









In part (a) of the second question, the students were given three graphs of functions and asked to write the rule of each function using transformation performed on the original one, while in part (b) of the second question the students were asked to write the transformations performed on the same three functions verbally. The three graphs given in the post experiment task are described in Figure 1.

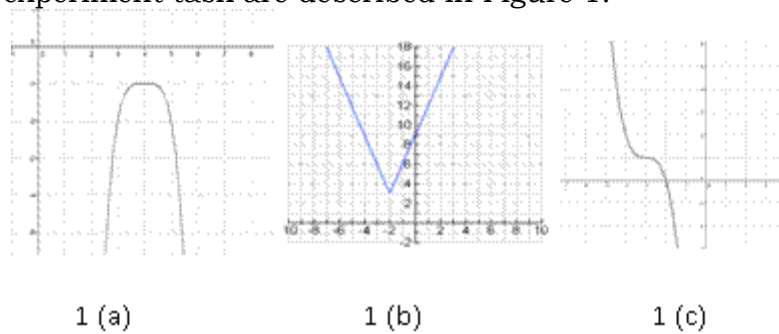


Figure 1. The three given graphs in the second task

In both questions, the three functions were ones that the students had worked before with, where the functions were a quartic function, an absolute function and a cubic function.

### Analysis of the Data

To analyze the collected data we used deductive content analysis, where we performed constant comparisons between the units of gathered data (verbal sentences, graphs, algebraic rules or a combination of them) in order to decompose them into the different themes of the three representations of function transformations.

### FINDINGS

As described above, students, in the first question, were given three algebraic rules of functions and required to draw each function using transformation performed on the original one, as well as to write these transformations verbally. So the first question involved the recognition of graphical and verbal meanings of algebraic operations in terms of transformations, when the transformations should be performed on the original function.

### Recognition of the Graphical Meanings of Algebraic Operations in Terms of Transformations

The following mathematical actions are involved in drawing a function using transformations on the original one: recognizing the original function, recognizing the graphical meanings of the following algebraic operations: adding a positive/negative number to  $x$ , adding a positive/negative number to  $y$ , multiplying a function with a number, and multiplying a function with  $(-1)$ .

When drawing the function  $f(x) = -(x - 4)^4 + 7$ , all the nineteen students knew the graphical meanings of the algebraic actions in terms of transformations done on the original function  $y=x^4$ , but generally the students drew the resulting function only approximately and in varied extent of accurateness. Some of the students' graphs are shown below in Figure 2.

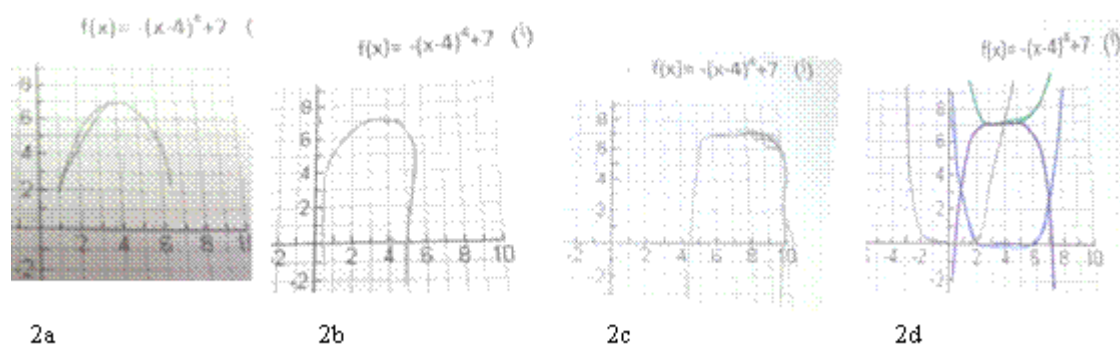


Figure 2. Students' drawings in the case of the function  $f(x) = -(x - 4)^4 + 7$

It could be seen that the students preferred to draw the resulting function, but not to draw manually and graphically the transformations performed on the original function (i.e. they drew the transformations virtually). Only three students showed the transformations as in the case of the student who drew graph 2d, while the rest (16 students) drew just the resulting function.

Drawing the function  $f(x) = 4|x-3|+8$ , seventeen students knew the graphical meanings of the algebraic actions in terms of transformations done on the original function  $f(x) = |x|$ , while one did not know these meanings and one did not answer the question. Some of the students' graphs are shown below in Figure 3 when they knew the graphical meanings.

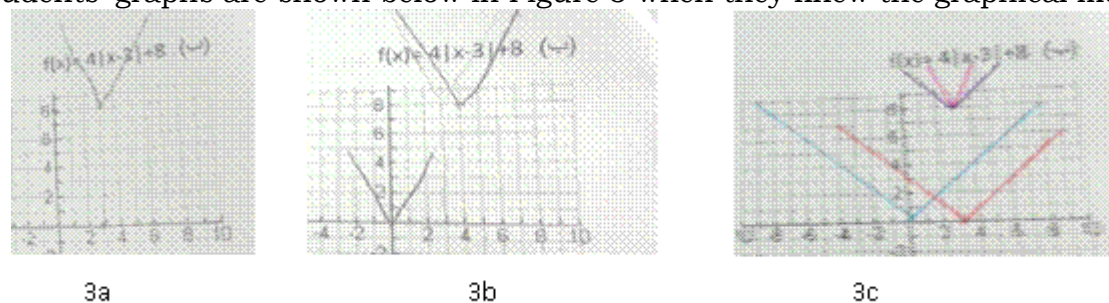


Figure 3. students' drawings in the case of the function  $f(x) = 4|x-3|+8$

In this case the students had three ways of arriving at the required function through transformations: twelve students imagined the original function and performed virtually the transformations, as in case 3a, three students drew the original function, performing virtually the transformations and drawing the resulting function (the required one), as in the case of 3b, and two students drew the original function, performing manually and graphically the transformations, and drawing the resulting function, as in the case of 3c.

The student who did not answer the question drew only the original function without performing any transformations on it, as if he did not know the graphical meanings of the algebraic operations in terms of transformations. The student's graph is shown in Figure 4.

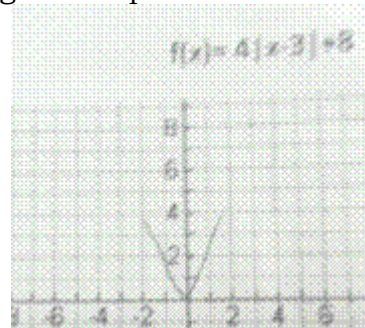


Figure 4. Student's graph when not recognizing the graphical meaning of algebraic operations in terms of transformations – the case of the function  $f(x) = 4|x-3|+8$

Drawing the function  $f(x) = -2(x + 3)^3 + 4$ , fourteen students knew the graphical meanings of the algebraic actions in terms of transformations done on the original function  $f(x) = x^3$ , while three knew only part of these meanings and two did not answer the question.

Some of the students' graphs are shown below in Figure 5 when they knew the graphical meanings.

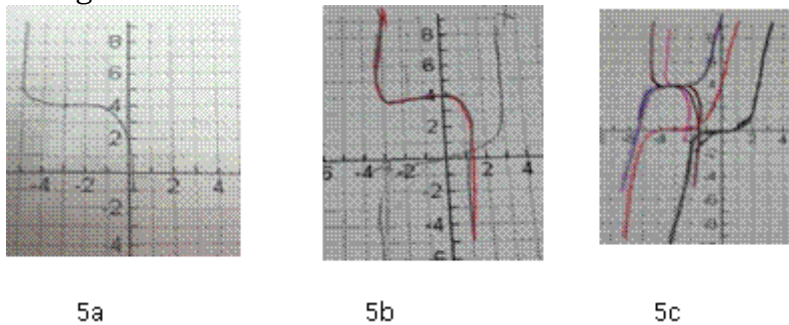


Figure 5. Students' drawings in the case of the function  $f(x) = -2(x + 3)^3 + 4$

In this case too, the students had three ways to arrive at the required function through transformations performed on the original function. Eleven students imagined the original function and performed virtually the transformations, as in case 5a, two students drew the original function, performing virtually the transformations and drawing the resulting function, as in the case of 5b, and one student drew the original function, performing manually and graphically the transformations, and drawing the resulting function, as in the case of 5c.

As described above, three students knew only part of the graphical meanings of algebraic operations in terms of transformations. One of the students did not recognize the graphical meaning of multiplying with (-1), as well as of adding a number to x. This student's graph is in Figure 6(a), the second student did not recognize the graphical meaning of multiplying with (-1), as in Figure 6(b). The third student knew only the graphical meaning of multiplying with (-1), but did not know the graphical meanings of adding a number to x or adding a number to y, as in Figure 6(c).

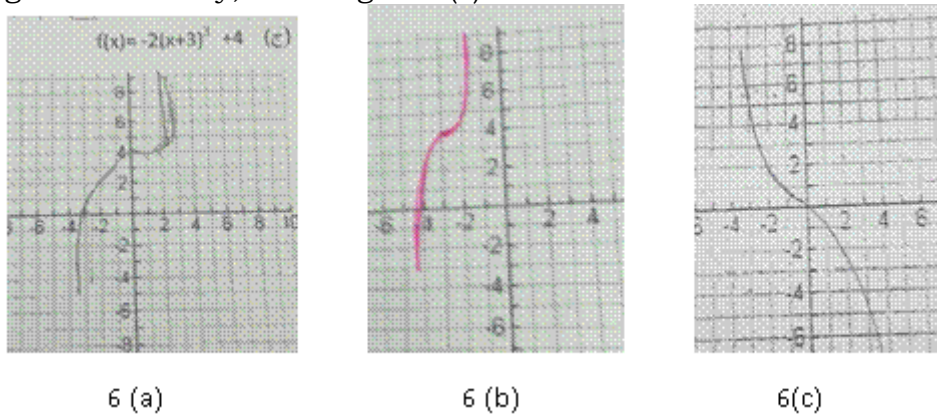


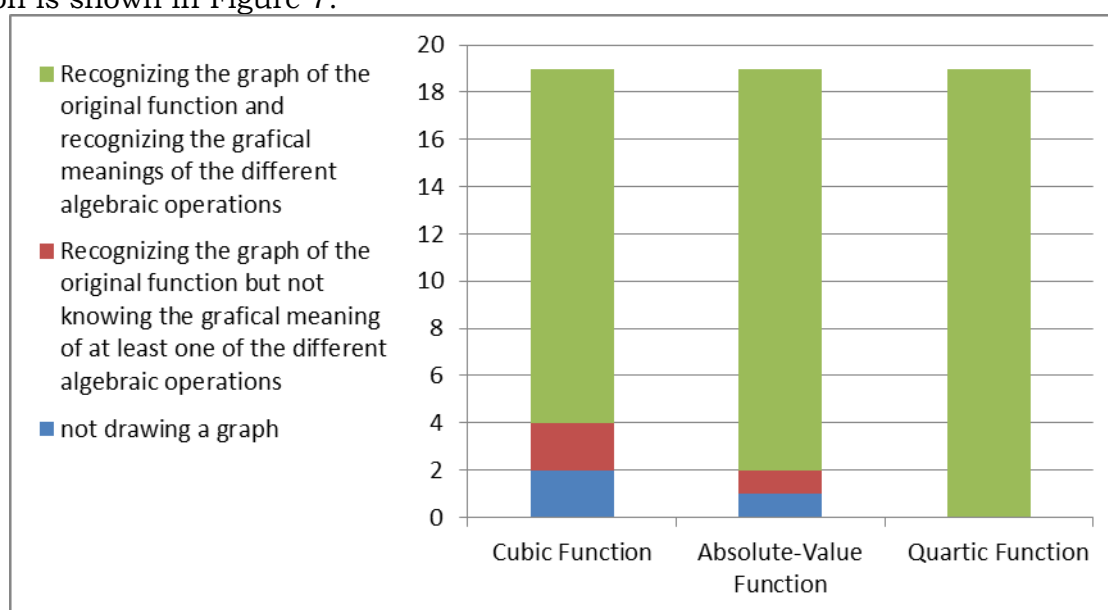
Figure 6. Graphs of students who recognized only part of the graphical meanings of algebraic operations in terms of transformations

Table (1) shows the frequency of students' drawings in terms of the recognition of transformations (associated with algebraic operations) performed on the rule of the original function to get the graph of the given function.

**Table 1.** Frequency of students' graphs in terms of their recognition of transformations done on the algebraic rule of an original function to arrive at the graph of the given function (N=19)

Algebraic rule	Recognizing the graph of the original function and recognizing the graphical meanings of the different algebraic operations	Recognizing the graph of the original function, but not knowing the graphical meaning of at least one of the different algebraic operations	Not drawing a graph
$f(x) = -(x - 4)^4 + 7$	19		
$f(x) = 4 x - 3  + 8$	17	1	1
$f(x) = -2(x + 3)^3 + 4$	15	2	2

The frequency of students' answers, in terms of recognition of graphical meanings of algebraic operations performed on the original function to get the graph of the given function is shown in Figure 7.



**Figure 7.** Frequency of students' answers, in terms of recognizing the graphical meanings of algebraic operations in terms of transformations

In the second part of the first task the students were asked to describe verbally the transformations performed on the original function to get the given one. Below is a description of students' recognition of the verbal meanings involved.

### Recognition of Verbal Meanings of Algebraic Operations in Terms of Transformations

The following mathematical actions are involved in describing verbally a function in terms of the transformations performed on the original one: recognizing the original function, recognizing the verbal meanings of the following algebraic operations: adding a positive/negative number to x, adding a positive/negative number to y, multiplying a function with a number, multiplying a function with -1, the order of the transformations. Also involved are recognizing the reflection axis when a reflection transformation is involved.

Describing verbally the transformations performed on the rule of the original function in the case of each one of the three given rules, the students behaved in one of three ways: (1)



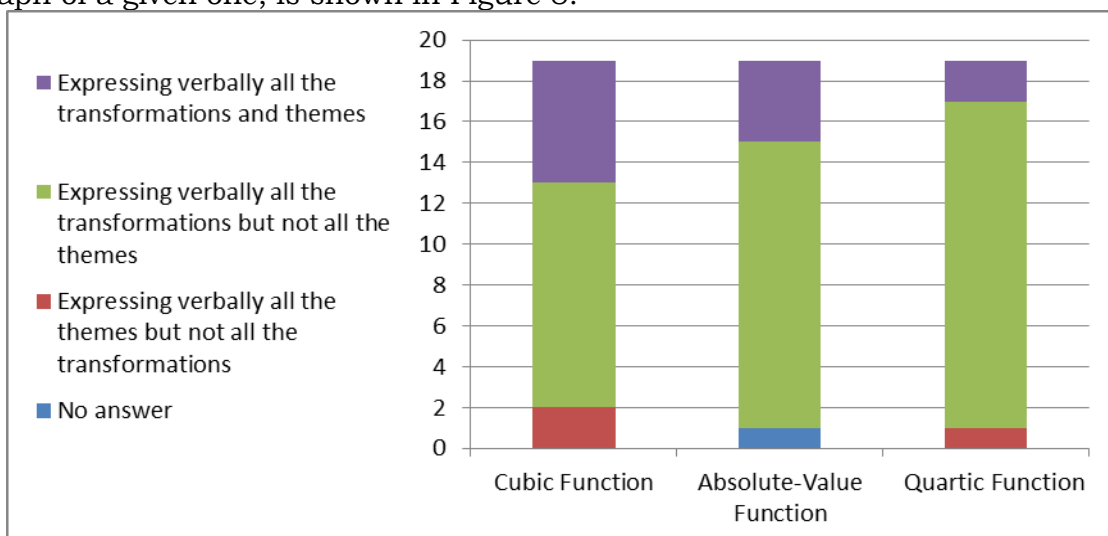
expressing verbally all the transformations (translations, reflections and compression) and themes (direction of a translation, magnitude of a translation or a compression, the original function and the reflection axis) associated with the algebraic actions done on the original function to get the given one, (2) expressing verbally all the transformation but not all the themes associated with the algebraic actions done on the original function to get the given one, and (3) expressing verbally all the themes but not all the transformations associated with the algebraic actions done on the original function to get the given one.

Table (2) shows the frequency of students' verbal expressions in terms of their recognition of transformations performed on the algebraic rule of the original function to get the graph of the given one.

**Table 2.** Frequency of students' verbal expressions in terms of their recognition of transformations done on the algebraic rule of the original function to arrive at the graph of a given one

Algebraic rule	Expressing verbally all the transformations and themes	Expressing verbally all the transformations but not all the themes	Expressing verbally all the themes but not all the transformation
$f(x) = -(x - 4)^4 + 7$	2 students	16 students forgot to mention the reflection axis and missed the accurate order of the transformations	1 student did not express the reflection transformation: in place of it, the student mentioned a compression
$f(x) = 4 x - 3  + 8$	4 students	14 students did not express the reflection axis and missed the accurate order of the transformations 1 student did not express the original function	
$f(x) = -2(x + 3)^3 + 4$	6 students	11 students did not express the reflection axis and missed the accurate order of the transformations.	2 students did not express the reflection transformation

The frequency of students' answers, in terms of recognition of the verbal meanings of function transformations done on the algebraic rule of the original function to arrive at the graph of a given one, is shown in Figure 8.



**Figure 8.** Frequency of students' answers, in terms of in terms of recognition of the verbal meanings of function transformations

In the first part of the second question, the students were given three graphs of functions and required to write the rule of each function using transformation performed on the graph of the original one, as well as to write these transformations verbally. So, the second question involved the recognition of algebraic and verbal meanings of graphical operations in terms of transformations performed on the graph of the original function.

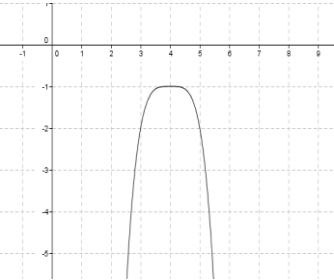
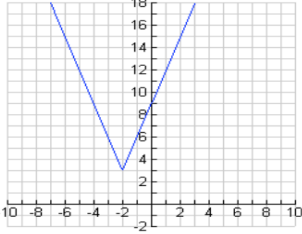
### Recognition of the Algebraic Meanings of Graphical Operations in Terms of Transformations

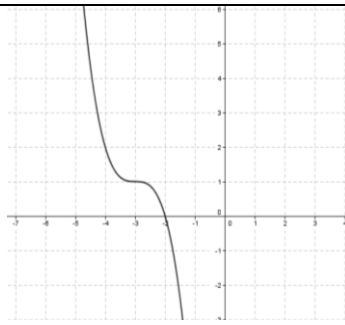
The following mathematical actions are involved in writing the rule a function using transformations on the graph of the original one: recognizing the original function, recognizing the algebraic meanings of the following graphical operations: translating a graph horizontally with attention to the direction and magnitude of the translation, translating a graph vertically with attention to the direction and magnitude of the translation, reflecting a graph of a function with attention to the reflection axis, performing expansion or compression on the graph of a function with attention to the magnitude of the expansion or compression.

Writing the rule of a given graph through considering the transformations performed on the graph of the original function of each one of the three given graphs, the students behaved in one of three ways: (1) took into consideration all the appropriate transformations and related themes, (2) took into consideration all the appropriate transformations but not all the appropriate themes, and (3) took into consideration all the appropriate themes but not all the appropriate functions.

Table (3) shows the frequency of students' algebraic behaviors in terms of their recognition of transformations performed on the graph of the original function to get the algebraic rule of the given one.

**Table 3.** Frequency of students' verbal expressions in terms of their recognition of transformations done on the algebraic rule of the original function to arrive at the algebraic rule of the given one

Graph of function	Expressing algebraically all the transformations and themes	Expressing algebraically all the transformations but not all the themes	Not expressing all the transformations
	17 students wrote the rule $f(x) = -(x-4)^4 - 1$		2 students did not express the reflection transformation, writing the rule: $f(x) = (x-4)^4 - 1$
	18 students wrote the rule: $f(x) = 3 x + 2  + 3$	1 student did not express accurately the magnitude of the compression	



13 students wrote the rule  $-(x+3)^3+1$   $f(x)=$

6 students did not express the reflection transformation, writing the rule:  $f(x)= (x+3)^3+1$

The frequency of students' answers, in terms of correctness and exactness is shown in Figure 9.

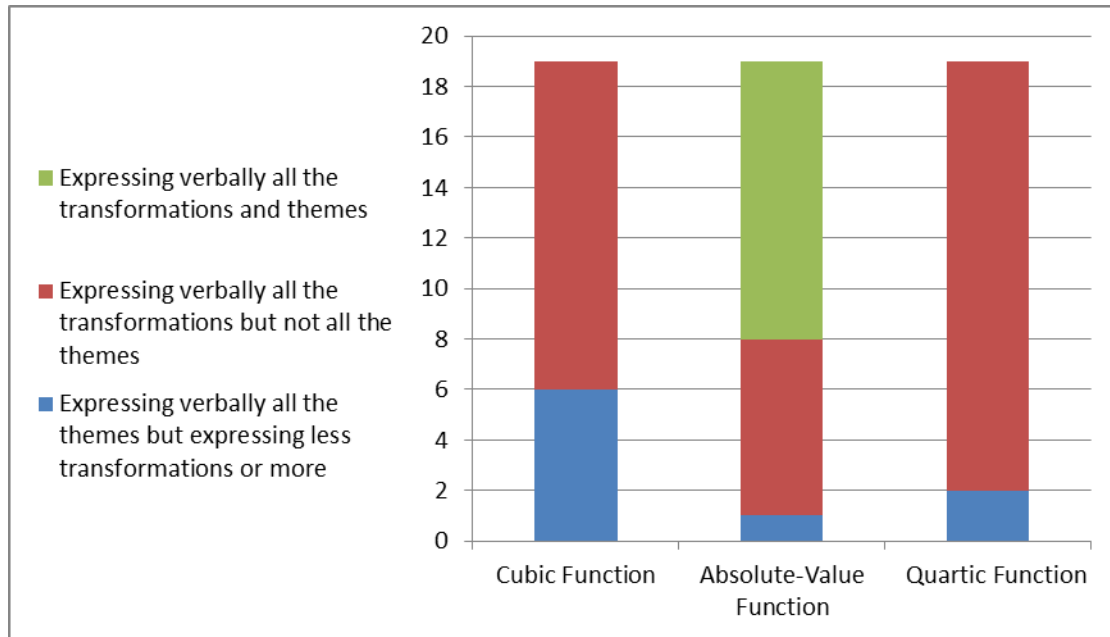


Figure 9. Frequency of students' recognition of transformations and related themes, when performed on the graph of an original function to get a given one

### Recognition of the Algebraic Meanings of Graphical Operations in Terms of Function Transformations

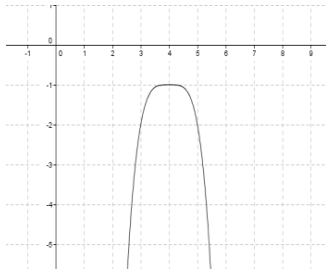
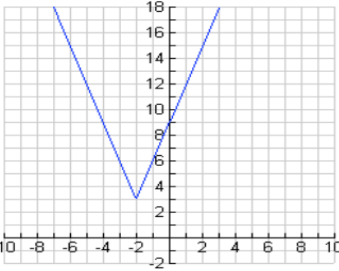

The following mathematical actions are involved in the verbal description of transformations performed on the graph of the original one to get the rule of a given function: recognizing the original function, recognizing the verbal meanings of the following graphical operations: translating a graph horizontally with attention to the direction and magnitude of the translation, translating a graph vertically with attention to the direction and magnitude of the translation, reflecting a graph of a function with attention to the reflection axis, performing expansion or compression on the graph of a function with attention to the magnitude of the expansion or compression, and paying attention to the order of the transformations.

Describing verbally the transformations performed on the graph of the original function of each one of the three given graphs, the students behaved in one of three ways: (1) expressing verbally all the transformations and themes associated with the graphical actions done on the graph of the original function to get the rule of the given one, (2) expressing verbally all the transformation but not all the themes associated with the graphical actions done on the graph of the original function to get the given one, and (3)

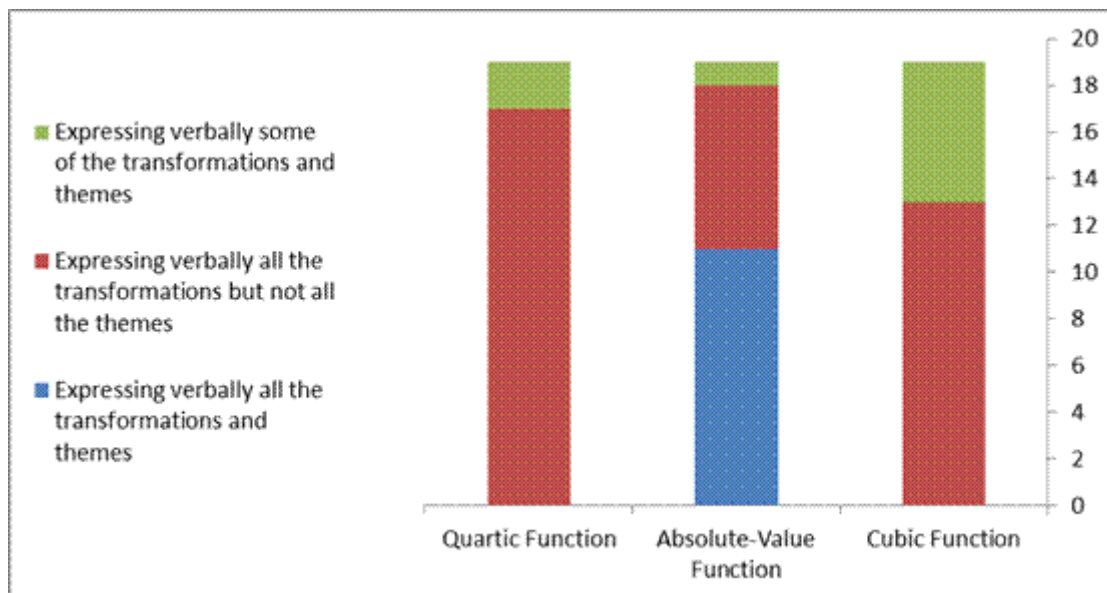
expressing verbally all the themes but not all the transformations associated with the graphical actions done on the graph of the original function to get the given one.

Table (4) shows the frequency of students' verbal expressions in terms of their recognition of transformations performed on the graph of the original function to get the algebraic rule of the given one.

**Table 3.** Frequency of students' verbal expressions in terms of their recognition of transformations done on the graph of the original function to arrive at the algebraic rule of a given one

Graph of function	Expressing verbally all the transformations and themes	Expressing verbally all the transformations but not all the themes	Expressing verbally all the themes but expressing less transformations or more
	11	17 did not express the reflection axis, where 8 of them did not express the original function too	2 did not express the reflection transformation
	11	7 did not express the original function	1 student added a compression transformation which is not needed
		13 did not mention the original function, or the reflection axis too	6 did not express the reflection transformation

The frequency of students' answers, in terms of correctness and exactness is shown in Figure 10.



**Figure 9:** Frequency of students' recognition of transformations and related themes, when performed on the graph of an original function to get a given one

## DISCUSSION

Learning the topic of function transformations is essential for mathematics students because understanding this topic makes it easy to understand other main math topics as functions (Lage & Gaisman, 2006). Furthermore, students encounter difficulties learning the topic, such as conceiving the horizontal translation (Eisenberg & Dreyfus, 1994; Zazkis, Liljedahl & Gadowsky, 2003). In this research we expected that teaching function transformations to middle school students by using technology, specifically Geogebra, would help them conceive better the different themes associated with this topic. We will discuss the current research results by referring to students' recognition of the meanings of mathematical operations in the different representations of function transformations.

The participating students succeeded generally in working with the algebraic and graphical transformations, where this success stems from two reasons: working with a dynamic tool and working with exploration tasks, where they investigated by themselves and gradually the transformation. The type of the activity being explorative is acknowledged in the literature as contributing positively to students' learning (see for example Bishop, 1993; Nunes, 1993). This is also the case with the positive influence of dynamic tools, specifically GeoGebra on students mathematics learning (Diković, 2009; Karadag & McDougall, 2009a, 2009b).

Regarding the graphical meanings of the algebraic operations, the participating students generally succeeded to recognize the graphical meanings of algebraic operations in terms of transformations in the three familiar functions (cubic, quartic and absolute), but they had relative difficulty when treating the cubic function. This difficulty appeared in students' attempts to perform reflections on the cubic function; probably due to the similarity between the original function and the reflected one (the two parts of the each function are rotationally symmetric). It could be said that students' difficulty, in the case of the cubic function, is attributed to the complexity of mental construction needed to process the reflection of the cubic function. This complexity of mental construction was mentioned by Baker, Hemenway and Trigueros (2000) as lying behind students' difficulty with horizontal transformation. The students had no or little difficulties in recognizing the graphical meanings of algebraic operations in terms of transformations in the case of the absolute and the quartic functions, probably because of their similarity to the quadratic function

which the students were used to (a reflection axis, extreme points, increasing and decreasing domains).

The participating students generally succeeded to recognize the algebraic meanings of graphical operations in terms of transformations in the three familiar functions (cubic, quartic and absolute). Furthermore, the students had relative difficulty in treating the cubic function. This difficulty appeared specifically in students' attempts to recognize the algebraic meanings of the reflection transformation. As we described above, this difficulty is associated with students' inability to recognize the relation between the cubic function and its reflected one.

It could be claimed that students' relative successful work and moving between the algebraic and graphical representations of function transformations is a result of the availability of both the algebraic and graphic representations in the interface of geogebra (Diković, 2009). The relative successful work of the participating students did not continue when the students came to represent verbally the required transformations, for most of them did not succeed in recognizing the verbal meanings of algebraic operations in terms of transformations. Generally, the students encountered difficulties mentioning the reflection axis and its related theme, namely the order of transformations (the non-mentioning of the reflection axis makes it difficult to evaluate the accurateness of the order of transformations). Students' difficulties working verbally with transformations could be due to their inability to transfer their knowledge regarding function transformations in the graphic or the algebraic context to function transformations in the verbal context. This inability to transfer knowledge across representations is reported in previous researches, for example Nguyen and Rebello (2009) reported that students encountered difficulties when attempting to transfer their problem solving skills across problems in different representations. Students' difficulties working verbally with transformations could also be due to the absence of the verbal representation in the Geogebra interface and, at the same time, due to the absence of exploration activities in which students were asked to work with various transformations given verbally in order to investigate the resulting function. Specifically, the previous reasons were behind students' inattention to the reflection axis and the order of the transformations. These explanations are supported by Consciência and Oliveira (2011) who argue that using exploration tasks focusing on one representation of transformations leads to students' difficulties in understanding transformations' concepts, so the emphasis on the verbal representation in the exploration activities would lessen students' difficulties with this representation. In our case, giving students verbal representation of transformations that include reflections of the same function, but with a different reflection axis, would make the students aware of the influence of the axis on the transformed function. The same thing is true regarding the related theme, namely the order of transformations.

More than half of the participating students succeeded to recognize the verbal meanings of graphic operations in the case of the absolute function in terms of transformations. This relative success could be related to the fact that moving from the original absolute function to the given one did not involve performing the reflection transformation (in our case). So, it seems that the students had difficulty with the reflection transformation in general, but this appeared more when the reflection was performed on the cubic function. This students' difficulty was represented in their inability to recognize the verbal meanings of some graphic operations (in terms of transformations) in the case of the quartic function, as well as the cubic function. Specifically, this inability appeared in their non-mentioning of the reflection axis and the original function. Students' difficulty with the reflection axis is also mentioned by Lage and Gaisman (2006) who reported students' difficulty with the reflection axis, considering it always to be the x-axis. Moreover, Panaoura, Elia, Stamboulides and Spyrou (2009) found that students encounter difficulties in giving a proper definition for the concept of the reflection axis and resolving tasks involving conversions between diverse modes of its representations. Furthermore, it could be said that students' difficulties here were due to the same reasons mentioned above regarding the difficulties in recognizing the

verbal meanings of algebraic operations in terms of transformations, i.e. the relatively little involvement of the students with activities dealing with the verbal representation of the transformations, the unavailability of the verbal representation in the interface of the technological tool, and students' inability to transfer their knowledge regarding function transformations in the graphic or the algebraic context to the verbal context.

We have been aware that students behaved differently in performing transformations to arrive at the graph of the transformed function given its rule, where they had three different ways in doing so: drawing just the transformed function, drawing the original function and the transformed one and drawing the transformed function step by step; i.e. drawing the functions resulting from all the transformations involved. Most of the students behaved in the first way, drawing just the transformed function, sometimes drawing marks to help them draw the transformed function accurately. One reason for this behavior could be due to students' mental following of functions' movement in the interface of Geogebra, where they did not require that trace of the function movement to be shown.

## CONCLUSION

Working with Geogebra, students generally succeeded to work with function transformations in their algebraic and graphic representations. This success could be attributed to the properties of the technological tool, as well as to the properties of the given task.

The technological tool helped the students work successfully with transformations because its interface includes both the graphic and the algebraic representations. At the same time, the participating students had some difficulty working verbally with function transformations, especially when the reflection transformation was involved. This could be due to the absence of the verbal representation from GeoGebra interface. On the other hand, the tasks being explorative influenced positively students' understanding of mathematical concepts related to transformations, where this type of tasks is reported in the literature to have such influence on students' learning (see for example Bishop, 1993; Nunes, 1993).

Various researchers were aware of the importance of the visual aspect to the teaching of function transformations, and thus experimented with visual tools in teaching students the topic. Some reported students' difficulties with some aspects of the topic, as the horizontal transformations, even with the presence of technology. We too experimented with using a technological tool, namely Geogebra, to teach the topic. The results of our research point at the need of students' involvement with three representations to conceive deeply and correctly the different themes of the topic, which are associated with the algebraic, the graphic and the verbal representations. Educators have been aware in the last two decades of the importance of the first two representations, but not the third one, namely the verbal representation. It's time to take care of this representation primarily in the activities given to students, but also in the interface of the technological tool. Taking care of the verbal representation would help students overcome their difficulties in paying attention to the reflection axis and its related theme, namely the order of transformations.

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