meanwhile, the table shows that it was unlikely for participants to move from naïve empiricism to other types of structural reasoning, such as reasoning in terms of general structures and process pattern generalization. The three examples below illustrate the most prominent types of connection between naïve empiricism and structural reasoning.

Table 1. Connections between naïve empiricism and structural reasoning

	Task 1	Task 2	Task 3	Task 4	Total
Naïve empiricism disconnected to structural reasoning	4	2	6	7	19
Naïve empiricism → Result pattern generalization	9	3	8	10	30
Naïve empiricism → Process pattern generalization	0	0	0	0	0
Naïve empiricism → Recognizing and operating with structure in thought	7	8	1	2	18
Naïve empiricism → Reasoning in terms of general structures	0	0	0	0	0
Total	20	13	15	19	67

Example 1: Naïve empiricism disconnected to structural reasoning

When asked to find the number of unit squares whose interiors a diagonal of the rectangular grid would pass through (Task #4), Skylar attempted to spot number patterns based on a very few cases and used them for prediction. The following transcript excerpts demonstrate how Skylar imposed number patterns and used them to predict the number of interior crossings to new cases.

"3 by 5, I am going to say it's 8. It looks there is a nice little pattern here (referring to the pattern that a 3×2 rectangular grid has 4 interior crossings and a 3×4 has 6 interior crossings in **Figure 3**). (Count the number of interior crossings in the 3×5 rectangular grid) 1, 2, 3, 4, 5, 6, 7, it is actually 7."

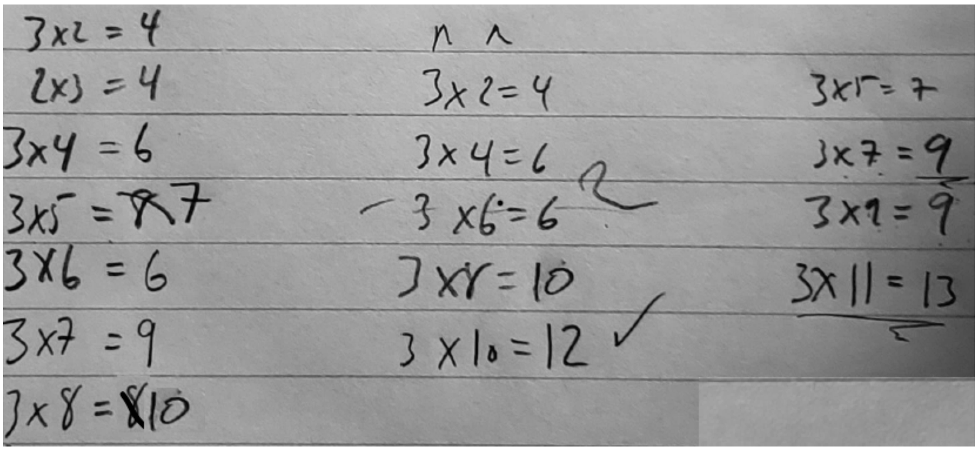

Figure 3. Skylar's number patterns

"3 by 8, I think this is going to be 8 because we have 3 by 2 equals 4, 3 by 4 equals 6, we just forget about 3 by 6, 3 by 8 should equal 8. And also we have 3 by 5 equals 7, 3 by 7 equals 9, 3 by 9 equals 9, I dare say 3 by 11 is going to be 11 plus 2, which is 13 (**Figure 3**). Now it is going to just follow this pattern. I wager 3 by 10 is going to be 12."

In this example, Skylar first imposed a number pattern to the problem situation solely based on the number of interior crossings in the cases of 3×2 and 3×4 rectangles. His prediction that the number of interior crossings in the case of 3×5 was 8 indicates that the prediction was made based on that perceived number pattern. After counting the number of interior crossings in the cases of 3×6 and 3×7 rectangles, Skylar predicted that the number of interior crossings in the cases of 3×8 rectangle was 8 because "we have 3 by 2 equals 4, 3 by 4 equals 6". While making the prediction, Skylar treated the number of interior crossings in a 3×6 rectangle as an anomaly of the number pattern. In a similar fashion, Skylar predicted that the number of interior crossings in a 3×11 was 11 plus 2 because in the case of a 3×5 rectangle it was 5 plus 2, and in the case of a 3×7 it was 7 plus 2. It seemed that the number of interior crossings in the case of a 3×9 rectangle was treated as an anomaly of the number pattern. Skylar then noticed that the same pattern existed for the case where m was an even number, in which the case of a 3×6 rectangle was again treated as an anomaly. The way in which Skylar detected number patterns provides some evidence that naïve empiricism dominated his thinking. His subsequent exploration showed that none of the perceived number patterns led Skylar to develop a generalizable number pattern or to notice a structure element inherent in the problem.

Example 2: Connection between naïve empiricism and result pattern generalization

This example came from Cameron's exploration of the trace of the midpoint of a segment drawing by connecting a point on a circle and a point outside the circle (Task 2). Cameron started by tracing the midpoint while dragging the point on the circle. He observed that the trace formed a circle. Cameron then drew a circle and fit it on top of the trace. He then measured the radii of the two circles on the screen and calculated their ratio. The calculation yielded a number close to 2. Cameron then drew a circle of different size and a point outside the circle. By repeating the above process, Cameron obtained another ratio of the radii of the circles, which was also close to 2. Cameron then checked the third case and concluded that the ratio of the radii of any given circle and the circle formed by the trace would always be 2. **Figure 4** is a snapshot of the three cases Cameron considered.

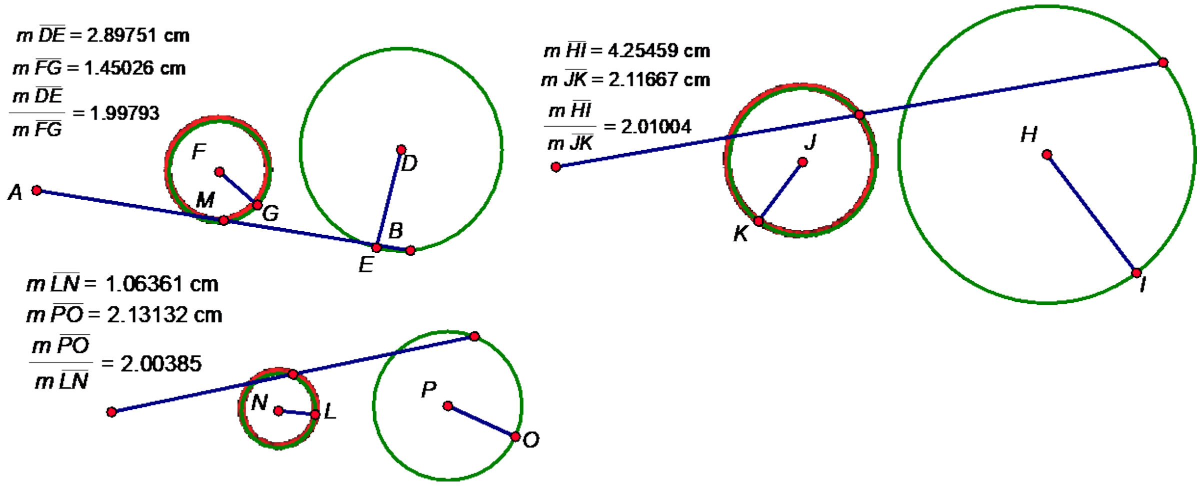

Figure 4. The three cases Cameron considered when exploring the trace of the midpoint

Although Cameron gained confidence in a claim by checking it with specific examples his choice of examples was not based on conceptual considerations. Moreover, Cameron did not gain any mathematical insight from this process apart from verifying that the ratio of the radii of the given circle and the circle formed by the trace was always 2. All the above suggests that Cameron's reasoning was dominated by naïve empiricism. The results from the three cases led Cameron to generalize that the ratio of the radii of any given circle and the circle formed by the trace of the midpoint would always be 2. The generalization was a result pattern generalization in that it was constructed based on the results of the calculations in the three different cases. Therefore, the above example illustrates the movement from naïve empiricism to result pattern generalization.

Example 3: Connection between naïve empiricism and recognizing structure

When asked to determine the condition under which the area of the largest square would be the sum of the area of the other two squares drawing from each side of a triangle (Task 1), Jordan started by measuring the areas of the three squares and then calculating the sum of the areas of the two smaller squares in the diagram. By dragging a vertex of the triangle Jordan created a few instances that satisfied the problem condition (**Figure 5a**), through which he conjectured that the right angle in the triangle might affect the relationship among the areas of the square. To verify this conjecture, Jordan constructed a right triangle and then drew a square from each side of the triangle by using the square tool in the GSP. He measured the areas of the three squares and calculated the sum of the areas of the two smaller squares. The measurement data confirmed that the sum of the areas of the two smaller squares was the area of the third square (**Figure 5b**). Jordan then wondered whether isosceles triangles would also work. He measured the sides of the triangle and adjusted the triangle to make it look like an isosceles triangle (**Figure 5c**). Based on measurement data in different cases of isosceles triangle, Jordan concluded that isosceles triangle did not work. Jordan then tested scalene triangles and concluded that they did not work either. As a result, Jordan concluded that the sum of the areas of two small squares would be the area of the third square drawing from each side of the triangle when there is a right angle in the triangle.

In this example, Jordan relied on the measuring and dragging features of GSP to create instances that satisfied the problem condition. Jordan's heavy reliance on dragging and measuring features as well as the feedback from the GSP screen indicates that his reasoning was dominated by naïve empiricism. Meanwhile, the visual-spatial clues in each instance created by measuring and dragging allowed Jordan to observe a common property that the triangle was a right triangle when the area of the largest square was the sum of the areas of the two smaller squares. This property was then verified through construction and measurement data. Once verified, this property became a structural element of the desired configuration of the diagram. The way in which this property was observed indicates the occurrence of transition from naïve empiricism to recognizing structure.

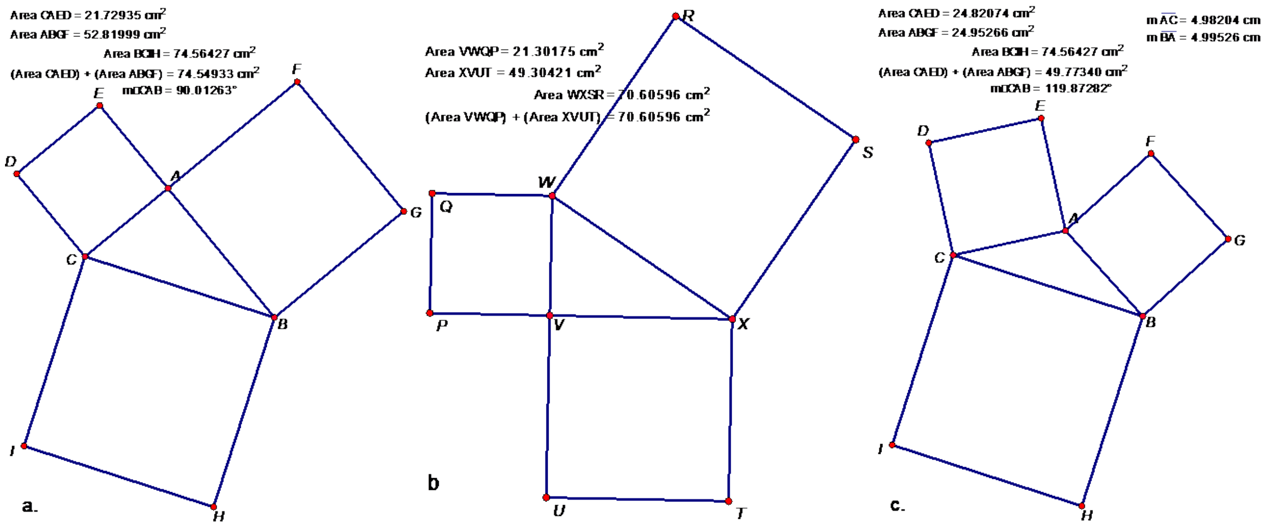

Figure 5. Jordan's exploration in the triangle

As shown in the above examples, various tools in the dynamic geometry software allowed the participants in this study to easily create a multitude of examples, obtain measurement data, and compare and contrast visual images on the screen, all of which might support the participants' observation of numerical patterns and geometric relation. The perceived numerical patterns and geometric relations might lead the participants to attend a structural element of a problem. Meanwhile, as seen in the data, the participants sometimes became so obsessed with numerical data and visual clues that they imposed patterns and relations that were not inherent to the problem.

Connections between Crucial Experiment and Structural Reasoning

A participant was said to engage in crucial experiment when the participant drew on deliberately chosen examples to reach a general conclusion. It is different from naïve reasoning in that the choice of examples in crucial experiment is intentional and based on conceptual considerations. **Table 2** shows that crucial experiment occurred only 18 times in the four tasks and there was no instance of crucial experiment that did not lead to some sort of structural reasoning. Compared with naïve empiricism and reasoning by generic example, crucial experiment was the least frequently observed in this study. **Table 2** also shows that the connection between crucial experiment and result pattern generalization was most prominent. This indicates that crucial empiricism often led to the discovery of a number pattern. Meanwhile, the table shows that it was less likely for the participants to transit from crucial experiment to other types of structural reasoning, such as reasoning in terms of general structures and process pattern generalization.

Table 2. Connections between crucial experiment and structural reasoning

	Task 1	Task 2	Task 3	Task 4	Total
Crucial experiment → Result pattern generalization	2	1	1	5	9
Crucial experiment → Process pattern generalization	0	0	1	0	1
Crucial experiment → Recognizing and operating with structure in thought	1	0	1	2	4
Crucial experiment → Reasoning in terms of general structures	0	0	0	0	0
Total	3	1	3	7	14

Since crucial experiment frequently led to result pattern generalization, we share one example from the data to demonstrate this connection. The connections between crucial experiment and other types of structural reasoning are not shared because they were less frequently observed.

Example 4: Connection between crucial experiment and result pattern generalization

This instance occurred during Cameron's exploration of the number of unit squares whose interiors a diagonal of the rectangular grid would pass through (Task #4). Cameron first considered the case where m was 2 and changed n from 1 to 10. When considering different values of n, Cameron organized the data in a table. He observed a pattern and then generalized that when n was even the number of interior crossings was n and when n was odd the number of interior crossings was $n+1$. Cameron then considered the case where m was 3 and made a similar table to find patterns. By manipulating the numbers in the table, Cameron started to observe patterns. He observed that if n was 2 mod 3 the number of interior crossings was $n+2$, if n was 1 mod 3 the number of interior crossings was $n+2$, and if n was 0 mod 3 the number of interior crossings was n. Cameron then considered the case where n was 4 and found similar patterns. Cameron then looked across the tables and observed that the number of interior crossings could be grouped into quadruple when $m=4$, triple when $m=3$, and double when $m=2$. Based on this observation, Cameron conjectured that a similar pattern (i.e., quintuple) would exist for $m=5$ and concluded that he could use this pattern and modular arithmetic to figure out the number of interior crossings. **Figure 6** is a reproduction of Cameron's data tables.

$n \times 2$	I. C.	Observations
1	2	When n is even,
2	2	I. C. = n;
3	4	When n is odd,
4	4	I. C. $= n + 1$.
5	6	
6	6	
7	8	
8	8	
9	10	
10	10	

$n \times 3$	I. C.	Observations
1	3	If $n \equiv 0 \bmod 3$,
2	4	I. C. = n;
3	3	If $n \equiv 1 \bmod 3$,
4	6	I. C. $= n + 2$;
5	7	If $n \equiv 2 \bmod 3$,
6	6	I. C. $= n + 2$.
7	9	
8	10	
9	9	
10	12	
11	13	
12	12	

$n \times 4$	I. C.	Observations
1	4	If $n \equiv 0 \bmod 4$,
2	4	I. C. = n;
3	6	If $n \equiv 1 \bmod 4$,
4	4	I. C. $= n + 3$;
5	8	If $n \equiv 2 \bmod 4$,
6	8	I. C. $= n + 2$;
7	10	If $n \equiv 3 \bmod 4$,
8	8	I. C. $= n + 3$.
9	12	
10	12	
11	14	
12	12	

Figure 6. Reproduction of Cameron's data tables in Task 4

The example shows that Cameron kept fixing one side length while altering the other side length of the rectangular grid. Cameron was very systematical during the exploration in that he deliberately broke the problem into different case scenarios and organized the data in each case into a table to search for patterns. Since Cameron used this purposeful strategy rather than randomly examining cases it is clear that he engaged in crucial experiment.

By purposefully examining cases in groups sharing a specific property Cameron was able to observe and extend patterns in each group of cases. Moreover, by looking at the tables across each group Cameron observed a pattern that he could use to figure out the number of interior crossings for m=5, which indicates that Cameron started to generalize across case scenarios. Meanwhile, the generalizations that Cameron made were based on numerical patterns. These numerical patterns were not connected to the structural elements inherent in the triangular grid. In other words, these generalizations are result pattern generalizations.

Although the participants who engaged in crucial experiment also relied on technology to generate examples, there was a difference in technology usage between naïve empiricism and crucial experiment. When engaging in crucial experiment the participants were more intentional in terms of what kinds of examples to consider and how to organize data obtained from these examples.

Connections between Reasoning by Generic Example and Structural Reasoning

A generic example in mathematics is an example that illuminates a general property rather than a particular relation within the example. The participants in this study were said to engage in reasoning by generic example when the participant used examples as a tool to communicate abstract thought or the general structure of what is occurring. **Table 3** shows that reasoning by generic example occurred 26 times in the four tasks and there was no instance of reasoning by generic example that did not lead to some sort of structural reasoning. Reasoning by generic example was less frequently observed than naïve empiricism but more frequently observed than crucial experiment. **Table 3** also shows that the connections between reasoning by generic example and process pattern generalization and between reasoning by generic example and reasoning in terms of general structure were the most prominent types of connection to structural reasoning. This indicates that reasoning by generic example often led to the generalization of a pattern based on invariant relationships in the process rather than regularity in the numerical results. It could also likely lead to reasoning in terms of general structure, in which the participants generalized a general property, a method, or a chain of reasoning to a set of new problem situations.

Table 3. Connection between reasoning by generic example and structural reasoning

	Task 1	Task 2	Task 3	Task 4	Total
Reasoning by generic example → Result pattern generalization	0	0	0	0	0
Reasoning by generic example → Process pattern generalization	3	5	2	1	11
Reasoning by generic example → Recognizing and operating with structure in thought	2	2	1	1	6
Reasoning by generic example → Reasoning in terms of general structures	4	3	0	2	9
Total	9	10	3	4	26

As shown in **Table 3**, connections between reasoning by generic example and process pattern generalization and between reasoning by generic example and reasoning in terms of general structures were the most prominent types of connection to structural reasoning in the data. The two examples below illustrate the two types of connections.

Example 5: Connection between reasoning by generic example and process pattern generalization

This instance occurred when Jordan was exploring the number of unit squares whose interiors a diagonal of the rectangular grid would pass through (Task #4). When considering the rectangular grid with distinctive side lengths, by adjusting values m and n to create a few cases, Jordan noticed that the diagonal has to pass through at least m squares, wherein this case m is equal to the larger side length of the rectangle. In the case of 3×6 rectangle in which the diagonal only passes through the interiors of 6 unit squares, Jordan reasoned that it was because there were "perfect intersections", by which Jordan meant that the vertices of

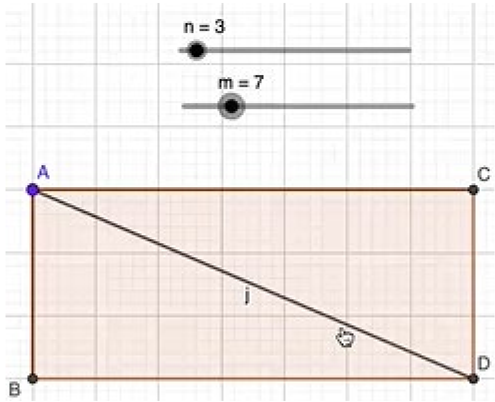

Figure 7. Jordan's generic example

the unit squares the diagonal passed through. Jordan then moved to consider the situation where no "perfect intersections" exists. Jordan altered the rectangle to be a 7×3 rectangle (**Figure 7**) and made the following discovery:

"…here it (the diagonal) starts with 7 because it has to go across, but then it's making steps down. So, it (the diagonal) has to go all the way across and it has to go down two intersections. So, would it be like m plus n minus 1? … 1, 2, 3, 4, 5, 6, 7, 8, 9. So that's nine yeah. So I think in the case like I'm saying where um where it has no perfect intersections I think it will cut through m plus n minus 1 where m is the greater side."

"So, I'm just thinking in this case it obviously has to go through at least m to get to the other side of the rectangle. But it also has to jump down two. Yeah, it has to jump down two, so n-1 in general"

By looking at the verbal disclosure we can see that while Jordan was using an example (i.e., the case of a 7× 3 rectangle) to explain his reasoning, the reasoning was not tied to the specifics of the example. Jordan used the example to describe his thinking, which can be seen through statements mentioning that the diagonal has to move each possible column and row in order to reach the other side of the rectangle. By using a specific example Jordan was able to describe a structural element inherent in the problem. Since Jordan was able to reason about and described his general mathematical thinking through the example it is clear he used this example as a generic example.

This segment also demonstrates how Jordan was able to use the property in the generic example to develop a process pattern generalization. Based on the observation that the diagonal has to pass through each column and row if there are no vertices of the unit squares the diagonal passes through (i.e., "no perfect intersections" in Jordan's words), Jordan concluded that the number of unit squares whose interiors the diagonal passes through would be $m + (n - 1)$, where m is the number of steps to go across and $(n - 1)$ is the number of steps to go down. It is a process pattern generalization because it was developed as a result of attending to an invariant relationship between the number of "perfect intersections" and the number of steps to go down in the process. By using this relationship, Jordan then further generalized that for any rectangular grid the number of unit squares whose interiors the diagonal passes through would be $m + ((n - 1) - \text{ the number of "perfect intersections"})$.

Example 6: Connection between reasoning by generic example and reasoning in terms of general structures

This instance occurred during Jesse's exploration of Task 1. After Jesse connected Pythagorean theorem to the problem and realized that the area of each square is the square of a side length of the triangle, the interviewer prompted Jesse to explore the conditions under which the area of the largest square is the sum of the areas of the other remaining squares drawing from each side of a quadrilateral. After extensive exploration, Jesse was able to use the Pythagorean theorem to create the quadrilaterals that satisfy the relationship. When asked to further extend a similar relationship to other polygons, Jesse constructed an octagon in which the area of the largest square is the sum of the areas of the remaining squares drawing from each side of the octagon. The following excerpt shows Jesse's reasoning activities with the octagon.

Interviewer: Now let's think a little bit of what we have done here (referring to the octagon in **Figure 8a**). What if it is a nonagon, decagon, or an n-sided polygon, how can you create the polygon such that the area of the largest area is equal to the sum of the areas of the other squares drawing from each side of the polygon?

Jesse: From one of the vertices of the octagon, the vertex on the largest square, I need the side of each square and the line connecting A to each vertex of the octagon or the n-gon to form a 90-degree angle. So, you need to make $n - 2$ right angles because the only ones that aren't (right angles) are the two vertices from the largest square.

Interviewer: Very nice. What if I want the sum of the areas of two squares equal to the sum of the areas of the remaining squares drawing from each side of a polygon, how would you adjust your diagram?

Jesse: I think I need a right angle here (pointing to angle ABC). From here then I think it is kinda like the same, just draw five right angles, so $n - 3$ (Jesse deletes AB and BC in **Figure 8a** and creates configuration in **Figure 8b** to demonstrate the idea).

The above excerpt demonstrates that Jesse used the octagon case to communicate a more general relationship. Although Jesses frequently referred to elements of the octagon (e.g., vertices of the octagon, specific number of right angles in the octagon, and angle ABC) in his thinking, the relationship that Jesse perceived holds beyond octagon. Indeed, Jesse's thinking often switched from the specific case to the more general situation. For instance, Jesse stated that "from here then I think it is kinda like

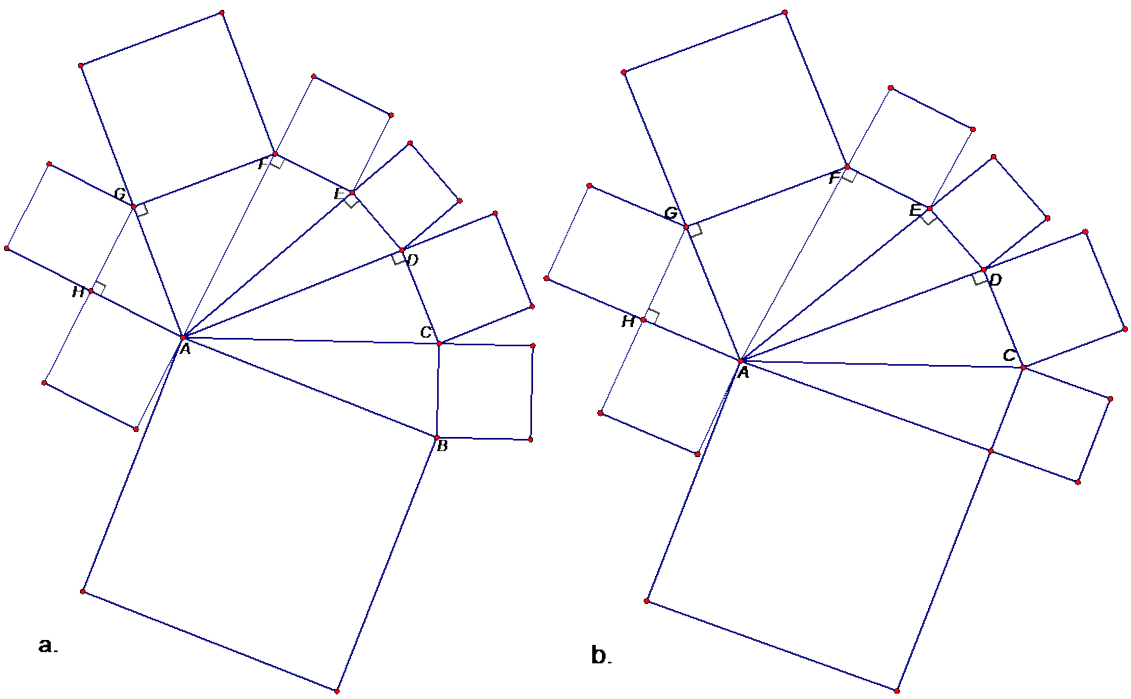

Figure 8. Jesse's generalization of the Pythagorean theorem

the same, just draw five right angles, so n-3". Since Jesse was describing more general mathematical relations through the octagon case it is clear he used the octagon case as a generic example.

The above excerpt provides evidence that Jesse extended the Pythagorean theorem to any polygon and made the generalization that the area of the largest square is equal to the sum of the areas of the $\boldsymbol{n-1}$ squares drawing from each side of an n-sided polygon when the polygon is created by sequentially drawing n-2 right angles from a vertex of the polygon to the sides of the polygon. Symbolically, $\mathbf{(\overline{A_1A_2})^2 + (\overline{A_2A_3})^2 + \cdots + (\overline{A_{n-2}A_{n-1}})^2 = (\overline{A_1A_n})^2}$ **if** $\mathbf{\angle A_1A_2A_3 = \angle A_1A_3A_4 = \cdots = \angle A_1A_{n-1}A_n = 90°}$. Moreover, Jesse further extended this generalization to the situation where the sum of the areas of 2 squares would be the sum of the areas of remaining squares created from each side of an n-side polygon, which implies a more general statement that $\mathbf{(\overline{A_1A_2})^2 + (\overline{A_2A_3})^2 + \cdots + (\overline{A_{m-1}A_m})^2 = (\overline{A_1A_n})^2 + (\overline{A_nA_{n-1}})^2 + \cdots + (\overline{A_{m+1}A_m})^2}$ **if** $\mathbf{\angle A_1A_2A_3 = \angle A_1A_3A_4 = \cdots = \angle A_1A_{m-1}A_m = 90°}$ **and** $\mathbf{\angle A_1A_nA_{n-1} = \angle A_1A_{n-1}A_{n-2} = \cdots = \angle A_1A_{m+1}A_m = 90°}$. These generalizations were made by reasoning with the Pythagorean theorem rather than by searching for and extending a numerical pattern.

Analysis of the instances of connections between reasoning by generic example and structural reasoning revealed that the participants at this level of reasoning mostly used tools in GSP or GeoGebra to produce generic examples that they could use to communicate the more general idea. Various tools in the dynamic geometry environment made it easy for the participants to quickly produce generic examples. For instance, Jordan created the case of 7×3 rectangular grid by adjusting m and n and used the example communicate the notion of "perfect intersection" (Example 5). Jesse used the various construction tools in GSP to create an octagon and used it to communicate a generalization (Example 6). The analysis also showed that the dynamic nature of GSP and GeoGebra could facilitate the participants' observations of the general properties inherent in the examples. As shown in Example 5, it was likely that the adjustment of m and n to different values supported Jordan to formulate the notion of "perfect intersection".

DISCUSSION AND CONCLUSION

This study examined the connections between empirical and structural reasoning when a group of preservice mathematics teachers engaged in solving mathematical problems that were designed to promote learners to generalize mathematical ideas. We reported the instances of connections between empirical reasoning (i.e., naïve empiricism, crucial experiment, reasoning by generic example) and different types of structuring reasoning. The results revealed that naïve empiricism was the dominant type of empirical reasoning, which is consistent with the existing literature on pattern generalization (e.g., El Mouhayar, 2018; El Mouhayar & Jurdak, 2016; Küchemann, 2010; Küchemann & Hoyles, 2009). Although the inductive nature of the dynamic geometry environment made it relatively easy for the participants to observe, conjecture, validate, and generalize mathematical relations on the basis of perception and numerical patterns, identifying mathematical structure underlying these relations and using them to generalize the relations to broader contexts proved to be challenging for them. Learners need additional support to choose examples based on conceptual considerations and to use generic examples in their reasoning.

The results also revealed connections between certain types of empirical reasoning and certain types of structural reasoning. More specifically, the following connections between a specific type of empirical reasoning and a specific type of structural reasoning occurred more frequently: connection between naïve empiricism and result pattern generalization, connection between naïve empiricism and recognizing and operating with structure in thought, connection between crucial experiment and result pattern generalization, connection between reasoning by generic example and process generalization, and connection between reasoning by generic example and reasoning in terms of general structures. By identifying these specific connections, this study deepened our understanding of the complex relationship among the various types of empirical and structural reasoning in the process of generalizing.

Pattern generalization is a typical generalization activity in school mathematics, in which a figurative, numerical, or tabular pattern is usually presented in the form a systematic sequence of elements, and learners are expected to generate a systematic set of ordered pairs from which an empirical relationship can be induced. This approach often promotes learners to identify and express a numerical relationship (i.e., a result pattern generalization) without necessarily seeing the mathematical structure that produces it (Küchemann, 2010). The results from this study showed close connections between result pattern generalization and naïve empiricism and between process pattern generalization and reasoning by generic example. In doing so, this study provides empirical evidence that the ability to generalize based on perception and numerical patterns does not necessarily lead learners to generalize on the basis of mathematical structure. We argue that one plausible reason that many participants in this study were not able to create generalizations based on mathematics structure is that they were not provided sufficient opportunities to engage in this way of thinking in their own mathematical learning experiences. To promote process pattern generalization, the pattern tasks should be designed in a way to encourage reasoning by generic example. Pattern tasks that only include one single example seem to be more likely to promote reasoning by generic example than tasks in which pattern elements are presented sequentially.

The results also showed close connections between naïve empiricism and recognizing a structure in thought and between reasoning by generic example and reasoning in terms of general structures. Although various tools available in GSP or GeoGebra supported the participants to recognize mathematical properties based on empirical cases, it rarely occurred that the participants moved directly from naïve empiricism to reasoning in terms of general structures. Meanwhile, it was quite common for the participants to move from reasoning by generic example to reasoning in terms of general structures. This implies that a learner might be able to recognize a structural element based on naïve empiricism but unable to reasoning with the perceived structure. For instance, in this study many participants recognized the Pythagorean theorem in Task 1 but only a very few of them were able to reason with it as Jesse did in Example 7. This is consistent with the differentiation described in Mason et al. (2009) between perceiving properties and reasoning on the basis of the identified properties while discussing different states of learner’s attention to mathematical structure. Perceiving properties occurs when a learner perceives the discerned relationships as instantiations of general properties which can apply in many different situations. It involves the transition from seeing something in its particularity to seeing it as representative of a general class. Reasoning based on the identified properties involves extending the perceived structure to novel contexts and transforming inductive and abductive reasoning about specific objects into deductive reasoning. Learners should be provided with opportunities to engage in both recognizing structures and reasoning with the perceived structures.

In addition, this study showed that there seem to be at least three prominent types of generalization based on the connections between empirical and structural reasoning. They are result pattern generalization based on naïve empiricism (Example 2) or crucial experiment (Example 4), process pattern generalization based on reasoning by generic example (Example 5), and generalization through generalizing the reasoning (Example 6). These different types of generalization have been discussed in the literature. This study provides a way to organize the different types of generalization by the types of mathematical reasoning that contribute to their generation. Current literature identified different forms of generalization based on, for instance, object of abstraction (Dörfler, 1991), level of connection to context (Radford, 2003), or change of cognitive schema (Harel &Tall, 1991). This categorization is different from these existing categorizations in that it foregrounds reasoning in differentiating forms of mathematical generalization.

This study has a few limitations. First, the results of this study were based on a series of task-based interviews with a small number of learners. Although we deliberately chose tasks from different mathematical domains, the tasks cannot represent all the wide variety of contexts in which mathematical generalizations occur. Larger scale studies (in terms of both the number of participants and mathematical contexts) are needed to validate the findings from this study. Second, we limited our analysis to instances of transitions from empirical reasoning to structural reasoning in technology-aided generalization activity and did not consider the instances in which the participants moved from structural to empirical reasoning. Further research is needed to examine the dynamic relationship between empirical and structural reasoning and the mechanism for engendering and sustaining productive transition in reasoning.

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