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An analysis of gifted students' proof and reasoning process based on the Toulmin model

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ABSTRACT

This study aims to examine the mathematical reasoning and proof processes of gifted students within the framework of Toulmin's Argumentation Model. Grounded in the Rich Teaching Model, the instructional design included seven differentiated lesson plans. The research was conducted with two 9th-grade students in a Science and Art Centre over 20 weeks during the 2022–2023 academic year. Employing a design-based research methodology and a qualitative case study approach, the study focused on a non-routine problem from the product phase of the first lesson plan. This task was examined in depth through micro-level case analysis. Data were analyzed using descriptive methods aligned with Toulmin's model components. Findings indicate that students used diverse representations, models, and notations to make sense of abstract mathematical concepts and integrated dynamic tools such as GeoGebra into their reasoning. Nonetheless, challenges with formal notation were observed. The study underscores the value of structured, technology-supported instruction in enhancing gifted learners' mathematical thinking.

Keywords: gifted students, Toulmin model, mathematical proof

INTRODUCTION

In the modern era of rapid development in scientific research and technological innovations, education systems need to implement fundamental reforms to adapt to these dynamics. This change aims to ensure that curricula not only facilitate the transfer of knowledge but also equip individuals with the 21st century skills required in this age (Kaptan & Kuşakçı, 2002). Gifted individuals are the driving force behind social progress and a critical factor shaping the developmental trajectory of countries. Therefore, the education of these individuals should be prioritized at both national and international levels (Davaslıgil, 2004). However, findings in the literature reveal that current curricula in Turkey do not adequately address the individual characteristics and learning needs of gifted students (Sak, 2011). This deficiency hinders students from developing higher-order thinking skills and fully utilizing their potential. Consequently, it is crucial to provide differentiated teaching strategies and flexible, enriched learning environments for these students (Çetinkaya & Erden, 2017). In these environments, differentiation and enrichment based on content, teaching methods, and products, which take into account students' readiness levels, will enable them to enhance both their academic achievement and their critical, creative, and analytical thinking skills (Porter, 2005).

The main purpose of designing learning environments is to enable students to use their complex thinking skills, research and inquiry habits, and scientific reasoning capacities effectively. Such environments encourage students not only to arrive at the correct solution, but also to structure the solution process, to justify it, and to critically question it. In particular, teacher-guided learning environments facilitate the development of students' geometric understanding and strengthen their mathematical reasoning processes (Baig & Halai, 2006). Geometric thinking is not limited to the recognition or definition of shapes; it requires understanding the relationships between shapes, developing spatial reasoning skills, and supporting these processes with logical justifications. When students work on geometric problems, they not only reach results but also construct, express, and defend mathematical arguments consisting of claims, justifications, and conclusions (Demiray et al., 2023; Pedemonte, 2007; Yackel & Hanna, 2003). This approach supports the development of both mathematical and geometric thinking skills through critical inquiry and structured discussions (Verallo & Cajandig, 2024). Therefore, this process, which goes beyond the linear transfer of knowledge, requires the conscious and systematic teaching of students' logical reasoning skills.

Reasoning skills allow individuals to structure their thoughts logically, support them with evidence, and reach consistent conclusions. In this context, Toulmin's argumentation model serves as an important framework that contributes to a structured presentation of the reasoning process. This model enables individuals to present data to support a claim, justify that claim with reasons, and reconstruct their ideas by refuting or accepting opposing views (Toulmin, 2003). Unlike traditional cause-effect relationships, Toulmin viewed the argumentation process as a structure composed of justified claims. The model he developed includes six basic components, each serving different functions: data, claim, warrant, qualifier, backing, and rebuttal (Mahdiyyah & Susanah, 2022). Data, the starting point of the model, is the concrete information on which the claims are based. The claim is the central idea supported by this data and reflects the individual's perspective on the discussion topic. Although the backing data primarily determines the credibility of a claim, a warrant clarifies the logical basis of the link between the data and the claim. Reasons underpin the argument by revealing how the process of reaching the claim unfolds. Qualifiers indicate the situations and conditions under which the validity of the claim can be maintained; in this regard, they reflect the scope and strength of the argument. Backing elements provide additional information or evidence that reinforces and supports the data and warrants. The final component, rebuttal, addresses the conditions under which the claim may be invalid or how opposing views can be managed (Pedemonte, 2007). Considering these components, the Toulmin model is considered to be a model that includes all structural components and has advantages in analyzing formal and informal arguments (Demiray et al., 2023; Faizah et al., 2021; Suartha et al., 2020). If the discussion process includes all of the mentioned components, the process is considered to be well managed (Zulainy et al., 2021). A visual example of these components of the Toulmin model is presented in **Figure 1** (Driver et al., 2000).

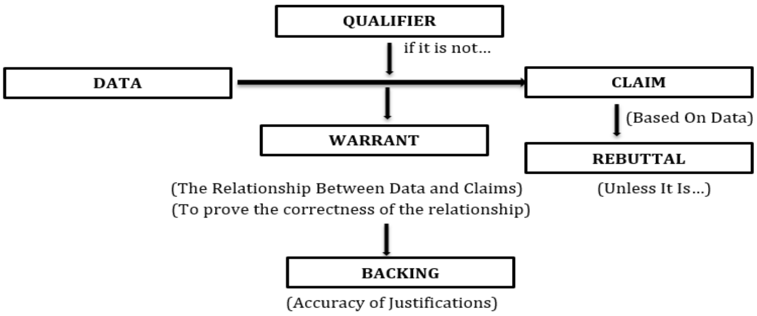


Figure 1. The Toulmin argumentation model (Driver et al., 2000)

The Toulmin Model, which allows students to develop their proof and mathematical reasoning skills systematically, stands out as an effective tool for the progression of geometric thinking (Toulmin, 2003). The components of this model prepare the ground for students to think systematically about geometric concepts and make logical inferences (Pedemonte, 2007). In this context, dynamic geometry software (GeoGebra, Cabri, Sketchpad, etc.) enable students to generate hypotheses, test these hypotheses, and justify mathematical relationships by visualizing them through interactive operations on geometric shapes (Mariotti, 2006). By bringing the argument structures presented by the Toulmin model into the digital environment, this software allows students to concretize their abstract mathematical ideas and conduct more in-depth analyses. The teaching process, structured with the Toulmin model and supported by dynamic software, contributes to the development of students' proof-making and mathematical and geometric thinking skills as a whole (Anderson-Pence & Moyer-Packenham, 2016).

Geometry software with dynamic features allows students to work effectively on abstract mathematical structures and adds a new dimension to the learning process. Thanks to this software, students can conduct research on given geometric situations, formulate hypotheses, test these hypotheses in various ways, reject or accept the hypotheses according to the results, and find the opportunity to develop new formulas and make explanations (Jones, 2000). In particular, steps such as analyzing the given information, defining appropriate justifications and hypotheses, organizing the intermediate steps in a logical order and reaching the conclusion, which are the basic components of the proof process, become more understandable thanks to the interactive environments offered by dynamic software. In this context, dynamic geometry software develops thinking skills by providing students with important opportunities in the transition between informal reasoning and formal proof (Mariotti, 2006). Dynamic geometry software provides multifaceted advantages such as concretizing abstract concepts, examining geometric shapes from different angles, making various drawings, sizing shapes and performing interactive operations by dragging. Therefore, dynamic software both contributes to a better understanding of concepts and increases students' participation in problem-solving and proof activities (Altun, 2010).

In geometry teaching, there are many studies based on various theoretical foundations and conducted with different teaching methods. However, when the existing literature is examined, it is seen that there is a limited number of studies focusing on proof and reasoning processes, and there is a significant gap in this field (Demiray et al., 2023; Kozlowski & Chamberlin, 2019; Pitta-Pantazi et al., 2011; Stylianides et al., 2023). However, the proof process not only tests the accuracy of mathematical expressions, but also supports the development of higher-order thinking skills such as explaining, discovering new concepts, drawing conclusions from definitions, thinking systematically, communicating, and reconstructing known knowledge (Sriraman et al., 2013; Yackel & Hanna, 2003). The basis of these skills is mathematical reasoning. Mathematical reasoning and geometry learning stand out as holistic structures that mutually support one another (Kilpatrick et al., 2001). In this framework, teaching approaches that use technological tools effectively and create meaningful learning environments based on reasoning have special importance in geometry education (Bergqvist & Norqvist, 2022; Özçakır et al.,, 2020). In this context, geometry stands out as one of the disciplines where gifted students can use their reasoning skills most intensively (Hollebrands, 2007; Sriraman et al., 2013).

It is noteworthy that the number of argumentation-based approaches focusing on proof and reasoning processes in geometry teaching for gifted students is insufficient (Demiray et al., 2023; Kozlowski & Chamberlin, 2019; Küçük-Demir, 2014; Pitta-Pantazi et al., 2011; Stylianides et al., 2023; Ysseldyke et al., 2004). Some existing studies have examined students' argumentation skills within the framework of the Toulmin Argumentation Model to analyze their thinking processes (Dinçer, 2011; Le Roux et al., 2004; Sampson et al., 2011; Stephan & Rasmussen, 2002; Urhan & Bülbül, 2016). This model effectively develops students' ability to justify and defend their claims, providing structured explanations within a logical context (Conner & Singletary, 2021; McNeill et al., 2006). In this context, recent international studies have made the role of argumentation in mathematics teaching more visible. In this context, various studies conducted in recent years have made the place of the Toulmin Argumentation Model in the mathematics teaching process more visible. For example, in the study conducted by Pramesti and Rosyidi (2020), students' responses were analyzed within the framework of the Toulmin argumentation model and the findings revealed that students were able to formulate their claims by associating data with appropriate justifications. In the study conducted by Umah et al. (2016), students' argumentation structures were examined using the Toulmin argumentation model and it was determined that these structures mostly consisted of data, claim, justification and support components, while refutation or qualifying elements were rarely included. The distinctive aspect of this study is that it analyzed students' arguments in geometric proofs using the Toulmin model and then classified their geometric argumentation skills at different levels. When the studies in the literature are examined, most of these studies are limited to pre-service teachers or general student groups and there are very few studies focusing on gifted individuals (Aydoğdu & Keşan, 2016; Aygün, 2019; Dinamit, 2020; Eraky et al., 2022; Kanbur Teker & Argün, 2022; Lee, 2005; Sriraman, 2005; Vatandaş, 2022; Ysseldyke et al., 2004). This suggests the need for more research in mathematics education to address the needs of gifted students. In particular, García-Martínez et al.'s (2021) systematic review highlights that educational interventions for gifted students worldwide are limited and that more individualized and high-quality research is required in this area. Similarly, Trpin (2024) highlights the importance of differentiated instructional methods and the use of digital tools to improve the mathematical abilities of gifted students. These findings reveal the necessity for further research focusing on the mathematical proof and reasoning processes of gifted students. Considering these needs, the Rich Teaching Model developed for gifted students was designed using GeoGebra software based on international differentiation approaches and implemented in the 9^{th} grade mathematics course "Triangles" unit. The students participated in the analysis and argumentation processes through the non-routine problem named "Dronefest Final", and the arguments they formed were analyzed within the framework of the Toulmin Argumentation Model. This case study provides a data-based response to the question "How can the proof and reasoning processes of gifted students be analyzed according to the Toulmin Model?" and at the same time presents a unique model of how argumentation-based instruction can be adapted to this student group. In this context, in addition to supporting the development of higher-order thinking skills of gifted students, the study provides a theoretical and practical basis for argumentation-based mathematics teaching.

METHOD

In this section, the research model, study group, data collection tool, data analysis, preparation of the teaching module, and validity and reliability studies are discussed in detail.

Research Model

The design of this research involves the implementation and evaluation of an instructional model developed to enhance the mathematical proof and reasoning skills of gifted students. The study is structured within the framework of the Design-Based Research (DBR) approach; the development process of the model follows a cyclical structure that includes the stages of analysis, design, and redesign (Wang & Hannafin, 2005). The Rich Teaching Model, created through this approach, is designed to support the high-level cognitive skills of gifted individuals. The model is analyzed not only at the instructional planning level but also at the micro level, based on classroom interactions. During the data collection process, the study employed the "case study" method, a qualitative research design. This method provides context-specific, in-depth data by allowing for a detailed examination of a specific phenomenon (Creswell, 2016). The reason for choosing a case study is to gather multi-layered data on the argumentation-based learning processes of gifted students. Thus, in this study, a non-routine problem in the product phase of one of these lesson plans is examined in detail at the micro level using a case-based analysis method. The research, structured this way, contributes to the design of innovative learning environments for gifted students based on both theoretical and practical foundations; it also reveals original findings on the effectiveness of argumentation-based approaches in mathematics teaching. Accordingly, the research process is planned and implemented within the framework of a three-stage design cycle as shown in **Figure 2**.

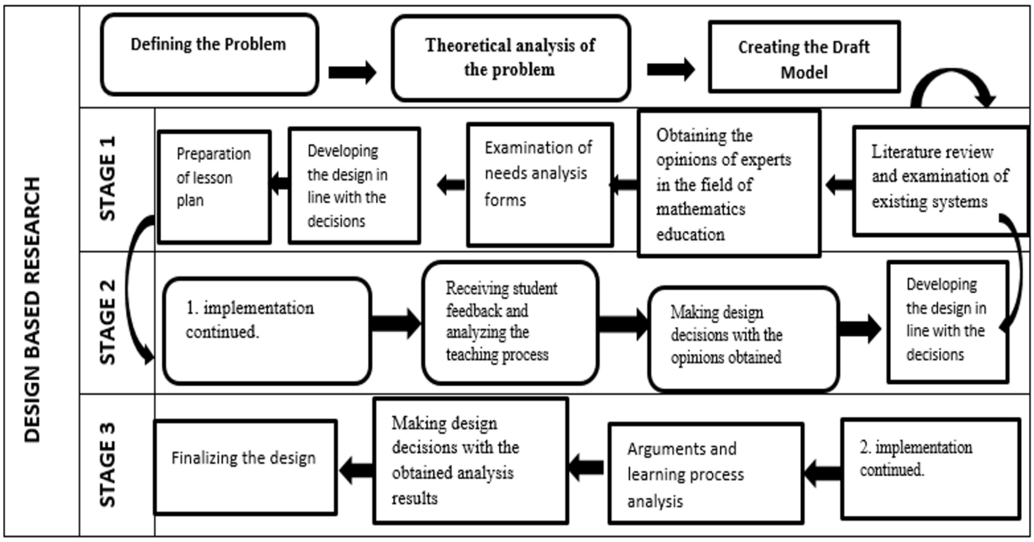


Figure 2. Flowchart of the research process (Source: Authors' own elaboration)

During the design process of the Rich Teaching Model, textbooks for gifted education, academic studies in national and international literature, and differentiation approaches recommended for gifted individuals were examined in a multidimensional manner. The model aims to provide gifted students with in-depth learning experiences, enhance their comprehension of abstract mathematical concepts, and foster high-level reasoning and proof skills. The developed model comprises four main components:

1) Outcome identification,
2) Content design,
3) Structuring the learning process, and
4) Product development.

In the learning outcome identification stage, the basic learning outcomes for the 9th grade "Triangles" topic were identified and restructured using Bloom's Revised Taxonomy (Anderson & Krathwohl, 2001) to align with higher cognitive levels. In the content creation process, based on the Parallel Education Model (Tomlinson et al., 2002), Pardue Model (Feldhusen & Kolloff, 1986), Grid Model (Betts & Kercher, 1999), and Curriculum Compression Model (Reis & Renzulli, 2010), the content was restructured through the dimensions of reorganizing the general scope, simplifying, linking concepts, transforming them into practice, and creating awareness. This process presented the content in a multi-layered structure, taking into account students' individual learning speeds, interests, and readiness levels. When structuring the learning process, the Pardue and Grid models were utilized to develop technology-supported learning activities designed to enhance students' basic mathematical skills, as well as their research, questioning, and creative thinking abilities. During the product development phase, the Maker Model (Maker, 1982) served as a foundation, and applications and design activities were created to enable students to transform the knowledge they learned into functional and creative products. Hence, the emphasis was placed on developing students' ability to apply what they have learned to real-life problems. In assessing all these processes, both process-oriented and result-oriented measurement and evaluation approaches were employed. The fundamental components of the model and its implementation structure are visually presented in **Figure 3**.

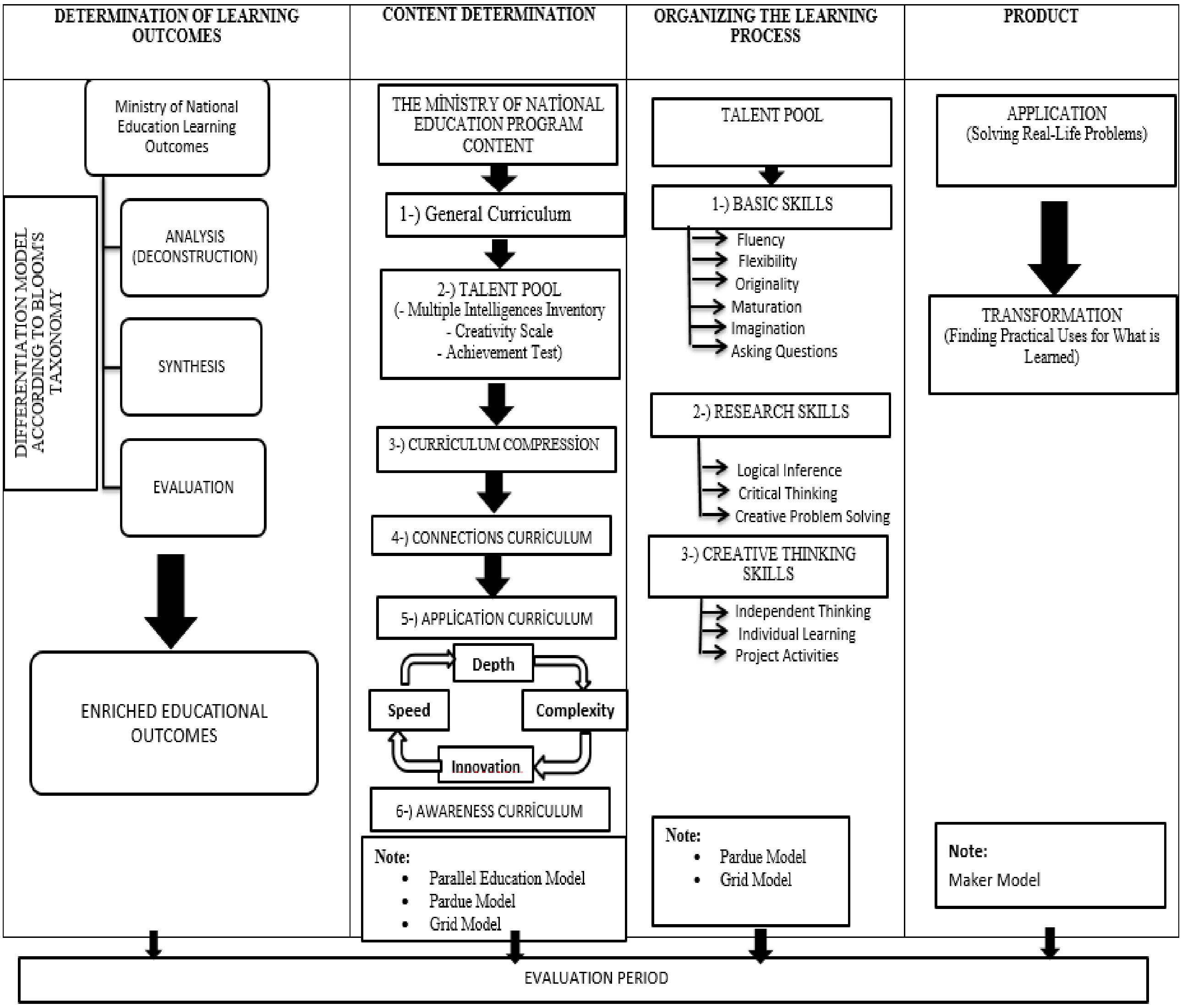


Figure 3. The rich teaching model (Source: Authors' own elaboration)

Within the scope of the Rich Teaching Model developed in this study, seven lesson plans were created by differentiated instructional approaches for gifted students. These lesson plans were designed to develop students' higher-order thinking, reasoning and proof skills.

Working Group

The research data were collected from two gifted ninth-grade students during a 20-week implementation period in the fall and spring semesters of the 2022 - 2023 academic year. The study was planned to be conducted for two class hours each week and was implemented for an average of 2 class hours (80 minutes). In some weeks, the duration increased to 3 class hours (120 minutes) due to additional activities. In this way, the implementation period was realized as a minimum of 40 hours and a maximum of 60 hours. The weekly content and duration information regarding the implementation process is presented in **Table 1**.

Table 1. Weekly implementation plan and duration information

Week range	Main theme	Mathematics subjects	Average time (min)	Activity type
1–3	GeoGebra Applications	Triangles	80–120	Introduction, GeoGebra applications
4–5	Problem 1: Dronefest Final	Basic Concepts of Triangles	80–120	Introduction of triangles, implementation of the 1st plan
6–7	Problem 2: The Peak of Two Mountains	Congruence and Similarity in Triangles	80–120	Studies of congruence and similarity, implementation of the 2nd plan
8–9	Problem 3: Suspension Bridge	Right Triangle and Trigonometry	80–120	Triangle types and trigonometry studies, implementation of the 3rd plan
10–11	Problem 4: Wall Painting	Auxiliary Elements of Triangles	80–120	Measurement, drawing and analysis studies, implementation of the 4th plan
12–14	Problem 5: Table Lamp	Trigonometric Ratios of acute angles in a right triangle	80–120	Measurement, ratio calculations, and implementation of the 5th plan
15–17	Problem 6: Ferris Wheel	Unit Circle and Trigonometric Ratios	80–120	Modelling with dynamic geometry, implementation of the 6th plan
18–20	Problem 7: Water Distribution	Area of a Triangle and Applications	80–120	Use of area formulas in problem solving application of the 7th plan

The participants were voluntarily selected from among the students at the Science and Art Centre where the researcher works by using the criterion-based purposive sampling method. This method focuses on including individuals who meet specific criteria, which the researcher may create to gather in-depth and detailed information or may rely on a pre-prepared list of criteria (Yıldırım & Şimşek, 2013). In this study, various factors were taken into account when determining the research group. Participant selection was based on having the technological infrastructure to work in dynamic geometry environments, voluntary participation in the study, and attendance at the science and art centre with which the researcher is affiliated. Pseudonyms were used to protect the privacy of the participating students; they were coded as "Francis (F)" and "Okan (O)." The researcher's teacher was identified as "T" in the data.

Data Collection Tool

In the study, a non-routine problem, video recordings of participants' argumentation processes, students' worksheets and researcher observation notes were used as data collection tools. These multiple data sources provided the opportunity to analyze students' problem-solving processes and argumentation structures in depth. The non-routine problem used in the study was chosen in a way to allow observation of students' reasoning, proof and argumentation processes. Non-routine problems are types of problems that require higher-order thinking skills where the solution path is not clearly defined, more than one strategy can be used, and a previously known formula or algorithm cannot be directly applied (Hiebert & Grouws, 2007). In this context, the main problem used to observe the components of the Toulmin Discussion Model was the "Dronefest Finale Problem", which was included in the first lesson plan developed and adapted from the Ministry of National Education Modelling Book and constructed in the context of daily life based on the topic of the triangle inequality. The problem used in the research is shown in **Figure 4**.

Figure 4. Problem 1: Dronefest final event (MoNE, 2017)

As seen in the figure, various competitions are organized for remotely controlled vehicles known as "drones". Within the scope of Dronefest, one of these competitions, information on the competition course and competition rules for the two teams that made it to the finals is presented below:

- The competition area consists of a triangular-shaped course.
- Each team participates with three drones.

- Each drone must pick up its payload from its starting location and deliver it to the next designated point by following the shortest possible path.
- The drones can either be positioned individually at each vertex of the triangle, with each directed toward a different vertex, or be placed at an arbitrary point inside the triangle, from which they will deliver their payloads to the three distinct vertices.
- At the end of the competition, the total distance travelled by the three drones of each team is calculated and used for evaluation.

According to the rules of the competition, the team that covers the longest total distance will be the winner. Accordingly, in order for the teams to succeed, they need to develop a strategy to maximize the total distance to be covered and determine the appropriate routes.

The problem focused on in this study was related to the "Triangles" topic in the 9th grade Mathematics course and required students to produce solutions by using triangle inequality, angle-edge relationships and different triangle properties; in this process, they needed to demonstrate their higher-order cognitive skills such as justification, proof and argumentation. In this context, the Toulmin Argumentation Model was used to help students systematically express their mathematical ideas about triangles (Toulmin, 2003). GeoGebra software was also utilized in problem-solving, which enabled students to concretize their arguments by dynamically examining triangle structures. The combined use of the Toulmin Model and GeoGebra contributed to students' in-depth development of proof and reasoning processes based on triangles (Herbst & Brach, 2006; Mariotti, 2006).

Data Analysis

The data in the study were collected to examine in depth the reasoning and proof processes of gifted students while solving a non-routine problem situation related to the topic of triangles. The research was carried out during the implementation process planned as part of a 20-week-long teaching process. In order to analyze the argumentation-based thinking structures of gifted students in depth among the various activities included in the comprehensive teaching plans, only the implementation based on the Dronefest Final problem situation is discussed in detail in this article. The application was carried out in a small group format with two students working together in a computer environment supported by Dynamic Geometry Software (DGS). Effective communication between the participants and the researcher contributed to both the preservation of data integrity and a detailed analysis of the students' cognitive processes. Throughout the research, the students' mathematical reasoning and proof skills were analyzed comprehensively using qualitative methods. The Toulmin Argumentation Model was utilized to analyze and interpret the collected data, which was also examined using descriptive analysis methods. During the argumentation process, students were encouraged to collaboratively consider the problem, develop alternative solutions, and discuss these solutions within the framework of group work. There were no time constraints during the implementation. As the students navigated the process, the researcher maintained a facilitative position, interacting only when necessary. The entire process was documented through video recording, and the students' arguments during problem solving were transcribed verbatim. Student statements were captured in their natural form without any editing or interpretation. This qualitative data was structured for analysis within the context of claims, data, warrants, backing, rebuttals, and conclusions, which are the foundational elements of the Toulmin Argument Model. Consequently, the students' reasoning processes were analyzed on multiple levels, and the findings were interpreted. This analytical process illuminated the intellectual development of gifted students in geometry, particularly about non-routine problems related to triangles, and demonstrated how the Toulmin Model provides a functional framework for structuring student thinking.

Validity and Reliability

In qualitative research, validity and reliability are the basic criteria for ensuring the accuracy and generalizability of the findings (Yıldırım & Şimşek, 2013). In this study, Lincoln and Guba's (1986) qualitative research-specific reliability and validity criteria of credibility, transferability, dependability and confirmability were taken as a basis, and various methodological strategies were used to support these elements. Dependability is not merely the reproducibility of the findings, but rather the internal harmony of the conclusions reached based on the data obtained (Merriam, 2013). In this direction, the research question was clearly defined; the role of the researcher, participant selection, content structure, data collection process and learning environment were constructed in integrity with the findings. In addition, the findings were compared with expert opinions to ensure consistency with the interpretations. Confirmability refers to the traceability and controllability of the research process (Yıldırım & Şimşek, 2013). In this context, data collection, analysis and interpretation stages were reported systematically. The codes created during the coding process were independently examined by an expert in the field, and the reliability was calculated as 85% using Miles and Huberman's (1994) consensus rate formula. This rate is above the 70% threshold value accepted in qualitative research and shows the consistency of the analysis process. Thus, it was accepted that the study could reach similar results by independent experts. Concerning the credibility, in qualitative research, strategies such as in-depth data collection, long-term observation, participant verification and expert opinion stand out (Yıldırım & Şimşek, 2013). In this study, screen and audio recordings were taken throughout the 20-week implementation process to prevent data loss; the researcher directly observed the process and the data obtained were analyzed immediately. In addition, the accuracy and validity of the comments were reinforced with the regular contributions of an academic expert in the field. Transferability refers to the adaptability of the findings to similar situations. In this study, participants who met the criteria were selected, the data obtained were presented in a clear, thematic and systematic way, and the participant statements were directly included. This approach aims to enable the reader to understand the context of the study and adapt the findings to their circumstances.

FINDINGS

In this section, various findings related to the problem question of the study are presented. The findings are presented under the following subheadings: Francis and Okan's argumentation process analysis, the representation of the argumentation using the Toulmin diagram, the analysis of the argumentation process, the results obtained from mathematical arguments, and the argumentation process analysis steps related to the problem.

Analysis of Francis' (F) and Okan's (O) Argumentation Processes

Under this subheading, students' mathematical argumentation was examined in detail within the framework of the elements of Toulmin's model.

T: Yes, what are your thoughts on this question? How might we solve it?

F: I've solved similar problems before, and I'm quite familiar with the topic, teacher. I think we first need to sketch the course to better understand the problem. *(Data)*

T: That's a good idea. How did you come to that? What was your reasoning? *(Teacher's supportive prompt)*

O: Since we're going to draw the course as a triangle, let's first look at the general form of the problem, then we'll fly the drones and navigate them accordingly. From what I understand, there are two teams: one will fly along the edges, and the other from a point inside the course. *(Data)*

F: Right, but it says above that each drone must carry the load from its starting point to the next point via the shortest distance. So we need to choose a point in the middle such that the distances are minimized. I think our solution plan should be based on that. *(Data)*

O: We can work on the edges later, but how do we decide on a point within the triangle first?

F: Should we choose the centroid of the triangle, or place the point in a way that makes the path shorter? Maybe we should try measuring?

O: I can construct an equilateral triangle and find the point. I can adjust it using GeoGebra. I believe that in this triangular course, the route inside the region for the drones could be shorter than the edge routes. (Claim)

The solution of O's mathematical claim using GeoGebra is shown in **Figure 5**.

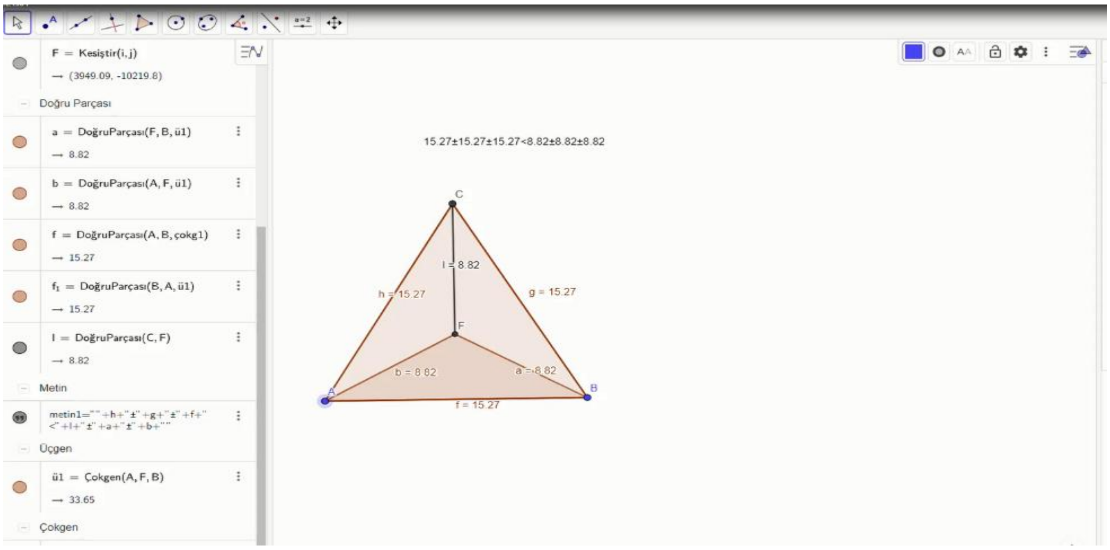


Figure 5. Solution of O's mathematical claim using GeoGebra (Source: Authors' own elaboration)

O: The center point I have chosen in the equilateral triangle is the common point of bisectors, medians, and elevations; it is also the center of the inner tangent circle. Therefore, the drone flying from this point takes the shortest path. I wrote the inequality above. (Warrant)

F: So you're saying the total distance travelled by the drones flying on the outside is less than the one flying inside? I don't think that's correct. (Rebuttal)

O: I wrote that part incorrectly; it should be the other way around. That is, the total path of the drones flying on the outside should be longer than that of the ones flying inside. So yes, that's the path I'm considering. What do you think?

F: That might be true, but what if the triangle isn't equilateral? That's where I'm stuck. (Qualifier)

F: Since we can use a circle when constructing the triangle, we could find the central point that way, too. Should we draw an incircle or a circumcircle? I don't think it's the incircle since we want to find a central point that encompasses the whole triangle, let's draw the circumcircle. I also think that, in the course, the drone's path inside the triangle may be shorter than or equal to the paths along the edges. (Claim)

T: Yes, that might be a much better approach. Go ahead and draw it. (Affirmation and guidance)

Solution of F's Mathematical Argument Using GeoGebra is shown in **Figure 6**.

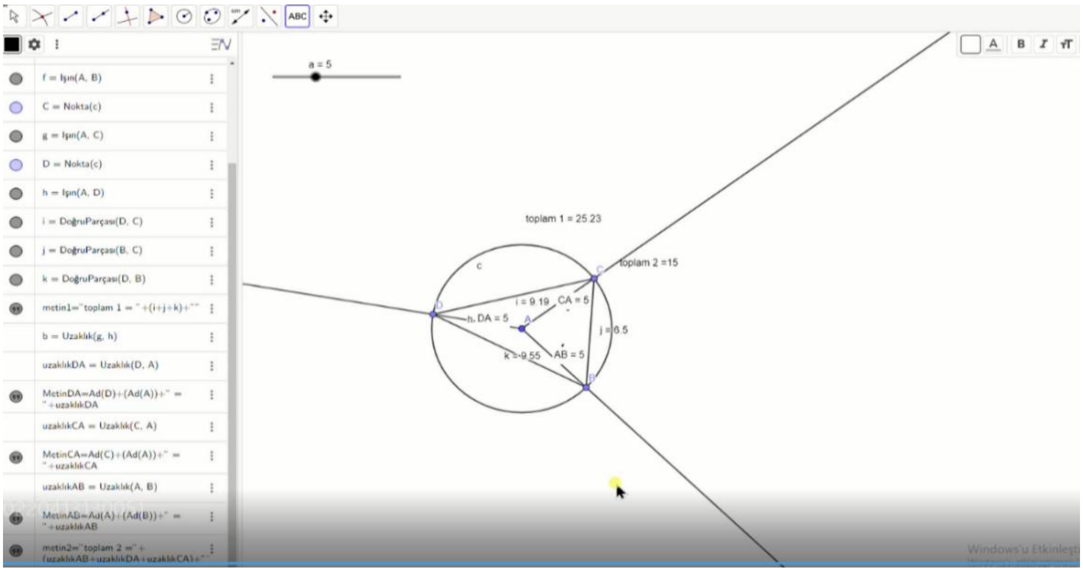


Figure 6. Solution of F's mathematical claim using GeoGebra (Source: Authors' own elaboration)

F: We set the distance from the selected central point A to each vertex as 5 cm, so if the total distance travelled by the drone inside is 15 cm, the distance along the edges turns out to be 25.23 cm. Therefore, the internal path is shorter. (Warrant)

O: That's the same result I found as well.

F: I moved the point around a bit earlier. This dynamic environment makes it so much easier; it's great, teacher. So I didn't just stick with point A. Subsequently, it was found that the sum of the lengths of any straight line from various points inside the triangle to its vertices was always less than the perimeter of the triangle. I tried it in every triangle and with every point. (Warrant)

O: I wonder if this could become or already is a general rule? We should research it.

F: Let's research it—I think such a rule must exist. We could also ask our teacher.

T: If you were to generalize this, what would you say? I think you're making great progress. (Affirmation and guidance)

O: I think we can formulate a generalization. This is easy to express as a formula. Let's say: "In a triangle ABC with side lengths a, b, and c, if we take a point within the triangle and call its distances to the triangle's vertices k, l, and m, then k + l + m is less than a + b + c. In other words, a + b + c > k + l + m." (Claim)

F: That's a pretty solid generalization you've come up with 😊 (Affirmation and guidance)

T: I agree you've arrived at a nice generalization. But there's an additional layer to it that you haven't noticed yet, which is normal. There's a known theorem called the Fermat–Torricelli Point Theorem. It states: "In a triangle ABC with side lengths a, b, and c, if a point P is taken inside the triangle and its distances to the vertices are x, y, and z, then the sum x + y + z is less than a + b + c (the perimeter), and greater than half of that. So if the perimeter a + b + c is denoted as 2u, then the inequality u < x + y + z < 2u holds." (Reference-based guidance)

F: When I checked, the total distance from the inside point was 15, and the perimeter of the triangle was around 25 cm. So it fits: 12.5 < 15 < 25. We could probably try many more examples. This is what I like most about GeoGebra. (Warrant)

O: Same here, I just checked, and it always turns out like that. But I think we could prove this beyond examples, too, Francis. Like, let me show you something in GeoGebra. I've got an idea. We could approach it that way. I mean, I could apply the inequality rule to every triangle. That could be another path to explore, teacher, what do you think? (Warrant)

The solution of Analysing the Reasoning of O's Argument Using GeoGebra is shown in **Figure 7**.

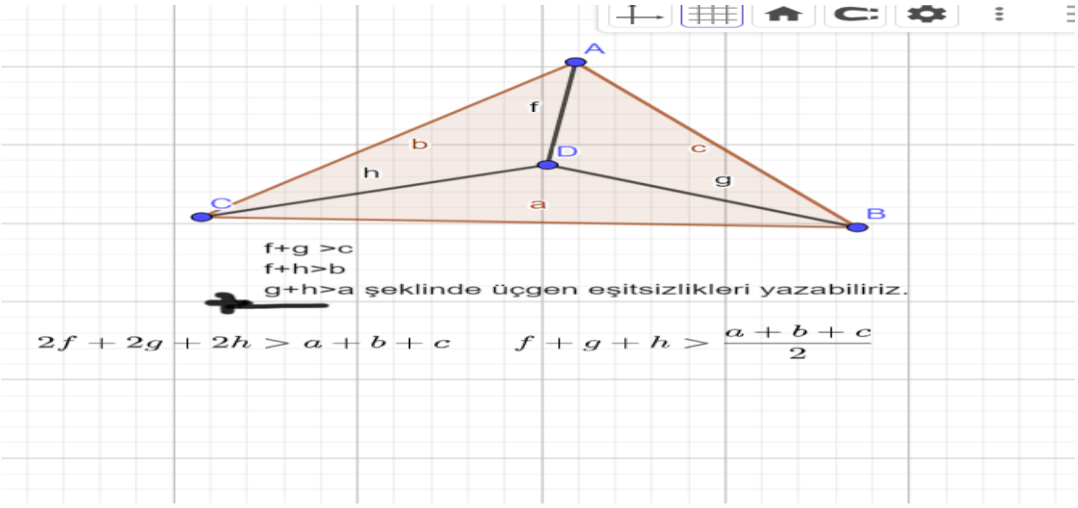


Figure 7. Analyzing the warrant of O's argument using GeoGebra (Source: Authors' own elaboration)

Ö: Very well done, children. (Concluding guidance support)

Analysis of the Discussion Process

The mathematical argumentation conducted by Okan and Francis during the discussion process was analyzed through specific codes. This analysis reveals how the students structured the stages of claim formation, justification, rebuttal, and conclusion.

- Codes 2, 4, and 5: In these codes, students referred to various data to support their emerging claims. Based on this information, they discussed how the course (track) should be constructed and the possible drone trajectories. As a result, the foundation for the discussion was gradually structured based on the students' statements.
- Codes 8 and 13: Analysis of these codes shows that Okan and Francis developed specific claims grounded in data. Both students, with similar reasoning, asserted that the drone routes within the inner part of the track could be shorter than or equal to those along the triangle's edges. These claims were further strengthened by the generalization proposed by Okan in Code 21.
- Codes 9, 15, 17,24 and 25: Okan and Francis made mathematical justifications through the solution paths they created using GeoGebra software; they started the argumentation process with an abductive (inferential) approach at the beginning and then proceeded with deductive reasoning. This indicates that the students systematically constructed their justifications and worked toward validating their claims through structured argumentation.
- Codes 3,14, 20 and 22: Teacher interventions in these sections were carried out for guidance. Positive reinforcement and instructional prompts were provided to support the students in substantiating their justifications and to maintain the constructive flow of discussion. The teacher offered affirmative guidance and support.
- Code 12: Francis presented a critical evaluation of Okan's solution. He emphasized that the fact that Okan's approach is limited to the equilateral triangle may limit the generalizability of the result.
- Code 10: Francis questioned the reasoning behind Okan's claim and attempted to refute it by indicating potential errors in calculating drone path lengths.
- Code 23: Within this code, the teacher introduced a scientific context to the discussion by referencing the Fermat–Torricelli point theorem, thereby providing the students with a conceptual anchor. This constituted reference-based guidance support.
- Code 25: Students elaborated on their arguments through the use of GeoGebra software and advanced their proof processes with mathematical expressions and reasoning.
- Code 26: The final stage of the discussion was marked by the teacher's comment, *"Very good, children,"* signalling the end of the session. This was interpreted as concluding guidance support.

Table 2 systematically presents the structure of the discussion conducted by Francis and Okan, offering a comprehensive perspective on how the students utilized their mathematical thinking, reasoning, and proof skills.

Table 2. Argumentation process of Francis and Okan within the framework of the Toulmin argumentation model

Component	Content
Claim	The distance is covered by drones. Moving along the inner track may be shorter than or equal to that of the drones on the outer edge. (Codes: 8, 13,21)
Data	The course planning was carried out, and distance measurements were analyzed based on drawings created using GeoGebra software. (Codes: 2, 4, 5)
Warrant	The presence of diagonal and side symmetries in the structure of an equilateral triangle supports distance equalities. (Codes: 9, 15,17,24,25)
Backing	Numerical and visual data obtained from the GeoGebra application; teacher's guidance and directions. (Codes: 3,14, 20, 22,23)
Rebuttal	Francis argued that the solution using only equilateral triangles cannot be generalized and that there are distance calculation errors in the solution. (Codes: 10)
Qualifier	This inference is valid only under specific geometric conditions (e.g., equilateral triangle). (Code: 12)
Conclusion	The teacher positively reinforced the process with the statement "Very good, children," and ended the discussion. (Code: 26)

In **Figure 8**, the summary of the discussion conducted by Francis and Okan is presented systematically in a Toulmin diagram.

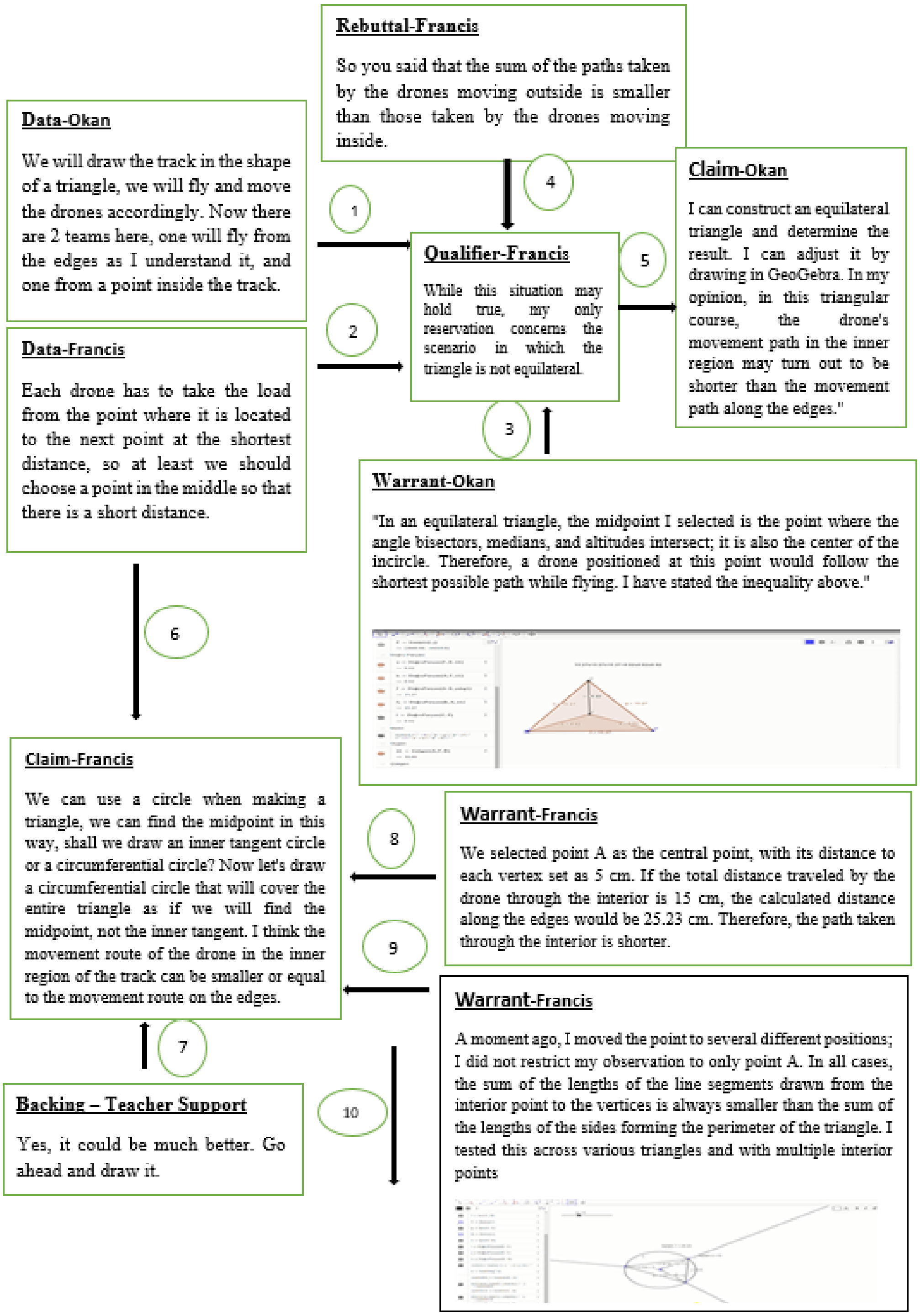


Figure 8. Representation of the discussion through the Toulmin diagram (Source: Authors' own elaboration)

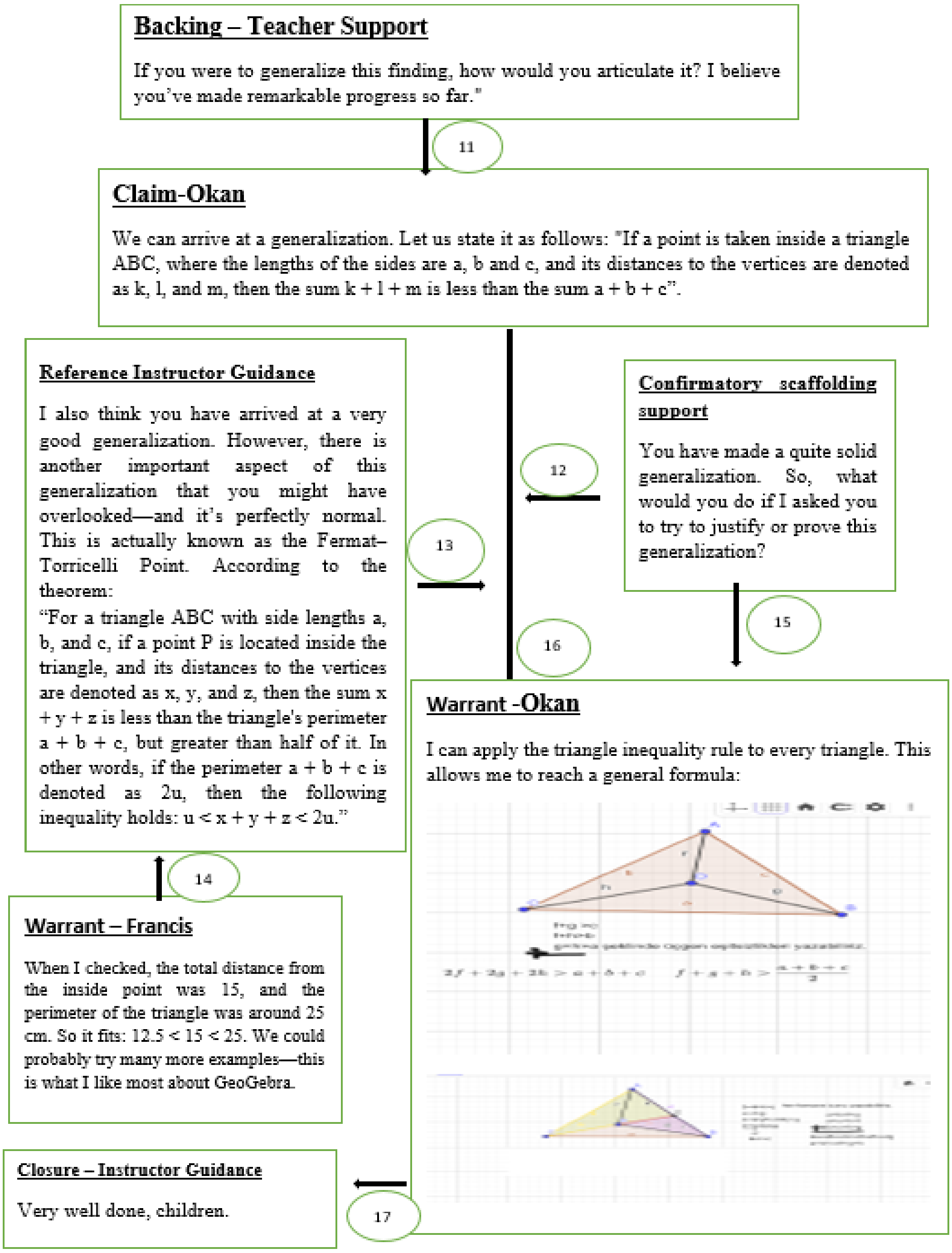


Figure 8 (Continued). Representation of the discussion through the Toulmin diagram (Source: Authors' own elaboration)

Findings Derived from Mathematical Arguments

The mathematical arguments related to the students' proof and reasoning processes in line with the Toulmin Argumentation Model and the findings obtained from the detailed analysis of these arguments are presented as a flowchart in **Figure 9**.

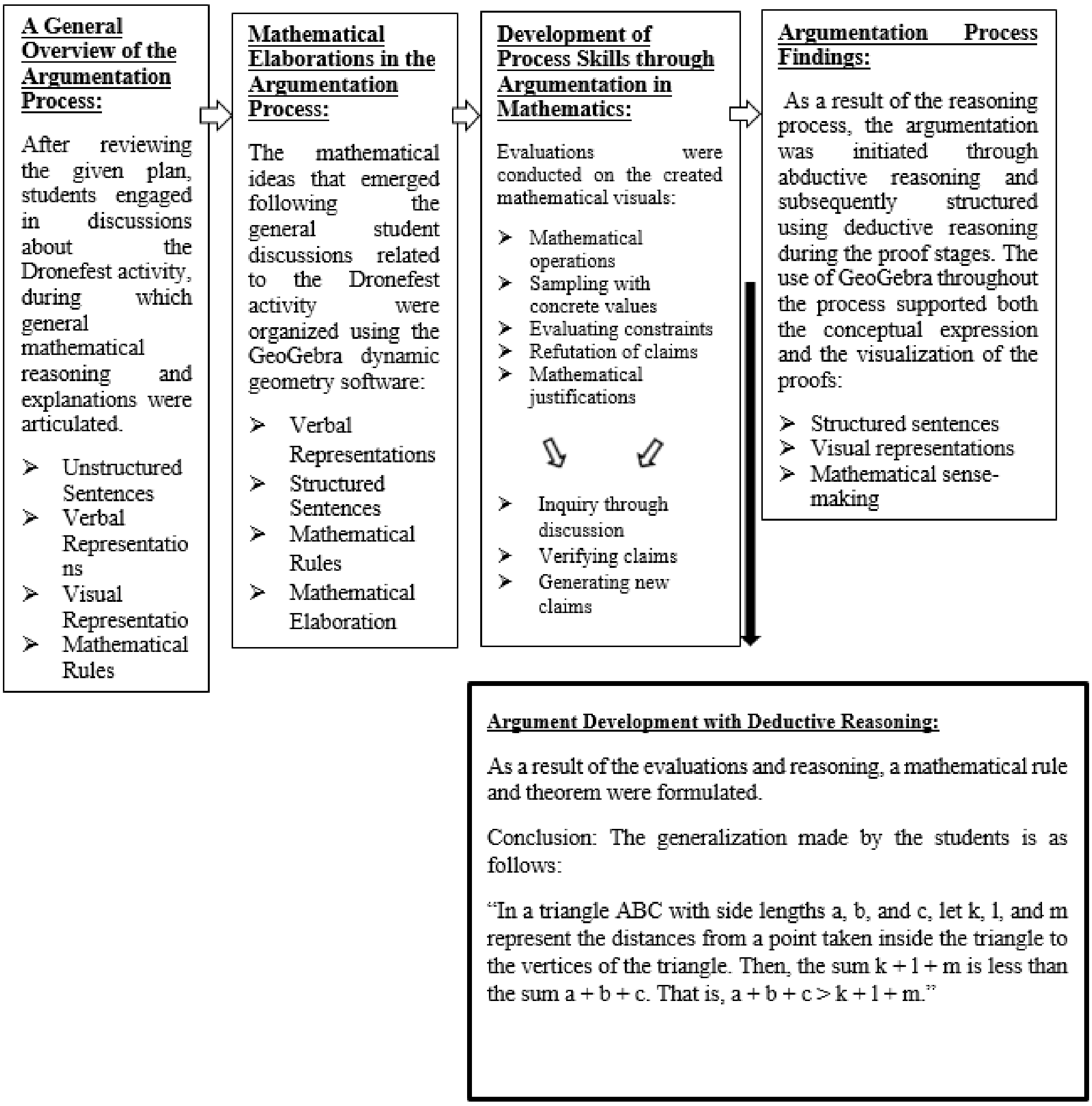


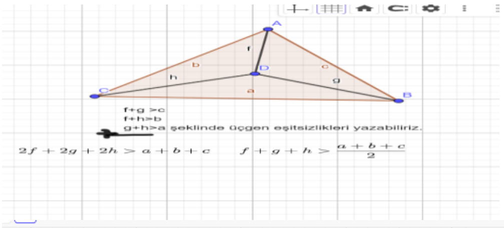
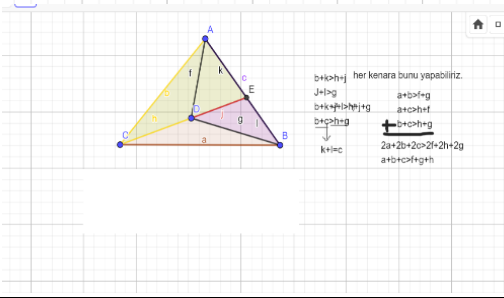
Figure 9. Flowchart of mathematical argumentation processes (Source: Authors' own elaboration)

The analysis of the data showed that the students had prior knowledge of the topic and had encountered similar problems before. It was observed that students started the proof and reasoning process with mathematical explanations, but at the beginning of the process, they used unstructured sentences, verbal and visual representations, and basic mathematical rules. The use of GeoGebra made it easier for them to switch between visual and verbal representations and to establish conceptual relationships. As the process progressed, it was noteworthy that students started to express their thoughts more systematically and made their explanations clearer through visual arrangements. Making mathematical connections in the context of the problem, students analyzed various mathematical visuals, made calculations, produced examples by giving values, evaluated problem conditions and rejected some claims. In this process, they made explanations with mathematical justifications, accepted some arguments and rejected others through interactive discussions, and developed new claims. It was found that students were able to develop various strategies for the proof process with the mathematical claims they produced. Throughout the process, it was observed that students started to use clear and precise mathematical expressions and were able to demonstrate certain principles and theorems through evaluation and reasoning. Students, who initially proceeded with abductive strategies, structured their proofs with deductive inferences over time and represented them in the GeoGebra environment. The findings revealed that students were able to comprehend the meaning behind the mathematical rules they used and reached the following generalization: "When the distances to the vertices of a point taken in the interior region of an ABC triangle with side lengths a, b, c are defined as k, l, m, $k + l + m < a + b + c$."

Analysis of Reasoning and Proof Processes Related to the Problem

In this section, students' problem-oriented reasoning and proof processes were analyzed in detail based on the Toulmin Argumentation Model. The findings of the process reflecting students' ways of thinking are presented in a structured way in **Table 3**.

Table 3. Analysis of students' reasoning and proof processes related to the problem

Student	Types of reasoning		Mathematical elaborations and process skills in the argumentation process	Structure of argumentation and reasoning processes
	Reasoning process	Proof process		
Francis & Okan	Abductive Reasoning	Deductive Reasoning	In the proof process, students expressed their statements using mathematical language and notation, free from examples. Students expressed their thoughts as "Let's call the distances of a point taken in the interior region of an ABC triangle with side lengths a, b, c to the vertices of the triangle k, l, m. For example, the sum of k + l + m is less than the sum of a + b + c. So a+b+c > k+l+m". They used the triangle inequality to prove this rule. The visual representations used by the students in the proof process are shown below.   	As a result of their reasoning, the students began the process with abductive reasoning and structured the subsequent proof process using deductive reasoning. At this stage, they employed verbal representations, visual representations, and mathematical elements throughout both the reasoning and proof processes. By integrating these components, the students demonstrated higher-order thinking skills and established cognitive coherence across the different stages of mathematical argumentation.

DISCUSSION AND RECOMMENDATIONS

The findings of the study revealed that gifted students have a high level of mathematical thinking capacity, but they experience certain limitations in using this potential in an argumentative context. The fact that students frequently use "claim" and "justification", but less frequently use "refutation" and "limitation", which increase the quality of arguments, shows that the culture of discussion and questioning is not sufficiently established. This situation shows that students mostly remain in one-way thinking structures and lack a mathematical discussion environment open to alternative perspectives. In this context, it is recommended that more argumentation-based activities should be included in mathematics and geometry courses, and students should be systematically introduced to argument elements such as claim, justification, refutation and limitation. These findings are in line with studies such as Simon et al. (2006) and Berland & Reiser (2009), who emphasized the need for a special focus on rebuttal-based arguments for the development of scientific argumentation skills. In parallel, Elballah et al. (2024) and Fabio et al. (2023) emphasized the importance of structured discussion environments to develop the multidimensional thinking skills of gifted students and concluded that the strategic use of rebuttal and limitation elements in particular enhances critical thinking.

An important source of the difficulties encountered by students in proof processes is the lack of formal instruction in mathematical language. The fact that students are not sufficiently familiar with mathematical symbols, terms and structures causes them to experience deficiencies in defining concepts and making logical inferences. Therefore, it is important for teachers to introduce students to the unique language of mathematics from an early age and to use this language effectively in lessons. Recent studies show that although gifted students have high potential in this area, the conceptual disconnects they experience in symbolic expression negatively affect their proof production (Inglis & Mejía-Ramos, 2019; Stylianides et al., 2017). Moreover, it is emphasized that such deficiencies in the use of formal mathematical language can limit students' reasoning levels and weaken their argumentative construction skills (Durand-Guerrier, 2020; Weber, 2021). Therefore, developing specific content that focuses on the use of language, notation and symbols in mathematics teaching would be an important step in supporting proof-based learning.

The study revealed that the current curricula, in which gifted students face mostly exam-oriented, routine problems, do not adequately support their creative and critical thinking capacities. This is in line with Çitil's (2018) criticism that education policies are not planned and differentiated. As a matter of fact, teaching content that is not suitable for students' potential causes them to move away from originality in problem-solving processes. Specially structured curricula are needed to develop the higher-order thinking skills of gifted individuals. Tomlinson (2014) and Reis et al. (2021) found that individualized instructional designs increase the learning motivation and academic productivity of gifted students. In addition, it has been observed that students learn more effectively in environments supported by visual and digital tools in the process of making sense of abstract concepts. Yao (2020) found that dynamic software such as GeoGebra significantly improved the algebraic thinking and visualization skills of gifted students. Therefore, it is important to integrate such technological tools into the teaching process.

Students' understanding of proof as not only a conclusion but also a process-based structure plays a significant role in developing conceptual depth and analytical thinking skills. Therefore, integrating proof into the educational process at an early age will support the shaping of students' thinking habits in this direction. Stylianides (2007) and Knuth et al. (2009) also emphasize the need to use proof as an effective learning tool not only at the advanced level but also at the early learning stages. In this context, it is clear that especially gifted students require structured, visual, dynamic, and differentiated learning environments to develop their argumentation, proof, and abstract thinking skills. Creating these environments effectively necessitates strengthening both the content knowledge and pedagogical competencies of teachers. The study by Ball et al. (2008) reveals that teachers' subject matter and pedagogical knowledge are critical in shaping students' mathematical abilities. Accordingly, it is recommended that teachers be supported with in-service training to effectively manage argumentation processes in the classroom and structure their lesson plans accordingly.

CONCLUSION

This study revealed that gifted students predominantly adopted deductive reasoning in the argumentation process, thereby enhancing the quality of the logical connections between claims and justifications. However, it was determined that students frequently used “claim” and “justification” elements in the argumentation process, while ‘refuting’ and “limiting” elements were utilized less often. This finding suggests that students lack sufficient strategic experience in considering opposing arguments, as stated in the studies of Berland and Reiser (2009) and Jiménez-Aleixandre et al. (2000). Similarly, Zohar and Nemet (2002) argued that the ability to address rebuttal arguments should be developed to cultivate higher-order thinking skills. Through interaction throughout the process, students were observed to develop argumentation skills that encompassed not only individual reasoning but also social dimensions, such as responding to counterarguments, defending their views, and reaching consensus. These interactions enabled students to articulate their mathematical ideas more clearly, coherently, and based on evidence, while also contributing to the development of higher-order cognitive skills such as critical thinking, listening, and rebuttal (Komatsu & Jones, 2022; Reuter, 2023).

The findings of the study revealed that gifted students employed different representations, models, and notations to comprehend abstract mathematical concepts. These tools played a crucial role in generating proofs, constructing arguments, and developing strategies. Previous studies have emphasized that representations as cognitive tools support meaningful mathematical thinking (Stylianou, 2011). Conversely, it has been observed that students sometimes struggle to use formal notation effectively, which is a common difficulty encountered in teaching proofs (Selden & Selden, 2003). Students typically initiate the reasoning process with abductive reasoning, and over time, they transition this process into a deductive structure before advancing to the proof stage. This structural transition reflects a progression from intuitive thinking to systematic proof skills (Mason et al., 2010; Reid & Knipping, 2010). Recent studies indicate that gifted students achieve a deeper conceptual understanding by creatively employing multiple representations with digital tools, which helps them externalize their thought processes (Hwang, 2024; Laina, 2024). Furthermore, it has been observed that these students can develop more flexible strategies in their proof processes, particularly in interdisciplinary enriched environments (Cisneros, 2022; Ramos & Camacho-Machín, 2021).

Using dynamic mathematics software, especially tools such as GeoGebra, allowed students to construct concepts in a more concrete and meaningful way by utilizing visual and auditory support in the proof process. This paved the way for students to interactively explore mathematical objects and construct proofs by testing their hypotheses. Similarly, studies by Arzarello et al. (2011) and Healy and Hoyles (2001) show that dynamic software environments support such cognitive processes. Furthermore, Hoyles & Noss (2003) emphasize that digital environments facilitate conceptual thinking by reducing students' cognitive load. Similarly, Vargas-Montoya et al. (2023) showed that information and communication technologies (ICT) are particularly effective in developing the mathematical argumentation skills of gifted students. In this direction, Rudenko et al. (2021) found that educational technologies contribute to gifted students' in-depth thinking on abstract concepts through visual modelling. Furthermore, Kovács et al. (2024) emphasizes that various features of Geogebra significantly develop mathematical argumentation skills by focusing not only on the conclusion but also on the logical reasoning process that leads to the conclusion. In conclusion, it is understood that gifted students need structured, dynamic and discussion-based learning environments to be more effective in mathematical argumentation and proof processes. Accordingly, it is very important to systematically include elements such as formal proof, mathematical notation, alternative argument constructions and the use of dynamic tools in mathematics curricula.

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AI statement: Authors stated that an AI-based grammar checker (e.g., Grammarly) was used for proofreading. No content generation was performed by AI.

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