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Combinatorial approach on the recurrence sequences: An evolutionary historical discussion about numerical sequences and the notion of the board

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ABSTRACT

The tradition of studies involving the combinatorial approach to recurring numerical sequences has accumulated a few decades of tradition, and several problems continue to attract the interest of mathematicians in several countries. This work specifically discusses the Fibonacci, Pell, and Jacobsthal sequences, focusing on Mersenne sequences. The often-used definition of board involves considering how to fill a specific regular surface -the board- with a limited quantity of regularly shaped tiles. On the other hand, an analogous problem can be generalized and exemplifies current research developments. Finally, the examples covered constitute unexpected ways of exploring visualization and other skills in mathematics teachers' learning, consequently inspiring them for their teaching context.

Keywords: numerical sequences, combinatorial approach, board and tiles, mathematics teacher education

INTRODUCTION

In general, we can see the advances in contemporary research around the study of recurring numerical sequences and their countless forms of approaches and generalization (Grimaldi, 2012; Lagrange, 2013), which, recurrently, tends to be neglected by book authors of the history of mathematics (Grimaldi, 2012; Gullberg, 1997; Stillwell, 2010; Vorobiev, 2000). We especially remember the Fibonacci sequence, whose recurrence relationship allows us to determine the Fibonacci numbers and is defined by the following relationship: $F_{n+1} = F_n + F_{n-1}, F_0 = 0, F_1 = 1$ (*).

On the other hand, we must register the tradition of works (Benjamin & Quinn, 1999, 2003a, 2003b; Koshy, 2001, 2014, 2019) that introduce a combinatorial approach to the numerical sequence indicated in (*). In these terms, in a pioneering way, Benjamin and Quinn (2003a) introduce the notion of n-n-board and another broad set of demonstration and proof techniques that allow us to verify and confirm various theorems and properties derived from the generalization of the sequence indicated in (*) (**Figure 1**).

Based on these and other arguments, in the following sections, we will present some examples that, from an evolutionary mathematical-historical perspective, highlight the origin and evolution of the combinatorial approach for some recurring numerical sequences (Spreacífico, 2014). First, we will discuss the most classic cases in the literature and other examples that involve numerical sequences disregarded by authors of the history of mathematics books based on generalized and unexpected forms of boards. In these terms, we will first see that the notion of board admits unexpected and pictorial forms of representation for the sequences of Fibonacci, Lucas, Pell, Jacobthal, and Padovan and, more recently, the Mersenne sequence.

COMBINATORIAL APPROACH TO THE FIBONACCI SEQUENCE

In Grimaldi (2012), we can see that the author considers a board 2×n and considers vertical dominoes (2×1) and horizontal dominoes (1×2) to fill the board.

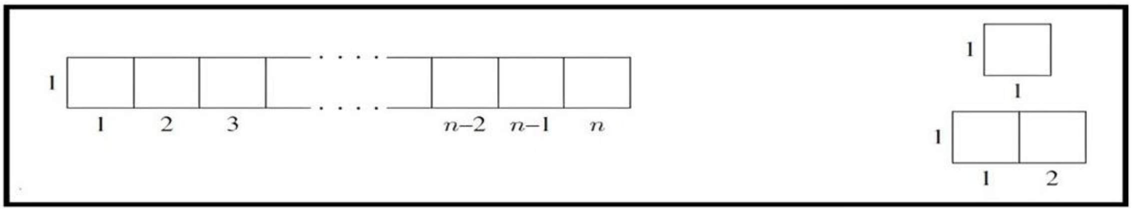


Figure 1. 1×n-type board related to Fibonacci sequence & numerous combinatorial properties (Adapted from Benjamin & Quinn, 2003a)

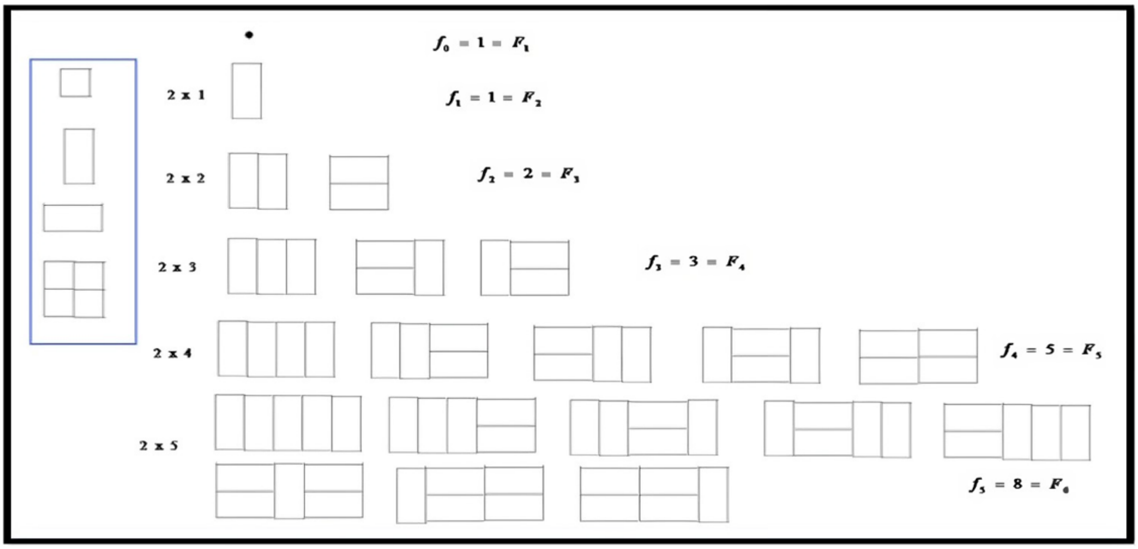


Figure 2. Examples of tiling related to Fibonacci, using horizontal & vertical dominoes to fill a 2×n-type board (Source: Authors' own elaboration)

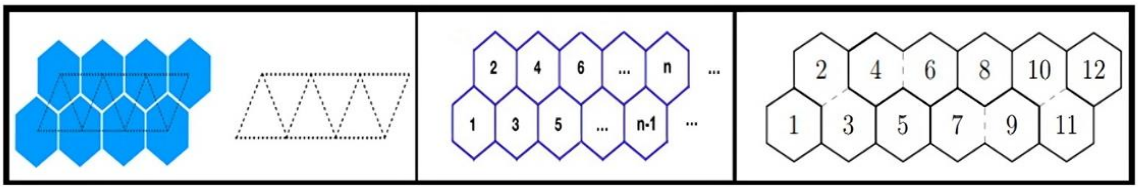


Figure 3. Board with hexagonal cells (Adapted from Došlić & Podrug, 2022 and Ziqian, 2019)

Figure 2 illustrates some particular cases disregarded by Grimaldi (2012). Considering that number f_n represents the tiling ways to fill the board 2×n, we will find the relationship $f_n = F_{n+1}$, for $n \geq 0$. It is worth noting that, based on particular combinatorial arguments, we can state theorem 1, whose detailed demonstration can be consulted in Grimaldi (2012) and other properties (Bicknell-Johnson, 1970; Feinberg, 1963; Feng, 2011).

Theorem 1: Considering board 2×n and fixing vertical dominoes (2×1) and horizontal dominoes (1×2), fill it by tiling. Defining the term f_n as the number of different ways to fill it, then $f_n = F_{n+1}, n \geq 0$ (Grimaldi, 2012).

Now, through the recent works of Došlić and Podrug (2022), Dresden and Tulskikh (2021), and Ziqian and Dresden (2022), we will discuss the case of hexagonal boards. In fact, these works generally introduce a board shape made up of regular hexagons, that is, a double-strip H_n constituted by n regular hexagons. To illustrate, **Figure 3** shows H_8 strip, made up of eight cells, which, in this case, are eight regular hexagons. Ziqian and Dresden (2022) define a double track with the so-called $n$regular and adjacent hexagons (right part in **Figure 3**). Ziqian and Dresden (2022) define an enumeration of cells (hexagons) as indicated and describe the possibility of covering the H_n range with horizontal and inclined dominoes.

On the other hand, Ziqian (2019) seeks to simplify the graph by introducing the dotted configuration to locate each hexagon. Thus, Ziqian (2019) uses a similar method when choosing a set of pieces to tiling the H_n strip using the dotted (triangular) diagram. On the other hand, on the right side of **Figure 3**, we note that, for each hexagonal cell, monomers or dimers can occur (contiguous and non-isolated hexagonal cells). For example, Došlić and Podrug (2022) comment that in the cell numbered '1,' we have a monomer, however, when we observe '2' and '3,' it corresponds to a dimer. For the same reason, cell '8' has a monomer, and '10' and '11' correspond to a dimer (right part in **Figure 3**). **Figure 3** establishes the ordering and numbering of H_n track. Next, we define the pieces that must be considered with the close interest of describing the tiles for each case indicated in **Figure 3**, for cases $n = 1, 2, \ldots, 8$. For our discussed instances, we will only consider isolated hexagonal cells (monomers only).

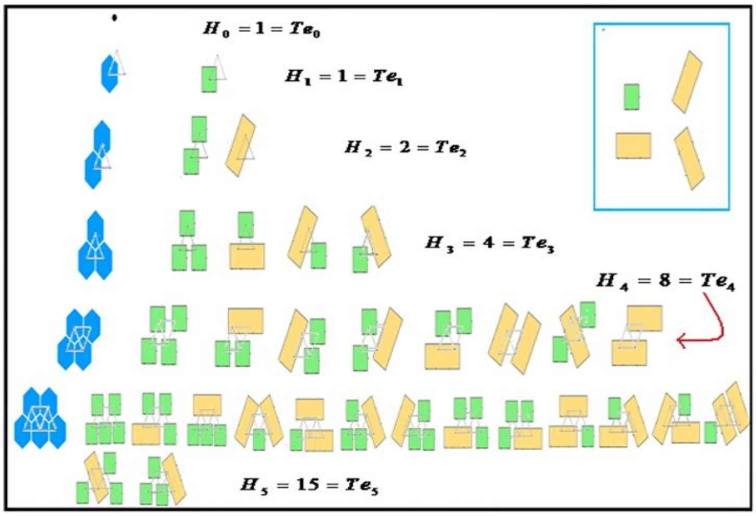


Figure 4. Visualization of particular cases proposed by Ziquian (2019) & hexagonal board with inclined tiles (dominoes & triminoes) (Source: Authors' own elaboration)

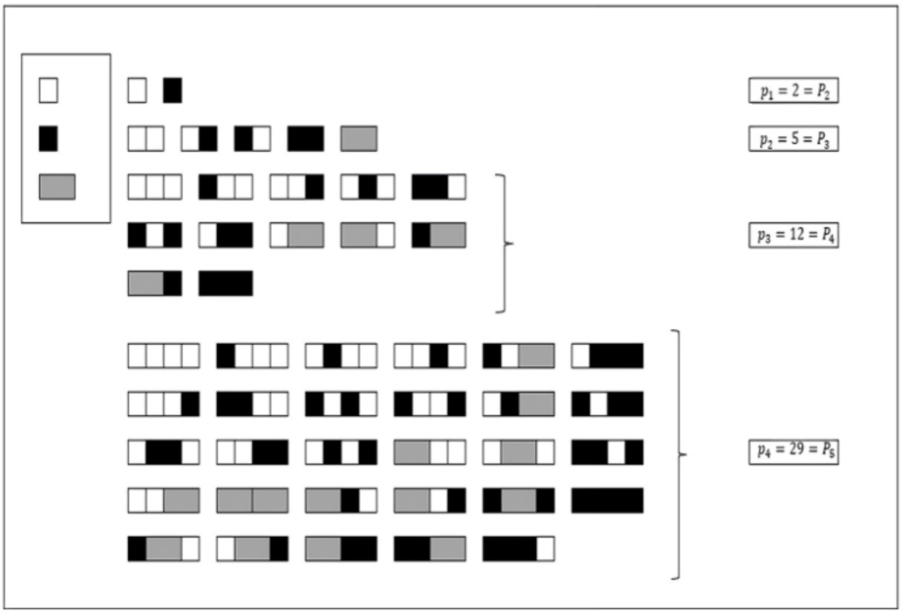


Figure 5. Benjamin et al. (2008) describe a $1 \times n$ board & rules that determine numerical relationships with Pell sequence P_n (Source: Authors' own elaboration)

Figure 4 depicts cases $n = 0, 1, 2, \ldots, 6$ and their numerical relationships with the Tetranacci sequence (0, 1, 1, 2, 4, 8, 15, 29, 56, 108, …), which we denote in **Figure 4** by $Te_0, Te_1, Te_2, Te_3, Te_4, Te_5$.

COMBINATORIAL APPROACH TO THE PELL SEQUENCE

Benjamin et al. (2008) address how to fill a board with the values corresponding to the Pell sequence, which, in a standard way, is described by relationship $P_{n+1} = 2P_n + P_{n-1}$, for the values $P_0 = 0, P_1 = 1$. So, let us consider a white square, a black square, and a gray domino. To this end, the authors denote $p_0 = 1 = P_1$, which corresponds to no initial tiling. For $p_1 = 2 = P_2$, which corresponds to filling a 1-board in two ways, with a white square and a black square. Next, for a 2-board, we can see in **Figure 5** the tilings for a 3-board and 4-boards that determine the relationships $p_2 = 5 = P_3, p_3 = 12 = P_4$, and $p_4 = 29 = P_5$ (**Figure 5**). Using a combinatorial argument, we will determine that the respective board can be related based on the number indicated by $p_n = P_{n+1}, n \geq 0$.

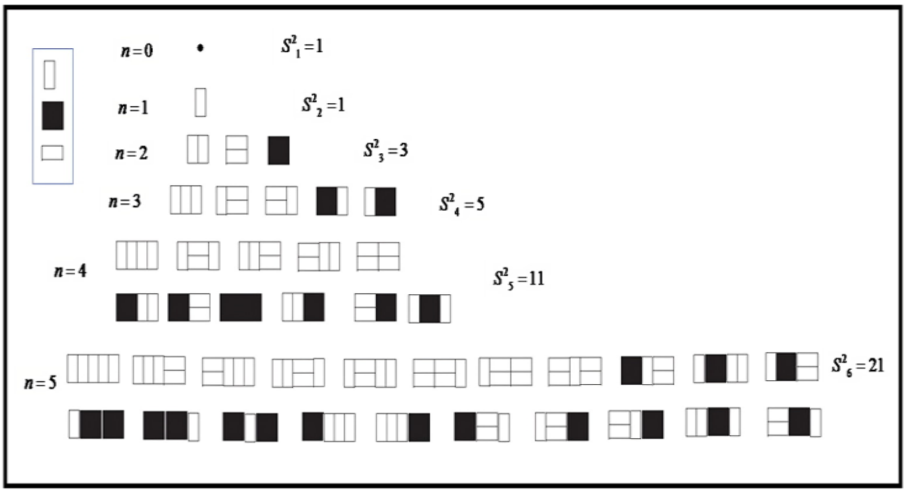


Figure 6. Koshy (2019) describes tiles & a board 2×n that corresponds to elements of Jacobsthal sequence (Source: Authors' own elaboration)

COMBINATORIAL APPROACH TO THE JACOBSTHAL SEQUENCE

In the case of the board related to the Jacobsthal sequence, we will first consider a one-dimensional board of the 2×n type, with the following pieces attached: horizontal dominoes 1×2 and vertical dominoes 2×1, both with weight 1, and black squares 2×2, with weight 1.

Next, in **Figure 6**, we will associate the weight defined by S_n for each board, considering the tiles described through the previously defined pieces. Thus, based on Koshy's (2019) thinking, we define that $S_0 = 1$. And, for the corresponding boards, we will determine the following arithmetic-algebraic relationships: $S_0 = 1 = J_1, S_1 = 1 = J_2, S_2 = 3 = J_3, S_3 = 5 = J_4, S_4 = 11 = J_5, S_5 = 21 = J_6$, etc.

From examining some particular cases through a visualization exercise, the more attentive reader acknowledges that only two forms of tiling occur when looking at the end. Tilings that end with vertical rectangles occur and, in this case, determine sub-tiling of length $n-1$. On the other hand, sub-tilings of length $n-2$ that end in black squares 2×2 or dominoes grouped in the form may occur. Let us look at the examples in **Figure 6**.

In fact, based on **Figure 6**, when we consider the tiling on the board for $n = 4$, we can count five tiles with vertical rectangles, three tiles that end in black squares, and three tiles that end with horizontal rectangles, i.e., $n = 4 \therefore 5 + 2 \cdot 3 = 11 = J_4$. Based on this particular analysis, we will generalize the respective argument for counting the tiles that fill a board 2 x n. Let us see the demonstration of the previous arguments, which can be consulted in more detail in Koshy (2019).

Theorem 2: Given board 2×n , and considering the horizontal dominoes 1×2 and vertically, both with weight 1, black squares 2×2, with weight 1. In light of the sum of the weights of length n, then the relation $S_n = J_{n+1}$, for $n \geq 0$ is valid (Koshy, 2019).

Proof: Let us consider a board 2×n. We will denote by S_n the number of different tile shapes on this board. Immediately, in cases $n = 0, 1$ we will define that $S_0 = 1 = J_1,\ S_1 = 1 = J_2$. In the following cases, we can examine the numerical relationships shown in **Figure 5**. Now, considering any arbitrary tile, which we will denote by T, we will consider the following cases:

(a). when it ends with a vertical domino ,

(b). when it ends with a square , and

(c). when it ends with the following configuration .

Note that when case (a) does not occur, we will only have two possibilities that do occur, which are the sub-tilings of the (n-2) form or sub-tilings of the (n-2) form. Thus, since the previous configurations are independent, the final total contribution of all cases, by an additive and counting principle, results in the expression $2 \times S_{n-2}$. And, adding the cases of the item (a), for the cases of sub-tilings of the (n-1) form , we will determine that: $1 \times S_{n-1} + 2 \times S_{n-2} = S_{n-1} + 2S_{n-2}$. Finally, since $S_0 = 1 = J_1, S_1 = 1 = J_2$ and the previous expression corresponds to the case and the same recurrence rule of the Jacobsthal sequence, we determine that $S_n = J_{n+1}$, for $n \geq 0$ ■.

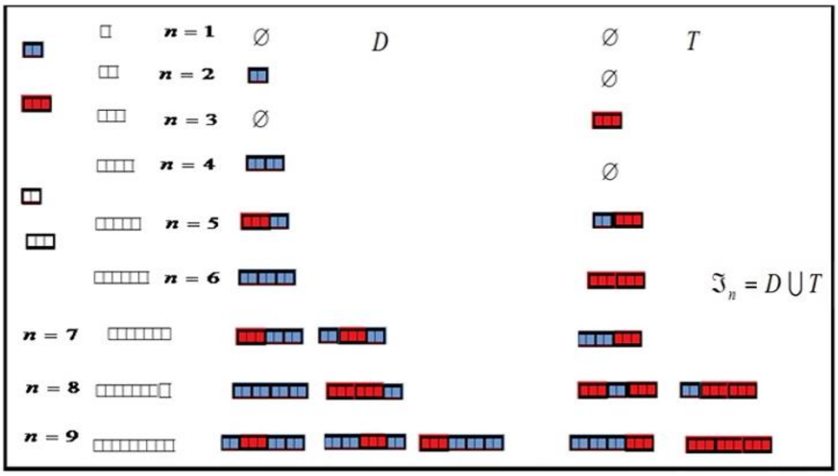
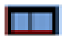
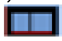


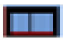
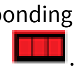
Figure 7. Description of cases $1 \leq n \leq 9$ based on combinatorial model, via board (Adapted from Tedford, 2019)

In the Jacobsthal sequence, we can see other forms of generalization that correspond, for example, with the introduction of weights for specific board pieces. In this case, based on the illustration of the previous cases, we can verify properties related to the polynomial Fibonacci sequence, the polynomial Pell sequence, and even the polynomial Jacobsthal sequence. On the other hand, it is worth noting that in the work by Soykan (2021), we come across the third-order Pell sequence, described by $P_{n+2} = 2P_{n+1} + P_n + P_{n-1}, P_0 = 0, P_1 = 1, P_2 = 2$. Employing, once again, the arrangement indicated in **Figure 5**, if we add a trimino  (in green), we will be able to determine the corresponding numerical correlation with the third-order Pell sequence. A similar strategy can be developed for Tribonacci, Tetranacci, etc.

COMBINATORIAL APPROACH TO THE PADOVAN SEQUENCE

Let us consider the following pieces of the board that we will indicate, according to Tedford (2019), for  (dominoes) and  (triminoes). Next, we contemplate some rules for the composition and description of tiling. In fact, we will represent the following element $\Im_n$, which defines a partition into two sets of dominoes (D) and triminoes (T): $\Im_n = D \cup T$. Then, according to Tedford's (2019, p. 291) description, we will consider the subset D of possible tiling that always ends with a domino . And on the other hand, the subset T of possible tiling always ending with a trimino .

For example, let's determine the initial elements denoted by $\Im_1, \Im_2. \Im_3, \Im_4. \Im_5$. Considering the previous definition $\Im_n = D \cup T$, in case $\Im_1$, we will assume that, by fixing a 1-board, it is not possible to perform any form of tiling, i.e., we will designate it as $\Im_1 = \emptyset$ and, numerically, we can indicate it by $|\Im_1| = 0 = P_{-1}$. On the other hand, setting a 2-board, we will naturally conclude that only $\Im_2 = \{1 \text{ dominó}\} \cup \emptyset$ occurs, that is, we see that $|\Im_2| = 1 + 0 = 1 = P_0$. In the next step, for a 3-board, we can infer that $\Im_3 = \emptyset \cup \{1 \text{ trimino}\}$; that is, we determine that $|\Im_3| = 0 + 1 = 1 = P_1$. Repeating the previous arguments, setting a 4-board, we will see that $\Im_4 = \{1 \text{ domino}\} \cup \emptyset$, that is, we can determine that $|\Im_4| = 1 + 0 = 1 = P_2$. Our last case that exemplifies the previous arguments, when we set a 5-board, we have the possibilities $\Im_5 = \{\text{ tridomino,domino}\} \cup \{\text{domino,tridomino }\}$. Therefore, we will conclude that $|\Im_5| = 1 + 1 = 2 = P_3$.

In **Figure 7**, we exemplify the cases of $1 \leq n \leq 9$. In the middle column, we indicate by D the tiling configurations corresponding to the tiling ending up in a domino . In the last column on the right, we can identify the tiling that end in a trimino . For example, we see the following numerical correspondence: $|\Im_6| = 1 + 1 = 2 = P_4$, $|\Im_7| = 2 + 1 = 3 = P_5$, $|\Im_8| = 2 + 2 = 4 = P_6$, $|\Im_9| = 3 + 2 = 5 = P_7$, etc…

In **Figure 8**, we provide examples of particular cases for $\Im_{10}, \Im_{11}, \Im_{12}$ and we can perform the following numerical correspondences when we view the configurations indicated in columns D and T. Thus, we establish $\Im_{10} = 4 + 3 = 7 = P_8, \Im_{11} = 5 + 4 = P_9, \Im_{12} = 8 + 4 = P_{10}$.

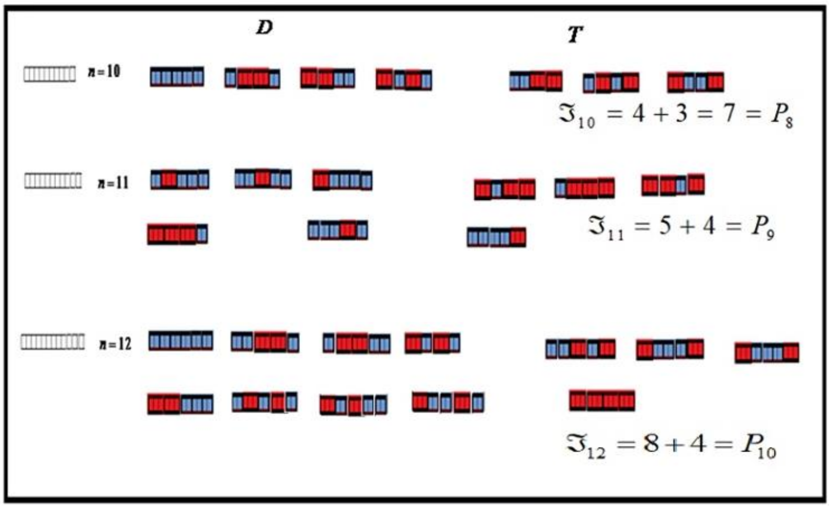


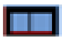
Figure 8. Description of cases based on $10 \leq n \leq 12$ model (Adapted from Tedford, 2019)

Table 1. Set of recurring numerical sequences & their corresponding formation rule

Sequence	Recurring rule	Numerical values
Fibonacci	$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1$	$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$
Tribonacci	$T_n = T_{n-1} + T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 2$	$0, 1, 1, 2, 4, 7, 13, 24, 44, \ldots$
Tetranacci	$Te_n = Te_{n-1} + Te_{n-2} + Te_{n-3} + Te_{n-4}, Te_0 = 0, Te_1 = 1, Te_2 = 2, Te_n = 4$	$0, 1, 1, 2, 4, 8, 15, 29, 56, 108, \ldots$
Pentanacci	$Pent_n = Te_{n-1} + Te_{n-2} + Te_{n-3} + Te_{n-4}$	$1, 1, 2, 4, 8, 16, 31, 61, 120, 236, \ldots$
Pell	$P_{n+1} = 2P_n + P_{n-1}, P_0 = 0,\ P_1 = 1$	0, 1, 2, 5, 12, 29, ...
Tri-Pell	$P_{n+2} = 2P_{n+1} + P_n +\ P_{n-1}, P_0 = 0,\ P_1 = 1,\ P_2 = 2$	$0, 1, 2, 5, 13, 33, 84, \ldots$
Jacobsthal	$J_{n+1} = J_n + 2J_{n-1}, J_0 = 0,\ J_1 = 1$	$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, \ldots$
Lucas	$L_{n+1} = L_n + L_{n-1}, L_0 = 0,\ L_1 = 3$	$1, 3, 4, 7, 11, 18, 29, 47, \ldots$
Padovan	$C_{n+1} = C_{n-1} + C_{n-2}, C_0 = 1,\ C_1 = 1,\ C_2 = 1$	$1, 1, 1, 2, 2, 3, 4, \ldots$
Perrin	$Q_{n+1} = Q_{n-1} + Q_{n-2}, Q_0 = 3,\ Q_1 = 0,\ Q_2 = 2$	$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, \ldots$
Tridovan	$Tri_{n+1} = Tri_n + Tri_{n-1} +\ Tri_{n-2}, Tri_0 = 0,\ Tri_1 = 1,\ Tri_2 = 0,\ Tri_3 = 1$	$0, 1, 0, 1, 1, 2, 2, 4, 5, 8, 11, 17, \ldots$ (Vieira & Alves, 2019)
Mersenne	$M_{n+2} = 3M_{n+1} - 2M_n, M_0 = 0,\ M_1 = 1$	$0, 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, \ldots$
Jacobsthal	$J_{n+1} = J_n + 2J_{n-1}, for\ initial\ values\ J_0 = 0,\ J_1 = 1$	$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, \ldots$
ꬵOresme	$O_{n+2} = O_{n+1} - (1/4)O_n, O_0 = 0,\ O_1 = 1/2$	$0, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \ldots$
Narayanna	$N_{n+1} = N_n + N_{n-2}, N_0 = 1,\ N_1 = 1$	$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, \ldots$
Leonardo	$Le_{n+1} = 2Le_n - Le_{n-2}, Le_0 = 1,\ Le_1 = 1$	$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, \ldots$
Ernst	$E_n = E_{n-1} + 2E_{n-2} + 1, E_0 = 0,\ E_1 = 1, n \geq 2$	$0, 1, 2, 5, 10, 21, 42, 85, \ldots$
Francois	$IF_n = IF_{n-1} + IF_{n-2} + 1, IF_0 = 2,\ IF_1 = 1$	$2, 1, 4, 6, 11, 18, 30, 49, 80, 130, 211, \ldots$

To verify the following theorem, we will define the terms p_n, which involves considering all the ways to fill the board with tiles indicated previously (see **Figure 7** and **Figure 8**). Therefore, when we consider an n-board, we can write that $p_n = |\Im_n|$.

Theorem 3: For every whole $n \geq 1$, the following relation holds $|\Im_n| = P_{n-2}$, where the numbers $\{P_{n-2}\}$ describe the Padovan sequence, defined by $P_{n+1} = P_{n-1} + P_{n-2}, P_0 = 1, P_1 = 1, P_2 = 1$ (Tedford, 2019).

Proof: We will consider the numerical values corresponding to the terms indicated in the sequence $\Im_1,\ \Im_2 . \Im_3,\ \Im_4 . \Im_5,\ \Im_6, ..$ and that, from the examples noted above, we find the relations $p_1 = |\Im_1| = 0 = P_{-1}, |\Im_2| = 1 = P_0, |\Im_3| = 1 = P_1, |\Im_4| = 1 = P_2, |\Im_5| = 2 = P_3$. We now observe that the numbers determined by the initial values coincide with the initial values of the Padovan sequence, at least of the order. Now, for $n \geq 3$, we will consider the decomposition of sets into two subsets that we can visualize in **Figure 7** and **Figure 8**, which we will designate as $\Im_n = D \cup T$. The first set D of the tiling that end up as one piece  (dominoes), and we can determine the following quantity $p_{n-2} = |D|$. The second set T of the tiling that end up as one piece  (triminoes), and we can write that $p_{n-3} = |T|$. We have that $p_n = |\Im_n| = |D| \cup |T| = p_{n-2} + p_{n-3}$ and that, from the initial values, such recurrence coincides precisely with the Padovan sequence, i.e., it is worth that $|\Im_n| = P_{n-2}$, for all $n \geq 1$ ■.

LITERATURE REVIEW: BOARDS OF OTHER NUMERICAL SEQUENCES

The diversity and other unexpected forms of tiling make it possible to establish a numerical correspondence with specific recurring numerical sequences. For example, in the case of the Jacobsthal sequence, whose recurrence relation we indicate in **Table 1**, we must consider a board of dimensions 3×n. In **Figure 9**, on the left side, we establish the tiles as white squares 1×1 and red squares 2×2, as described in Craveiro (2004).

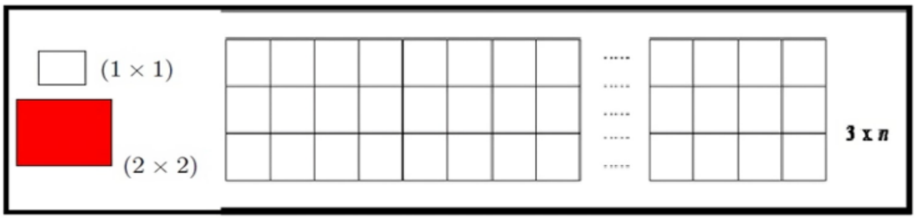


Figure 9. Viewing a $3 \times n$ board related to numerical values of Jacosbsthal sequenceAdapted from Craveiro, 2004)

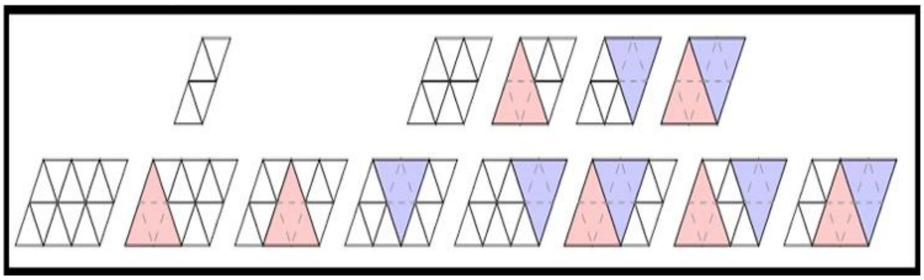


Figure 10. Representation of a triangular board (Adapted from Bodeen et al., 2014)

Next, we can verify a theorem established by Koshy (2019) and, some time later, by Craveiro (2004), considering the distinction between the ways to fill and the tiles chosen, and the different dimensions of each board.

Theorem 4: Considering a 3×n board, with only two types of tiles, a tile 1×1 in white  and a tile 2×2 in red . Then, j_n represents the number of possible tiles for the board and is determined by $j_n = J_n, n \geq 1$ (Koshy, 2019; Craveiro, 2004).

Bodeen et al. (2014) consider a board of triangular cells, as shown in **Figure 10**. From fixing four pieces , the authors describe the following recurring numerical sequence, starting from the following initial values: $I_n = I_{n-1} + 3I_{n-2} + I_{n-3}$, $I_1 = 1, I_2 = 4, I_3 = 8$. From the previous relationship, we can determine the numerical list: 1, 4, 8, 21, 49, 120, 288, 697, 1681, 4060, 9800, ... Theorem 5 expresses the results discussed by Bodeen et al. (2014).

Theorem 5: Denoting the element I_n that represents the ways of filling a triangular board composed of a double strip of the $2 \times n$ type and using triangular pieces of the  type. Thus, it is true that $I_n = I_{n-1} + 3I_{n-2} + I_{n-3}, I_1 = 1, I_2 = 4$, $I_3 = 8$ (Bodeen et al., 2014).

Furthermore, given a wide range of other recurring numerical sequences that we indicate in **Table 1**, we highlight the natural interest in the possibility of highlighting the description of new boards and determination of tilings that produce the desired numerical relationships that characterize the numerical sequences. However, we highlight some particular cases of sequences whose representation via the board cannot yet be found and have not been introduced in the scientific literature, such as the Narayanna and Leonardo sequences (Catarino & Borges, 2020). To illustrate, in **Table 1** we provide a simplified set of eleven recurring numerical sequences that, for the most part, constitute examples disregarded by history of mathematics books (Vieira et al., 2022). However, when we consider its combinatorial properties and relationship with the notion of Board, we can verify that not all numerical sequences indicated in **Table 1** have a corresponding representation, via Board and rules for the tiles.

Based on the recent work by Soykan (2022), when he presents (in a pioneering way) and defines a recursive relationship that produces a numerical set called, by the author himself, as Ernst numbers and which, despite being little detailed and/or explained in a specialized scientific article on pure mathematics, we infer a historical relationship with the German mathematician and physicist Ernst Eduard Kummer (1810-1893).

Definition 1: The Ernst number sequence is defined by the following recurrence relation $E_n = E_{n-1} + 2E_{n-2} + 1$, with the following initial numerical values indicated by $E_0 = 0, E_1 = 1, n \geq 2$ (Soykan, 2022).

Definition 2: A sequência numérica de François é definida pela seguinte relação de recorrência $IF_n = IF_{n-1} + IF_{n-2} + 1$, com os seguintes valores numéricos iniciais indicados por $IF_0 = 2, IF_1 = 1, n \geq 2$ (Diskaya et al., 2023).

A partir da definição 1, considerando a recorrência não homogênea (com a presença de constantes) indicada por $E_n = E_{n-1} + 2E_{n-2} + 1$, vamos considerar ainda que $E_{n-1} = E_{n-2} + 2E_{n-3} + 1$. Next, we will consider the following difference $E_n - E_{n-1} = (E_{n-1} + 2E_{n-2} + 1) - (E_{n-2} + 2E_{n-3} + 1) = E_{n-1} + 2E_{n-2} - E_{n-2} - 2E_{n-3} = E_{n-1} + E_{n-2} - 2E_{n-3}$. Therefore, we find that $E_n + E_{n-1} = E_{n-1} + E_{n-2} - 2E_{n-3}$ and then we will determine that $E_n = 2E_{n-1} + E_{n-2} - 2E_{n-3}$, that is, we find a homogeneous recurrence relation (without the presence of non-zero constants).

About the set of sequences indicated in **Table 1**, we recall the case of the Mersenne sequence, indicated by $M_{n+2} = 3M_{n+1} - 2M_n, M_0 = 0,\ M_1 = 1$. On the other hand, we recall Catarino et al.'s (2016) work, which employs some elementary properties of the number theory and indicates the following initial relationship for the Mersenne sequence, which we designate by $M_{n+1} = 2M_n + 1$. Furthermore, Chelgham and Boussayoud (2021) describe the k-Mersenne sequence through the relationship $M_{k,n+2} = 3kM_{k,n+1} - 2M_{k,n}, M_{k,0} = 0,\ M_{k,1} = 1$. Thus, from the relationship $M_{n+1} = 2M_n + 1$, with initial values $M_0 = 0, M_1 = 1$, let us

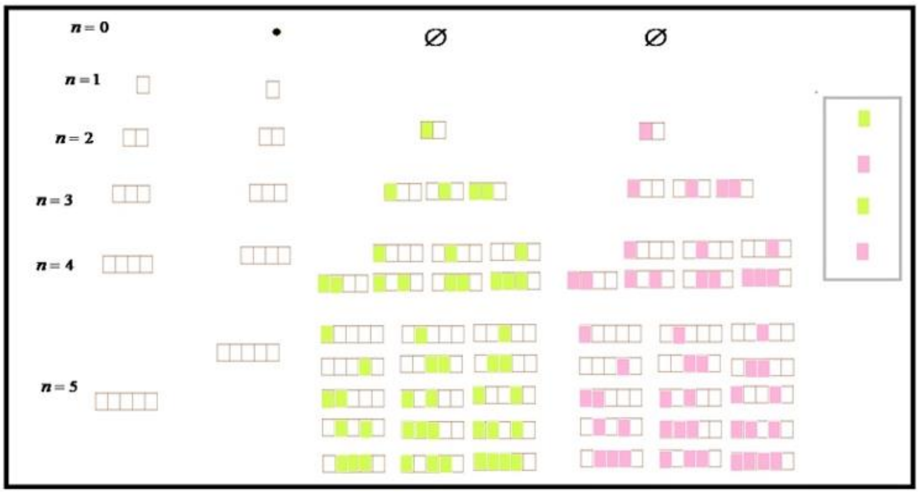


Figure 11. Determination of rules for using tiles corresponding to Mersenne sequence, according to previously defined rules (Source: Authors' own elaboration)

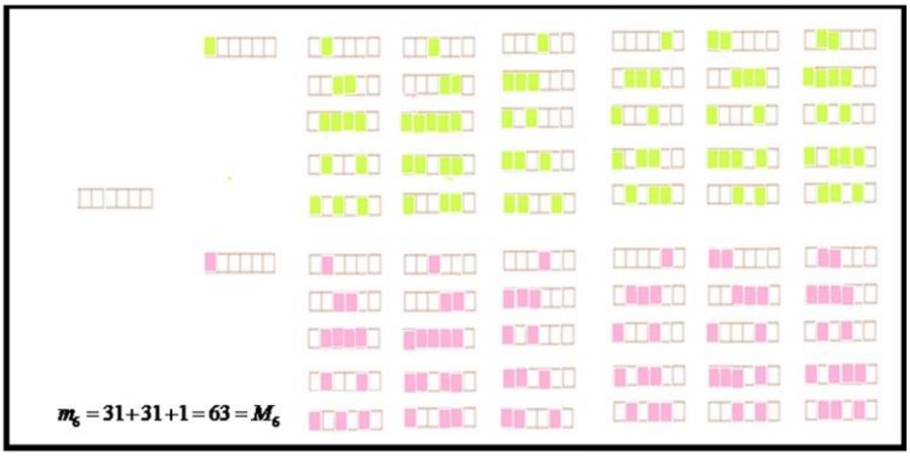
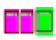


Figure 12. Determination of rules for using tiles corresponding to Mersenne sequence, n=6 according to previously defined rules (Source: Authors' own elaboration)

determine some rules aimed at filling out a 1×n board, with squares  (in pink) and squares  (in green color). In **Figure 11**, we exemplify the particular cases $n = 0, 1, 2, \ldots, 5$.

We will define the term m_n, which designates the total number of ways to fill a board of length n. For our correspondence, we will define $m_0 = 0$ and $m_1 = 1$. Furthermore, we will establish the following rules:

(a) in every tiling, there must be at least one square  (in white) in the highest order cell (i.e., always to the right of the board) and

(b) tiles that use green squares and pink squares at the same time cannot be used, for example, tiles of the type  or  cannot be employed.

Thus, with visual support from the diagram, we can verify that $m_0 = 0 = M_0$, $m_1 = 1 = M_1$, $m_2 = 3 = M_2$, $m_3 = 7 = M_3$, $m_4 = 15 = M_4$, $m_5 = 31 = M_5$, etc ...

In **Figure 12**, we bring another case $n = 6$ that corresponds to the following numeric values that correspond to the elements $m_6 = 31 + 31 + 1 = 63 = M_6$ determined by rule (a) and rule (b).

Based on the previous cases and emphasizing a visual verification of specific properties, we will establish the following theorem 6 and theorem 7, which corresponds to the board.

Theorem 6: Considering a board of order 1 x n, with squares  (in pink) and squares  (in green color). Considering that, in every tiling, there must be at least one square  (in white) in the highest order cell (i.e., always to the right of the board). Colored tiles of different colors cannot occur, and the term m_n designates the total number of ways to fill a board 1 x n of length n. Then $m_n = M_n$, for all $n \geq 0$ and the elements M_n satisfy the recurrence relation $M_{n+1} = 2M_n + 1$, with initial values $M_0 = 0, M_1 = 1$.

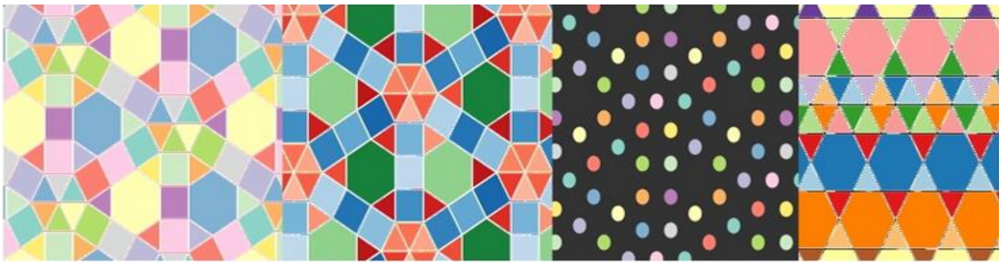


Figure 13. Unprecedented tiling patterns (Adapted from Sánchez, 2020)

Proof: Consider an n-order board and let us denote m_n the number of possible tilings of this board. According to the previously defined rule (a) and rule (b), considering that in every tile there must be at least one square (in white) in the highest order cell, when we fill it with only the tiles , we will have a total of $m_{n-1} = M_{n-1}$. Similarly, every tile must have at least one square (in white) in the highest order cell; when we fill it with only the tiles , we will have a total of $m_{n-1} = M_{n-1}$. Finally, in the case of only square (in white), we will have only one possibility, that is, resulting in a total of $m_n = m_{n-1} + m_{n-1} + 1 = M_{n-1} + M_{n-1} + 1 = 2M_{n-1} + 1 = M_n$, for every integer $n \geq 0$ ■.

Theorem 7: Considering a Board $1 \times n$ with the tiles: a green square $(1 \times n)$, a yellow domino $(1 \times n)$ a pink domino $(1 \times n)$,We will define the following rule: tiles made up only of restricted combinations of pink dominoes are disregarded. Defining the number e_n, which represents the number of tiles obtained by filling the Board, then $e_n = E_n, n \geq 0$.

Proof: Let's define the number e_n, which represents the number of tiles obtained when filling the Board, based on the rules and tiles defined as previously. Let's consider a table numbered with 'n' cells of the type $1 \times n$. Now let's consider two cases:

(a) suppose 'n' is even and considering fixing the tiles in the last positions of the Board, say in the positions $(n-2, n-1, n)$. Let's consider the case, where we fill with only green tiles , which is equivalent to +1. On the other hand, yellow dominoes (1×2), a pink domino (1×2),however, a tile with only pink dominoes is discarded. Fixing the dominoes in the last position $(n-1, n)$, we can count tiles in a quantity of e_{n-2}. The same argument applies to the case of pink dominoes with the domino fixed in the last position $(n-1, n)$. Thus, we will count a total of tiles $e_{n-1} + e_{n-2} + 1 = e_{n-1} + 2e_{n-2} + 1$.

(b) suppose 'n' is odd and considering fixing the tiles in the last positions of the Board, say in the positions $(n-2, n-1, n)$. Note that we automatically eliminate tiles completely filled with pink dominoes . In the case, where a green tile occurs in the last position and is fixed in position n, we will start counting in the form e_{n-1} for all tiles that end in a green square. When yellow dominoes occur (1×2), a pink domino (1×2), we will have a total amount of the $e_{n-2} + e_{n-2}$, with the positions fixed in $(n-1, n)$. In the case of a tile made up of green squares we will have the quantity $+1$. Again, we will find the total amount of $e_{n-1} + e_{n-2} + 1 = e_{n-1} + 2e_{n-2} + 1$. Finally, if we define that, for a Board with no cells, that is, $n = 0 \therefore e_0 = 0 = E_0$, an for the particular case $n = 1 \therefore e_1 = 1 = E_1$, we conclude that the total contribution of tiles coincides with the recurrence $E_n = E_{n-1} + E_{n-2} + 1$, for every integer $n \geq 0$ ■.

According to Sánchez (2020), periodic tilings and regular polygons have been studied since Archimedes (287 BC – 212 BC), which can be explored in the plane or on a sphere. The first attempt at a systematic study of tilings of the plane with regular polygons appears in Kepler's book, *Harmonices Mundi*, published 400 years ago (Sánchez, 2020, p. 1). Sanchéz (2020) discuss examples of flat periodic tilings and further explain that *tiling*, in mathematical terms, "is a subdivision of the plane into closed faces delimited topologically equivalent to a disk. We focus on periodic tilings whose faces are regular and periodic polygons."

Figure 13 shows examples of flat periodic tiling described, in an unprecedented way, in the doctoral thesis titled "*Sobre ladrilhamentos periódicos com polígonos regulares*" ["*On periodic tiling with regular polygons*"] and that the author describes a simple computational representation based on integers for periodic tiling of the plane with regular polygons using complex numbers, which constitutes the new state of the art for these objects. On the other hand, geometry problems consisting of covering a surface with a specific collection of tiling that is never repeated have intrigued mathematicians for decades!

FINAL CONSIDERATIONS

In the previous sections, we discussed properties and ways of representing recurring numerical sequences through the notion of board that come in different dimensions. We discussed the emergence and dissemination of the first works and some important books involving the combinatorial approach (Benjamin & Walton, 2009; Benjamin et al., 2011) and properties and representations of numerical sequences via boards using tiling. However, our work highlighted the importance of delimiting a historical and

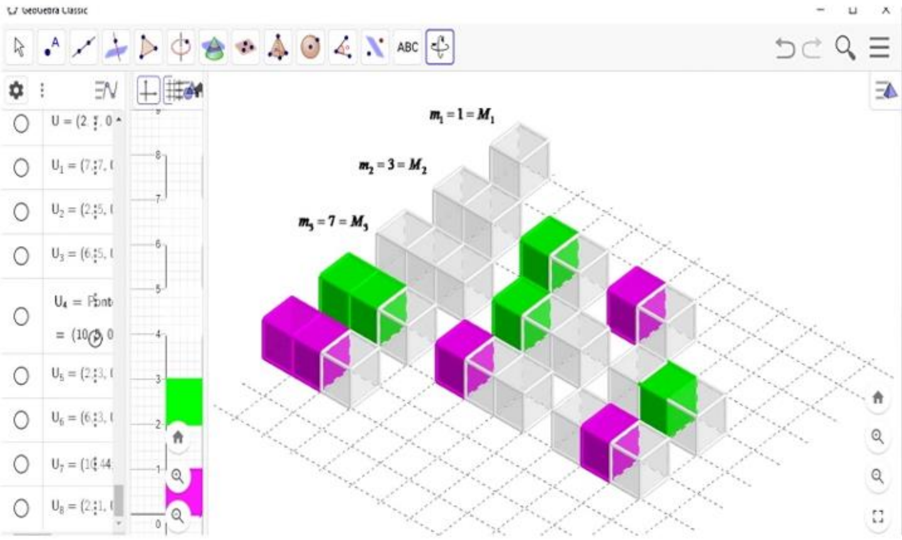


Figure 14. 3D view of tiling that correspond to numerical values of Mersenne sequence using GeoGebra software(Source: Authors' own elaboration)

evolutionary component (Bicknell-Johnson, 1987; Gould, 1991) that transmits strategic knowledge to the mathematics teacher (Alves, 2017, 2022).

Faced with an even larger set of examples of numerical sequences, as indicated in **Table 1**, we found that, in particular, the Fibonacci, Tribonacci, Pell, Jacobsthal, and Perrin sequences–and, more recently, Mersenne sequence and its representation via a 2D/3D board, which we introduced in this work–can indicate different learning itineraries for the mathematics teacher (**Figure 14**).

Finally, a perspective we incorporated in our work concerns understanding mathematical knowledge from a non-static and evolutionary perspective (Alves & Catarino, 2022). Thus, the examples we discussed demonstrate the vigor of contemporary research around combinatorial representations of sequences (Lagrange, 2013; Spivey, 2019), whose tradition and research in several countries have already accumulated at least two decades of tradition and consolidation of an area with broad interfaces with other branches of investigation, in addition to pure mathematics and its respective research in Brazil.

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