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A multi-faceted framework for identifying students' understanding of the generality requirement of proof

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ABSTRACT

The generality requirement, or the requirement that a proof must demonstrate a claim to be true for all cases within its domain, represents one of the most important, yet challenging aspects of proof for students to understand. This article presents a multi-faceted framework for identifying aspects of students' work that have the potential to provide insight into their understanding of the generality requirement. I use a single student's work across different snapshots in time to illustrate her understanding of the generality requirement as evidenced by the justifications, examples, variables, and diagrams present within her written and oral work when proving mathematical claims and her rationales when evaluating whether provided arguments were proofs. Analysis of the student's work across multiple categories at a single moment in time shows that students can exhibit understanding of the generality requirement for some categories while demonstrating limited or unclear evidence of understanding for others. By looking across the framework dimensions and types of tasks, researchers can identify patterns in student understanding and specific aspects that could be addressed through future instruction.

Keywords: reasoning and proving, example-use, teaching experiment, case study, secondary mathematics

INTRODUCTION

In order for a mathematical statement to be considered true, it must be demonstrated true for all cases for whatever domain the statement implies (Fischbein, 1982). Fulfilling this generality requirement involves different demands depending on whether it is a particular statement (e.g., a conjecture about a single triangle) or a general statement (e.g., a conjecture about all rectangles) (Otten et al., 2014a). For example, proving that triangle ABC is congruent to triangle DEF only requires verification that the corresponding sides and angles are congruent, whereas proving that, for any rectangle, its diagonals are congruent necessitates a deductive approach. It is unclear whether students recognize this distinction, especially given that proof exercises in textbooks tend to involve particular claims rather than general claims (Otten et al., 2014a). Furthermore, limited attention to or awareness of the number of cases indicated within the domain of the claim being proven may help to explain secondary students' use of examples to prove general claims (Mason, 2019). In order to explore this possible explanation for students' challenges with proof, it is important to be able to identify students' understanding of the generality requirement and the ways that understanding is reflected in their work on proof tasks.

The purpose of this paper is to further the notion of the generality requirement as a central component of proof by elaborating a framework that identifies aspects of students' work as they prove mathematical claims and evaluate provided arguments that provide insight into their understanding of the generality requirement. Specifically, I use a single student's work across different moments in time to illustrate her understanding of the generality requirement as evidenced by the justifications, examples, variables, and diagrams present within her work when proving mathematical claims and her rationales when evaluating whether provided arguments were proofs. This framework contributes to the field by providing a more robust way of thinking about the generality requirement and its impact in proofs through synthesizing prior literature, articulating its connection to this aspect of proof, and illustrating what it looks like for a student to demonstrate understanding or limited understanding of the generality requirement across each component of the framework. In the next section, I describe the theoretical background for this work by reviewing the literature on proof and the generality requirement, followed by students' understanding of the generality requirement across the different components of proof.

THEORETICAL BACKGROUND AND LITERATURE REVIEW

Proof and the Generality Requirement

Stylianides' (2007) definition of proof served as the foundational definition of proof used in this study. Stylianides (2007) defined proof as "a mathematical argument, a connected sequence of assertions for or against a mathematical claim" that uses accepted statements that are true without further justification, valid modes of argumentation, and representations that are appropriate and understood by the classroom community (p. 291). I interpreted the terms "accepted", "valid", and "appropriate" used in the definition according to both the classroom community we formed as a part of the study and the broader mathematics community. During the study, I served as a representative of the mathematics community and brought this perspective into dialogue with the students' expectations and ideas.

Although not explicitly stated within Stylianides' (2007) definition, the requirement that a proof must demonstrate the claim to be true for all cases indicated within the domain of the claim, which I refer to as the generality requirement, is implicitly embedded within the types of statements (e.g., definitions, theorems) considered acceptable in a proof. Fischbein (1982) referred to the generality requirement of proofs in the following way: "The level of generality of the theorem is then explicitly defined by the theorem itself and the proof refers exactly and clearly to that level of generality" (p. 15). In other words, all theorems and statements to be proven indicate the level of generality, or the set of mathematical objects to which the claim applies. This is accomplished by explicitly stating the level of generality (e.g., the numbers between 0 and 20) or through the use of quantifiers such as "all", "any", or "a" (i.e., an arbitrary case). Once the level of generality has been established, the subsequent proof must demonstrate the validity of the claim for all indicated cases. Afterwards, no additional work is needed to know with absolute certainty that the statement is true for all cases within the claim's domain (Ellis et al., 2012; Fischbein, 1982). Since the focus of this study is on one student's understanding of the generality requirement rather than on their ability to construct proofs, I use the term "argument" to broadly refer to the student's work on proof tasks and the qualifiers "empirical" or "general" to indicate whether their work adhered to the generality requirement. By doing this, the terminology used when analyzing the student's work focuses on whether their argument is general rather than whether it contains all of the characteristics needed to be considered a proof.

Students' Understanding of the Generality Requirement

In this section, I describe relevant literature associated with each of the components in the student understanding of the generality requirement in proof tasks framework: students' use of justifications, examples, and variables in the proving process; their use and interpretation of diagrams; and their rationale when evaluating provided arguments. While there have been some researchers who explicitly framed their discussion of student understanding in relation to the generality requirement (e.g., Knuth & Sutherland, 2004 when students evaluated provided arguments and Mason, 2019 in a commentary on students' use of examples in the proving process), I also describe findings from studies that focus on this component of proof without explicitly connecting it to what it reveals (or does not reveal) in terms of students' understanding of the generality requirement. These categories resulted from an iterative process of synthesizing prior literature and analysis of student work from the present study. Although each subsequent section focuses on a component of proof that either shows understanding or limited understanding of the generality requirement, it is also possible for students' work to reveal conflicting or unclear evidence of understanding. In other words, each of these components have the potential to demonstrate understanding of the generality requirement, but may not do so in every case.

Students' use of justifications in the proving process

One way to assess students' understanding of the generality requirement is through tracking the types of arguments they produce for general claims. For example, Knuth et al. (2009) analyzed approximately 400 middle school students' written arguments for three general claim tasks and found that between 42% and 59% of the student arguments were empirical (used specific examples as proof). Similar results were reported in large-scale studies involving high-attaining 14-15 year-olds in the UK (Healy & Hoyles, 2000) and high school geometry students in the US (Senk, 1985). With that said, caution should be exercised when using students' justifications as the sole measure of their understanding of the generality requirement. Researchers have speculated that some students may construct empirical arguments because it was the argument they were capable of producing and not based on their understanding of what constitutes a proof (Healy & Hoyles, 2000; Weber et al., 2020). Furthermore, Weber et al. (2020) argue that inferring students' proof schemes or levels of conviction solely through the types of justifications they use in their arguments is problematic and does not adequately capture students' reasons for why they produce empirical arguments on proof tasks. Thus, researchers should go beyond analysis of students' justifications in written arguments in order to ascertain their standards of conviction and understanding of proof more broadly.

Students' use of examples in the proving process

Students' use of examples can shed light into their understanding of the implications of the generality of proofs with respect to how examples can and cannot be used during the proving process. Students who use examples as justification that a general claim is true (e.g., Knuth & Sutherland, 2004) reveal a potential limited understanding of the generality requirement. In contrast, examples can also be used to investigate the structure of the mathematical relationships or convey a general argument (i.e., as generic examples), which can productively support students in developing proofs that adhere to the generality requirement (Aricha-Metzer & Zaslavsky, 2019; Knuth et al., 2019). In order for a specific case to be considered a generic example, the argument should only reference features of the example that are true for all cases or features that have been previously established true in the argument (Yopp & Ely, 2016). Aricha-Metzer and Zaslavsky (2019) found a strong correlation between students who used examples as generic cases and those who were able to produce a deductive argument. Similarly, Ozgur et al. (2019) analyzed middle school, high school, and undergraduate students' use of examples to identify potential relationships between the purpose of examples and students' success in constructing a proof, deductive argument, or justification leading to a proof. Whereas both successful and unsuccessful provers used examples to test the truth of the claim and explore the truth domain, successful provers were more likely to use examples to convey a general argument (81% vs. 14%), understand a representation (25% vs. 9%), illustrate a representation (19% vs. 5%), or understand why the mathematical relationship is true (15% vs. 5%). These studies demonstrate that the use of examples while engaging in proof tasks do not necessarily indicate limited understanding of the generality requirement; instead, emphasis should be placed on the purpose or role the examples take on during the proving process.

Students' use of variables in the proving process

Students' understanding of the generality requirement can also be evident through the way that they use and interpret variables when proving numerical claims or notating geometric diagrams. While "variable" broadly refers to a letter that represents a number(s) in mathematics, they implicitly take on different roles across contexts (Ely & Adams, 2012; Küchemann, 1978; Philipp, 1992; Schoenfeld, 1989, Schoenfeld & Arcavi, 1988; Usiskin, 1999). Two of those roles are placeholders (e.g., m in $y=mx+b$) and varying quantities. Variables act as varying quantities when they are "seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such set of values" (Küchemann, 1981, p. 104). Variables commonly take on the role of varying quantities in proofs. For example, in order to represent two consecutive whole numbers as n and $n+1$, students must recognize the relationship between the consecutive whole numbers and then represent that relationship using a single variable. Students who use variables in mathematical arguments as placeholders might think of a specific example (e.g., 3+4) and then replace the numbers with variables ($x+y$). When this occurs, it may be unclear whether the student recognizes that the variable is representative of all possible numbers instead of a single or finite amount of specific values.

While variables can be used to prove general number-based claims, this approach is not frequently used by students, regardless of their prior performance in Algebra courses. For example, Healy and Hoyles (2000) found that advanced 14-15 year olds were highly unlikely to construct an algebraic proof for the task, "prove that when you add any two odd numbers, your answer is always even," even though they overwhelmingly evaluated the provided algebraic responses as most likely to receive the highest scores by their teachers. Number theory proof tasks offer an opportunity for students to engage in proving general claims at the beginning of high school Geometry courses and have the potential to support students' understanding of both algebra and proof (Martinez & Superfine, 2012); thus, more research is needed to support students in learning how to effectively use variables in the proving process.

Students' use and interpretation of diagrams in the proving process

In order to understand the generality requirement with respect to diagrams, students must recognize that a given diagram is a specific instance that represents all possible cases (Fischbein, 1993). Subsequently, when using a diagram to construct a proof, students should only attend to the features of the diagram that hold true for all cases (e.g., 90° angles in a rectangle) and use variables to represent the features of the diagram that can vary (e.g., the length of the sides). When considering proofs that are accompanied by a diagram, some secondary students interpret the proof as only demonstrating the claim's validity for the specific diagram shown rather than for all cases indicated within the domain of the claim (Chazan, 1993; Hoyles & Healy, 2000; Martin et al., 2005). Since textbooks or teachers tend to provide students with labelled diagrams to use when proving mathematical claims (Cirillo, 2017; Herbst, 2002) and do not always explicitly unpack what information can be drawn from a diagram (Laborde, 1995), it is unclear to what extent students understand why variables are used in geometric diagrams and what values the variable can take on within each context.

Students' rationale when evaluating provided arguments

In order to reduce the potential influence of students' content or proof knowledge when assessing students' understanding of proofs, researchers have analyzed the criteria students at varying levels evoke when evaluating provided arguments (Bieda & Lepak, 2014; Sommerhoff & Ufer, 2019) or when evaluating their own argument (Stylianides & Stylianides, 2009a). While students' evaluations of provided arguments can be used to gain insights into their understanding of different proof components (e.g., ability to identify circular reasoning or beliefs about the form of proofs), their rationales when evaluating empirical arguments in particular can shed light on their understanding of the generality requirement. For example, when Bieda and Lepak (2014) asked seventh graders to compare an empirical and general argument for the same task, students who found the empirical argument more convincing cited one of the following rationales: "(a) examples enhance understanding of the argument, (b) examples provide more information, and (c) examples are essential in a mathematical argument" (p. 171). In contrast, students who stated the general argument was more convincing explained that either "(a) examples insufficient for general case and (b) [general] argument provides more explanation" (p. 172). Beyond assessing the number of students who recognized the need for a proof to demonstrate the general case, the students' rationales also shed light on the potential for teachers to leverage the explanatory power of deductive arguments as a way to highlight the limitations of empirical arguments. When used alongside proof construction tasks, students' evaluations of provided arguments can reveal their understanding of proof by allowing for analysis of the alignment (or mis-alignment) of students' work on the two tasks (Stylianides & Stylianides, 2009a).

In summary, the generality requirement has implications across multiple facets of students' arguments when constructing proofs, from the types of justifications used in their arguments and the intent behind the use of examples and variables to the notation used when labeling geometric diagrams. In addition to these components of students' oral and written work, students' rationales when evaluating provided arguments can shed light on their understanding of the components of proofs. Whereas prior literature has focused on describing students' understanding of proof across one or two of these components (e.g., justifications used when constructing and evaluating proofs), this study analyzed student work across all five components that each provide evidence of the student's understanding of the generality requirement. In doing so, I aim to show the way these components interact with one another and demonstrate that a student can show evidence of understanding the generality requirement in some areas, but not in others. In contrast to many proof studies (e.g., Healy & Hoyles, 2000; Knuth et al., 2009; Ozgur et al., 2019) that describe student understanding based on a single written assessment or interview, this study draws on both interview and classroom data across two months, collected while the student developed her initial understanding of the generality requirement. Subsequently, comparisons can be made across the different points in time without needing to account for other factors that could influence the differences in work, such as the student's content knowledge or ability to articulate their thinking.

METHODOLOGY

Setting and Participant

Ten ninth graders participated in the teaching experiment (7 females, 3 males; all approximately 15 years old). They were the only students enrolled in course 1 of the accelerated math program at their rural, public school in the Midwestern United States. Students had no high school Geometry instruction prior to the study. The teaching experiment contained 14 sessions, held twice a week outside of their regular mathematics class time, with each session lasting between 28 and 38 minutes. I led all of the sessions, but was not the students' regular mathematics teacher and did not assign grades to students during the study. Every student completed an initial and final interview to assess how their understanding of proof had changed over the study. Four focus students were also interviewed after sessions 5 and 11 to gain further insight into their current understanding of proof. The focus students were selected based on their range of responses during the initial interview. This manuscript presents the case of the focus student, Lexi (pseudonym), whose understanding of proof was typical of the broader group.

Lexi represents the case of a student who was beginning to develop an understanding of proofs, and in particular, of the generality requirement. Her mathematics teacher reported that she was a solid "A" mathematics student and considered her to be one of the strongest mathematics students in her grade. Additionally, Lexi regularly completed her work, actively engaged in class, sought clarification when needed, and generally learned new mathematical topics easily. As a result, Lexi represents a "best case scenario"; that is, a student who had a strong mathematical foundation, enjoyed mathematics, and experienced continued success in the mathematics classroom. She also demonstrated a willingness early on in the study to share her thinking with the whole group and was consistently vocal during small-group discussions. This resulted in additional information about her understanding across the sessions that was not available for some of the other, less vocal focus students.

Introduction to Proof Teaching Experiment

In this study, I conducted a 14-session teaching experiment (Cobb & Steffe, 1983; Steffe & Thompson, 2000) designed to introduce students to formal deductive reasoning (proof). The instructional sequence was guided by two main goals. First, I aimed to develop students' understanding of purpose of proof through use of tasks that emphasized the proving process as a means of (a) developing certainty that the given statement is always true and (b) understanding why it is always true[1] (de Villiers, 1990; Hanna & Jahnke, 1996).

In order to motivate the need for deductive reasoning, the instruction only used general claims so that students could not prove the claim by checking every possible case. As a result, the instructional sequence consistently provided opportunities for students to consider the generality requirement. Second, I designed the instruction with the goal of having students develop understanding of proof through actively engaging in repeated opportunities to construct and evaluate arguments for proof conjectures. A timeline of the instructional sequence is depicted in **Figure 1**; see **Appendix A** and Conner and Otten (2021) for additional details about the instructional sequence.

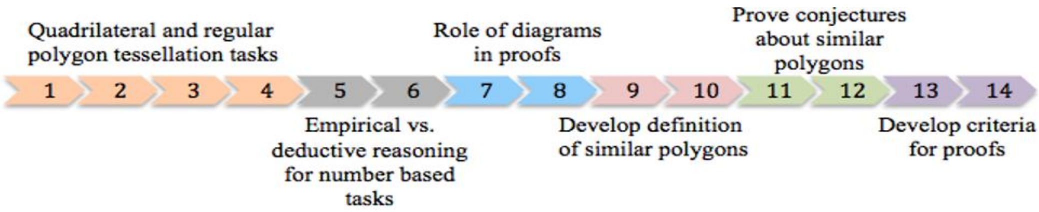


Figure 1. Overview of the instructional sequence

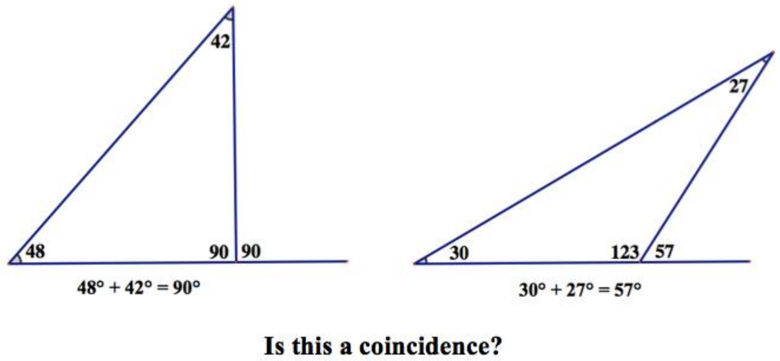
Data Collection and Analysis

The data set included transcriptions of audio and video recordings of the sessions and videos of four semi-structured interviews (Roulston, 2010) as well as the corresponding written work. The four interviews occurred prior to the start of the sessions, after sessions 5 and 11, and after the conclusion of the sessions. The purpose of the interviews was to assess Lexi's current conceptions of proof and ability to construct proofs on her own, without the outside influence of her classmates. Once Lexi

[1] Note that not all proofs are designed to explain *why* the conjecture is true (see e.g., Hanna, 2000; Weber, 2002). While there are meaningful proof tasks at the secondary level that do not have an explanatory component, I chose to specifically focus on ones that do in order to motivate a need for proofs.

constructed a written response for each prompt, she verbally explained her answer and then was asked follow up questions to gain further insight into her thinking. **Table 1** provides an overview of the tasks described in this article.

Table 1. List of tasks used during the interviews and relevant sessions

Interview or session	Proof tasks[2]
Initial interview	"Sarah said, 'If you add any three odd numbers together, your answer will be odd.' Is she right? Explain your answer."
Second interview	"Write an argument that proves all triangles tessellate."
Session 6	"9×11 equals 1 less than 10^2, 3×5 equals 1 less than 4^2. Is this a coincidence? If it is not a coincidence, how could you prove that this will always work?"
Session 11	Prove or disprove: "All squares are similar"
Third interview	"Prove that the sum of two consecutive numbers is an odd number."
Session 12	Prove or disprove: "All right triangles are similar"
Session 13	 *Note:* After students concluded it was not a coincidence, I verbally told them to prove the angle relationship.
Final interview	Task 1: "Sarah said, 'If you add any three odd numbers together, your answer will be odd.' Is she right? Explain your answer." Task 2: Prove or disprove: "All rhombuses are similar"

When conducting an initial analysis of Lexi's work in the study for the purpose of tracing changes in her understanding of proof over time, I realized that these changes were salient when viewed through the lens of her understanding of the generality requirement. Based on this narrowed focus, I then identified instances in the broader study data where she was asked to prove a mathematical claim or consider the validity of a mathematical claim or argument, as these tasks provided the clearest opportunity for Lexi to reveal her understanding of the generality requirement at a particular moment in time. Analysis focused on data that directly revealed Lexi's understanding of the generality requirement when engaging in the proving process and excluded small group work where her ideas could not be distinguished from her peers (Simon et al., 2010).

I analyzed Lexi's written and verbal work, in chronological order, for evidence of her recognition of and adherence to the generality requirement by (a) attending to her explicit and implicit justifications used in mathematical arguments; (b) categorizing the purpose of the examples she used, if any, during the proving process; (c) analyzing the way that she used and interpreted variables and diagrams in mathematical arguments; and (d) coding her justifications when evaluating provided student arguments. These categories were a culmination of multiple iterations of open and axial coding (Strauss & Corbin, 1998) alongside reviews of literature to provide a theoretical foundation and add relevalent terminology when describing the data.

In instances where Lexi did not use "by" or "because" to explicitly indicate a justification, I first looked for evidence that Lexi recognized the need to justify her statements (e.g., through explicitly stating a justification for another statement) and then hypothesized the implied justification. For example, when Lexi stated that the sum of two odds is even, I hypothesized the justification was an implicit reference to a previously established property of numbers that she did not believe needed further justification. After identifying the justification for each mathematical claim, I assessed whether it adhered to the generality requirement (i.e., whether it referred to all possible cases within the domain of the claim).

I tracked Lexi's use of examples during numerical tasks using the "purposes of examples" category of the criteria-affordances-purposes-strategies framework (Ellis et al., 2019). Specifically, I interpreted instances where the purpose of the example used was to confirm her belief in the validity of the claim as limited recognition of the generality requirement and instances where the example was used to convey a general claim or illustrate a representation as uses that acknowledged the generality requirement.

Analysis of Lexi's use of variables and diagrams in her arguments focused on whether the labels on her diagrams used variables to represent measurements that could vary and whether variables were used in a way that indicated an understanding of what it represented (e.g., varying quantities or generalized numbers vs. placeholders for specific values). I compared my interpretation of Lexi's use of variables when constructing arguments with her interpretation of provided arguments that contained variables to determine whether these interpretations aligned with one another.

Finally, I analyzed Lexi's rationale when evaluating provided arguments for proof tasks to gain insight into the criteria she was using to determining whether each argument was a proof. In particular, I attended to the way she interpreted variables and

[2] I use the term task to broadly refer to the mathematical conjectures and prompts given to students during the study that directed them to engage in some aspect of the reasoning-and-proving process. Note that these tasks were not accompanied by a series of questions or elaborative structure that is sometimes common in larger problem solving tasks.

diagrams as well as any references to the generality requirement when justifying why she thought a provided argument was or was not a proof. Lexi's rationales when evaluating the provided arguments were particularly insightful as there was very minimal direct instruction during the sessions about what a proof is or what it must contain in order to be considered valid. After analyzing each of the individual components, I made constant comparisons between each unit of analysis both within and across categories to look for similarities and differences in her understanding. I also noted instances where Lexi's statements could be interpreted in multiple ways and conducted multiple reviews of the data to look for disconfirming evidence and possible rival interpretations (Corbin & Strauss, 1998; Yin, 2014).

STUDENT UNDERSTANDING OF GENERALITY REQUIREMENT IN PROOF TASKS FRAMEWORK

The framework presented in **Table 2** provides a way to identify students' understanding of the generality requirement across different components of their work on proof tasks. The instances provided within the framework that demonstrate understanding or limited understanding of the generality requirement are based on analysis of student oral and written work from the teaching experiment. Subsequently, they are not meant to encompass all possible ways that students can demonstrate understanding or limited understanding. Finally, care should be taken when relying only on students' written work to assess their understanding of the generality requirement. As the data excerpts in the subsequent sections demonstrate, students' oral explanations for their work on proof tasks can provide evidence of understanding that contradicts evidence in their written work.

Table 2. The student understanding of the generality requirement in proof tasks framework

Type of proof task	Category	Instances that demonstrate understanding	Instances that demonstrate limited understanding
Student constructed arguments (proofs)	Justifications	Encompass all cases within domain of claim being proven (e.g., definitions, theorems)	Examples used to prove a general claim
	Usage of examples	Convey a general argument Refute validity of general claim	Justification for validity of claim
	Usage & interpretation of variables	Generalized numbers (generic examples) Varying quantities	Placeholder for specific values
	Diagrams	Used to notate measurements that vary Interpreted as representing all possible cases	Specific values used to notate measurements that vary Interpreted as showing a specific case
Students' evaluation of provided arguments (proofs)	Rationale	References need for argument to apply to all cases	References inclusion of examples as proof

Next, I use Lexi's work at different moments in time to illustrate her understanding or limited understanding across three categories: (a) use of justifications and examples when proving true mathematical claims and disproving false mathematical claims, (b) use and interpretation of variables and diagrams during the proving process, and (c) rationale when evaluating whether provided arguments are proofs. Although justifications, examples, variables, and diagrams were presented as four separate categories in the framework, they have been collapsed into two sections due to the way they were woven together in Lexi's work. In the final section, I provide a summary of her work across each of the dimensions, presented chronologically, in order to highlight the way in which the dimensions interacted at different moments in time.

Justifications and Examples

Use of justifications and examples that demonstrated understanding of generality requirement

From the second interview onward, Lexi demonstrated understanding of the generality requirement through her use of implicit and explicit justifications that applied to all cases within the domain of the claim and her use of examples to illustrate, or explain, the general claims in her arguments. For example, her verbal argument for the task, "Write an argument that proves all triangles tessellate" (second interview), is shown below:

> So I said I know that quadrilaterals tessellate, because angles add up to 360°, which that's just like a starting off, I guess. And then I said this works with triangles as well because the angles all add up to 180°. So anytime the shape adds up to a multiple of 180, it will tessellate. So like, if it adds up to 720°, that'll tessellate too. So like 6, like a hexagon.

Lexi began her argument by stating a fact previously established in session 4 ("quadrilaterals tessellate because angles add up to 360°"). Next, she justified her conclusion that triangles will tessellate by citing the triangle angle sum theorem. The final sentence in her written argument ("anytime the shape adds up to a multiple of 180, it will tessellate") did not include a justification. While her assertion that all shapes whose angles add to a multiple of 180 is incorrect, the statement does provide evidence to suggest she was considering the generality of the claim being proven. Lexi's oral argument continued beyond her written argument with the statement of an example ("if it adds up to 720°, that'll tessellate"), prefaced by the phrase "so like". While it is possible that her final statements were intended to serve as justification for her claim that all polygons whose angles add up to a multiple of 180° will tessellate, the phrase "so like" seems to signal a transition from justifications that referenced features of all shapes within the statement's domain to a statement that provided a specific example to further illustrate, or explain, her argument. In other words, Lexi appeared to use the example of hexagons to convey her general argument about the relationship between the sum of angles in a polygon and whether it will tessellate rather than as an empirical justification for her claim.

Lexi's understanding of the role of examples in the proving process was more clearly depicted in her argument for the *sum of two consecutive numbers* task (third interview). In this task, she began by trying two smaller examples (1+2 and 3+4) and one "larger" example (20+21) to test the validity of the claim. When verbally explaining her argument, she alternated between reading statements from her written argument that applied to all numbers and providing specific examples to help convey her general claims.

> Whenever you add an even and an odd, which is what you're going to do when you add consecutive numbers, you just add an even and an even, so like 1+1 or, 2+2 is 4, when you add like two numbers, and so like 1+2, for example, you add 1+1 that equals 2, but then since it has to be consecutive, 1+2 which equals 3.

Similar to the second interview, Lexi prefaced specific examples with the phrases "so like" and "for example" to indicate that they should be viewed as illustrations to represent the broader mathematical relationships she was trying to describe. After explaining her argument, she justified her decision to include the written statements ("Whenever you add an even and an odd…you just add an even and an even…when you add like two numbers…but then since it has to be consecutive") alongside the examples saying, "if you were to put just three examples like that no one knows what you're talking about instead of like adding something to it to show what you mean." Although this justification does not explicitly reference the need for the argument to encompass all consecutive numbers, it does demonstrate an awareness of the distinction between examples and proof.

Lexi also demonstrated understanding of the generality requirement through her use of a specific example (**Figure 2**) to disprove the false conjecture, "all right triangles are similar" (session 12). While she initially thought the claim was true based on the belief that the angles of a right triangle were 45º, 45º, and 90º, she decided to construct the diagram below in order to resolve her groups' debate about the acute angles in right triangles.

Figure 2. Lexi's drawing when investigating the conjecture "all right triangles are similar" during Session 12

It appears from this diagram (**Figure 2**) that Lexi purposefully drew the right triangles such that the vertical sides were the same length and the horizontal sides were different. Although this drawing did not resolve her group's question about the angle measurements, Lexi verbally concluded the two triangles were not similar since the sides were visibly not proportional. Later on, she decided that the acute angles did not have to be 45º because "if you keep stretching this (horizontal side) out, this (top acute angle) will be bigger and this (bottom acute angle) will get smaller." Lexi's contributions to the group's investigation demonstrated an awareness that a single counterexample was sufficient to disprove the claim. Notably, Lexi decided the conjecture was false based on the sides not being proportional *before* determining that the corresponding angles would not always be congruent, suggesting knowledge that one unmet criterion for similar polygons was sufficient for the claim to be false.

Use of justifications and examples that demonstrated limited understanding

In contrast to her later work, Lexi's use of examples as justification on the *sum of three odds* task (initial interview) suggested limited understanding of the generality requirement. Lexi's argument for this task consisted of two examples: one with smaller numbers (1+3+5) and another that contained a larger number (51+3+9). She then verbally concluded that the statement was true "because the two problems that I did that were different both added up to an odd number." Lexi's use of examples to "prove" the statement suggests either an unawareness of what constitutes proof or a lack of other strategies for proving the statement true for all possible odd numbers. When asked whether her answer proved that the claim was *always* true, she admitted it did not "because I only gave two examples. I could probably do more, but…" In this response, Lexi appears to be making a distinction between the word *any*, used in the original task, and the word *always*, used in the follow up question, suggesting a limited awareness that proving a claim about the sum of any three odd numbers implies that it must be true for all of them. During this interview, she also expressed doubt whether it was even possible to prove a claim true for all numbers. "I know in math we always have special cases, so probably not, but I think in the majority of the time, it'll probably end up odd." The disconnect between Lexi's work on the original task and her responses to the follow up questions suggests an unawareness of the generality requirement and a reliance on her experiences outside of mathematics to make sense of the terminology used in the questions.

Use of justifications and examples that demonstrated unclear understanding of the generality requirement

Whereas Lexi's use of a single diagram to disprove the conjecture "all right triangles are similar" in session 12 indicated clear understanding that a single counterexample was sufficient to disprove a general claim, her argument to disprove the conjecture, "all rhombuses are similar" during the final interview produced unclear evidence of understanding. For the similar rhombus task, Lexi's argument consisted only of the written statements below and did not include a diagram or specific counterexample.

> This is not a true statement. By the definition of a rhombus, we know that there are two parallel sides and the diagonal angles are the same. Also, all side lengths are the same. The reason this is not true is because a square is a rhombus but a rhombus is not a square.

The overall structure of her argument parallels the structure she used when proving true claims: namely, she began by stating the validity of the conjecture, followed by statements referencing definitions or properties. Her final statement provided a justification for her assertion that the claim was false. Instead of constructing a single counterexample, Lexi provided a class of counterexamples–the similarity between squares and (non-square) rhombuses. On its own, it is not clear whether she used a class of counterexamples because that was the first counterexample that occurred to her or because she did not think that a specific counterexample would suffice.

Although Lexi's written argument did not provide clear understanding of the generality requirement, she did clarify both how she was thinking about the task and her understanding of how to disprove general claims during the interview.

> Lexi: I don't really know how to explain it but like… A rhombus is not a square because, like, squares have to have the four, all the same angles, so therefore, like a square has diagonals are the same, and they have the same side lengths, and they both have two sets of parallel sides, so therefore, not all polygon or not all rhombuses are similar.
>
> Interviewer: Okay, so how do you prove that a statement in math is NOT true?
>
> Lexi: Well, I didn't include like a formula for it, which I could have done, but um… and I didn't like write the diagrams on here either, but I just found like alternate example that didn't work so I proved that it wasn't true.
>
> Interviewer: Okay. And how many alternate examples do you have to have to prove that it's not true?
>
> Lexi: Just one.

In her first response, Lexi referenced properties of rhombuses to establish that a square is a type of rhombus and then explained that not all rhombus were squares given the additional property that squares have "all the same angles," or 90º angles. This explanation suggests that she was not only attending to the overall appearance of squares and rhombuses, but also recognized the specific angle measurements of the two shapes were not congruent when using it as a counterexample. When asked how to disprove a mathematical statement, Lexi evaluated her own argument based on some of the classes' criteria for "good proofs" (session 14[3]) as well as her understanding of how to prove that a mathematical statement is false. In other words, it is possible that the lack of specific counterexample in her original written argument reflected a belief that *all* proofs must reference definitions and other justifications that apply to all cases, rather than just arguments for true claims.

Lexi's decision to provide a class of counterexamples (the similarity between squares and non-square rhombuses) rather than a specific counterexample can be interpreted in a few ways. One interpretation is that it provides evidence to suggest that she was not fully convinced that a single counterexample disproved the claim. A second interpretation is that she made a conscious choice to go beyond providing a single counterexample so that her argument also explained *why* the statement was false. Finally, it could be that this class of counterexamples was the first thing she came up with and she did not see the need to provide a single counterexample after she had provided a class of counterexamples. While there is certainly nothing wrong with her provided counterexample, it does not provide sufficient evidence either way regarding her recognition that a single counterexample is sufficient to disprove a general claim.

Use and Interpretation of Variables and Diagrams

Use of variables and diagrams that demonstrated conflicting understanding

Lexi's written argument for the exterior angle theorem (session 13) included the use of variables to label the angles measurements in her diagram that could vary, however; the equations she wrote to accompany the diagram suggested an interpretation of the variables as placeholders. As shown in **Figure 3**, Lexi used the variables *A*, *B*, and *D* to refer to the angle measurements and then used *x* to refer to 180 degrees in her constructed equations. Although Lexi's notation methods adhered to the generality requirement, her associated equations suggest this decision may have been based on her understanding that specific numbers should not be used in proofs rather than on a more nuanced understanding of which aspects in the context varied versus which ones remained the same. In particular, it is not clear from her equations whether she understood that variables should be used to represent the angles, since the angle measurements could all vary, but were *not* needed to represent the sums in her two equations, since (as she stated in her written argument) the sum of the angles in a triangle and the sum of angles in a line always equal 180 degrees.

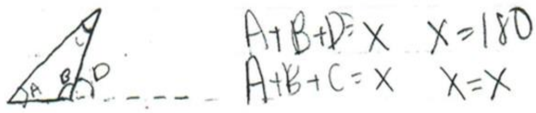


Figure 3. Lexi's diagram and associated equations on the exterior angle theorem proof task

[3] See **Appendix A** for more information about the tasks used during the teaching experiment.

Use of variables and diagrams that demonstrated limited understanding

When working on numerical proof tasks throughout the study, Lexi used variables as placeholders for specific values rather than as varying quantities that represented all possible numbers. For example, Lexi initially tried to prove the *sum of three odds* task during the final interview by defining *x* as an odd number and *y* as an even number (**Figure 4**). It is possible that she viewed x and y as representing any odd or even number respectively; however, the subsequent equations she wrote involving specific numbers (“1+1+1=3; 3=(2+1)”) reinforces the interpretation that she was using variables in place of specific values. Additionally, Lexi chose to represent the three odd numbers with the same variable even though she understood that the three odd numbers in the proof statement did not have to be the same number. Ultimately, Lexi’s use of variables as placeholders restricted her ability to prove the claim was true for all possible sums of three odd numbers, causing her to change approaches on the task.

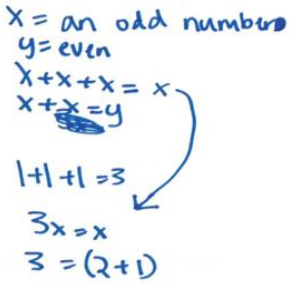


Figure 4. Lexi’s algebraic approach on the *sum of three odds* task during the final interview

In contrast to her use of variables when notating her constructed diagram for the exterior angle theorem, Lexi’s diagram to accompany her small group’s proof of the conjecture, “all squares are similar” in session 11 reflected limited understanding of the generality requirement. When discussing how to construct a diagram for their proof, Lexi suggested to her group members that they “should make a big square, and a smaller square, and a smaller square and then say all the angles are the same and all the sides have the same, not ratio, what’s the word?” This verbal argument not only addresses the two components of the definition of similar polygons, but also contains multiple references to the generality of the claim (“all angles…all sides…”). Lexi ultimately chose to draw two squares for the diagram (**Figure 5**). Although the statements in her oral argument explicitly referenced features of all squares, she labelled the side lengths in their diagram with specific measurements.

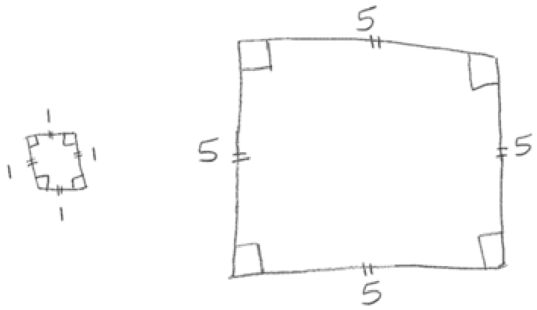


Figure 5. The initial diagram Lexi’s small group constructed for their argument of the claim, “all squares are similar”

Interpretation of variables and diagrams that demonstrated limited understanding

In addition to using variables as placeholders for specific values, Lexi also consistently interpreted provided arguments containing variables as placeholders and diagrams as a specific “example” rather than as representing all possible cases. Lexi’s evaluation of provided “student” algebraic arguments throughout the study suggested an interpretation of the variables as placeholders that could generate multiple examples. For example, she stated during the initial interview that Arthur’s answer for the *sum of three odds* task (**Figure 6**) was a proof “since it had like an equation with it…because you can always plug in numbers and see if it works out or not.” Similarly, she stated that Eric’s answer was a proof because it “had an equation and it looks like it worked out.” These justifications suggest an attention to the sheer presence of equations and variables rather than on why variables were used or what each equation represented in the argument.

Prove that “if you add any three odd numbers together, your answer will be odd”

Arthur’s answer

a, b, and c are any whole numbers.
Then 2a, 2b, and 2c are all even numbers and
2a + 1, 2b + 1, and 2c + 1 are all odd numbers.
(2a + 1) + (2b + 1) + (2c + 1) = (2a +2 b + 2c + 2) + 1
This is the same as 2(a + b + c + 1) + 1
2(a + b + c + 1) is even, and an even number + 1 is odd.
So, if you add three odd numbers together, your answer is odd.

Eric’s answer

x, y, and z are any whole numbers.
x + y + z = a
3x + 3y + 3z = 3a
3a is odd, so 3x + 3y + 3z is also odd
$\frac{(3x + 3y + 3z)}{3} = x + y + z$
So, if you add three odd numbers together, your answer is odd.

Figure 6. Two provided student responses for the task, “if you add any three odd numbers together, your answer will be odd.” Student work is a modified version of a task developed by Healy and Hoyles (2000, p. 400)

Although Lexi's rationale when evaluating Arthur and Eric's arguments changed between the initial and final interviews, her interpretation of variables remained fairly consistent. In the final interview, Lexi stated she was unsure whether Eric's argument was a proof, offering a specific example she used to test or verify the equations in the argument.

> "Well, I like for *x*, I did 1, *y* I did 2, and *z* I did 3, so 1+2+3 is 6 and then 3+6 is 9+9 is 18, so then *a* would have to be 6, and 3*a* is not odd, so I don't really think it is."

Even though her explanation provides a nice counterexample for why the argument is incorrect, she struggled to state that it was definitely not a proof, arguing that "it looks like it would be a proof, cause it looks like, very detailed and stuff." When justifying why Arthur's argument was a proof, she began by explaining that she was "doing different examples in [her] head" before providing a general argument for why 2n will always be even, regardless of the value of n. Collectively, Lexi appeared to evaluate algebraic arguments by using specific examples to check the accuracy of the equations rather than by considering whether the equations made sense for the claim being proven.

In addition to interpreting equations as generating examples, Lexi also appeared to interpret diagrams as examples in her evaluations of provided arguments during the second interview. Lexi argued that Terri's answer (**Figure 7**) was a proof because "it gave like a visual example"; similarly, she thought Rebecca's answer was not a proof because "she gave examples of one's she's tried but not like visual examples I guess." Across both justifications, Lexi appears to be indicating a belief that diagrams should be included in proofs because they provide a "visual example", which suggests that she is attending to the specific aspects of the diagram (e.g., the use of obtuse triangles) rather than viewing it as a specific instantiation that represents all triangles.

Prove that all triangles tessellate

Rebecca's answer

I tried ten different triangles – including obtuse, acute, right, and equilateral – and they all tessellated. Since I haven't found an example that doesn't work, all triangles must tessellate.

Terri's answer

If you take 3 copies of a triangle and place them together so that their sides match up but the angles that are touching are different, you'll get a straight line. I know this will work because the sum of the angles in a triangle is 180 and the sum of the angles in a straight line is also 180. So, all triangles tessellate.

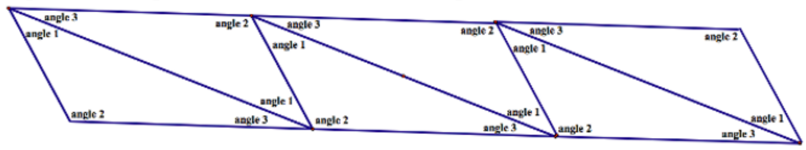


Figure 7. Two provided student responses for the task, "prove that all triangles tessellate"

Rationale when Evaluating Provided Arguments

Rationale indicating understanding of the generality requirement

Lexi's first explicitly referenced the generality requirement when evaluating provided arguments during the third interview. When justifying why Carter's empirical argument (**Figure 8**) was *not* a proof, Lexi explained, "You can have, like, different examples and it might not prove all of them once you're, well, it's not proving like the general, like I don't know how to say it, but it's not proving all of them." In this rationale, Lexi clearly differentiates between a proof that demonstrates the claim is true for all of the cases and examples, which only demonstrate the claim is true for the specific cases provided. Similarly, she thought Aaron's answer was a proof "because it explains like, it doesn't have any examples, but it does explain like how any number, or any like pair of consecutive numbers, can be or is going to be odd." Even though Lexi's rationale for Aaron's argument mentions the lack of examples, she ultimately concluded it was a proof due to the explanatory nature of the answer for *any* pair of consecutive numbers.

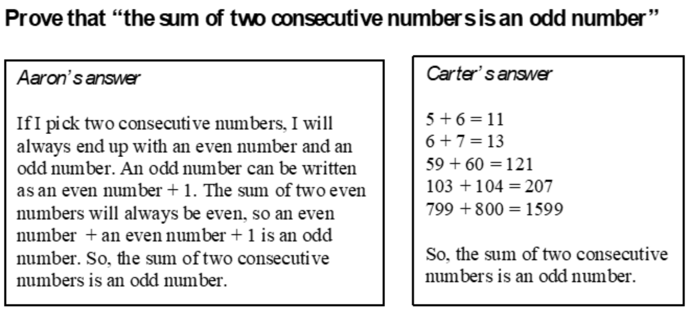


Figure 8. Two provided student responses for task, "prove that 'the sum of two consecutive numbers is an odd number.'" Student work is a modified version of a task developed by Healy and Hoyles (2000, p. 400)

Rationale indicating limited understanding

In contrast to her responses during interview 3, Lexi consistently referenced the presence of examples, or lack thereof, when evaluating the provided arguments for the *sum of three odds* task during the initial interview. Although she was unsure whether Beth's answer (**Figure 9**) was a proof "because it seemed kind of easy," she decided that Caleb's answer was not a proof, "because there's not many examples, or maybe cause um, they're like pictures?" Similarly, she was unsure whether Debbie's answer was a proof "because it doesn't like give any examples." Lexi's evaluation of the provided arguments during the initial interview consistently referenced the presence of examples, mirroring her own approach to the task.

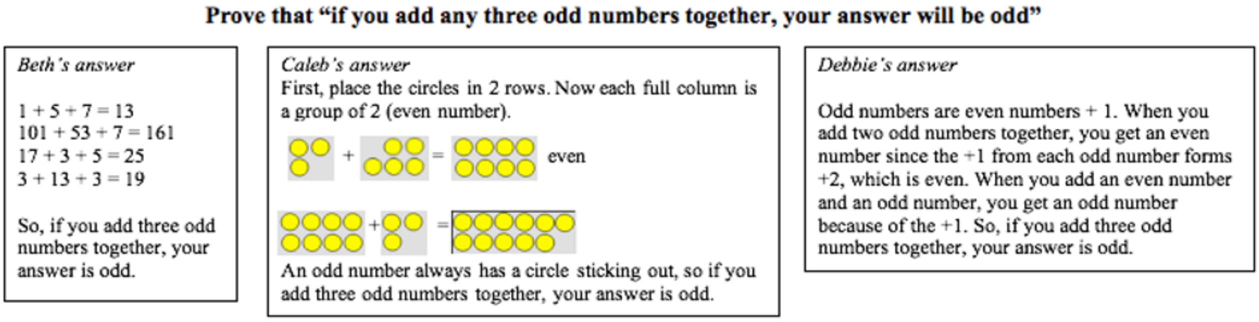


Figure 9. Three provided student responses for the task, "if you add any three odd numbers together, your answer will be odd". Student work is a modified version of a task developed by Healy and Hoyles (2000, p. 400)

Looking Chronologically Across Dimensions of the Generality Requirement

Lexi's work demonstrated varied understanding of the generality requirement in part because it was collected across different moments in time while she was participating in instruction specifically designed to focus on this element of proofs. In this final section, I provide a summary of Lexi's understanding of the generality requirement, presented chronologically, in order to highlight her understanding across different snapshots in time. **Table 3** allows for analysis of the extent to which her understanding aligned across the dimensions of the framework (shown horizontally) at a single moment in time as well as how her understanding of each dimension evolved over the study (shown vertically).

As shown in **Table 3**, Lexi did not exhibit any evidence of understanding the generality requirement during the initial interview. This was to be expected since she had not yet been introduced to proof in her regular math class. Over the course of the teaching experiment, Lexi began to use justifications in her written arguments that encompassed all cases within the domain (second interview onward), used rationale that explicitly referenced the generality requirement when evaluating provided arguments (third interview onward), and recognized that a single counterexample disproved a general claim (session 12). On the other hand, Lexi consistently used and interpreted variables as placeholders for specific numbers instead of using them as varying quantities that represented all possible numbers. Thus, while Lexi's work and explanations exhibited multiple indicators of understanding the generality requirement by the end of the study, this understanding did not extend to her use and interpretation of variables. Furthermore, the second interview in particular highlights a disconnection in evidence of understanding the generality requirement across the dimensions. Whereas Lexi constructed a written argument containing justifications that encompassed all possible cases and used examples in her oral argument to convey general claims, she continued to use the presence of examples as rationale when evaluating provided arguments and referred to the diagram within a provided argument as a "visual example."

Table 3. Overview of Lexi's understanding across multiple aspects of generality requirement during different points in the study

	Justifications	Example use	Variable & diagram use & interpretation	Rationale when evaluating provided arguments
Initial interview	Examples used as justification	Examples used as justification	Interpreted variables as placeholders	Proofs should contain examples
Second interview	Justifications applied to all cases within domain	Examples used to convey a general argument	Referred to diagrams as "visual examples"	Proofs should contain "visual examples" and explain why
Session 6			Interpreted variables as placeholders	
Session 11	Justifications applied to all cases within domain		Used specific numbers to label diagram	
Third interview	Justifications applied to all cases within domain			Proofs should explain why; prove statement for all cases
Session 12	Single diagram used to disprove claim			
Session 13	Justifications applied to all cases within domain		Used variables as placeholders; labelled diagram with variables	
Final interview	Justifications applied to all cases within domain Class of counter-examples used to disprove claim	Examples used to understand how conjecture works	Used variables as placeholders No diagram included for similar rhombuses task	Proofs should explain why; prove statement for all cases

DISCUSSION

This article presented a framework for assessing students' understanding of the generality requirement across multiple dimensions followed by examples of student work illustrating varied levels of understanding. Rooted in beginning students' work across different types of proof tasks and drawing from multiple bodies of literature related to specific aspects of proofs, this framework demonstrates how each dimension has the potential to provide insights into students' understanding of the generality requirement. At its core, the generality requirement dictates that a mathematical claim must be demonstrated true for all cases indicated within its domain in order to be considered a valid statement. Beyond impacting the types of justifications can be used in a proof, the generality requirement also influences other aspects of proofs, such as the purposes examples and variables can play in the proving process and the way that diagrams are notated and interpreted. In order to use and interpret diagrams, variables, and examples in a way that adheres to the generality requirement, a student must focus their attention on the general aspects and ignore the specific details of the given proof component. While mathematicians and math educators instinctively do this, the same cannot necessarily be said for novices (Mason, 1989). Thus, this framework can be used to draw teachers' attention towards the different aspects of the generality requirement as well as offer a more robust way to assess students' understanding of this aspect of proofs.

Although the framework presents the five dimensions as separate categories, Lexi's work highlights potential overlaps between dimensions and illustrates that one dimension can impact another. For example, the diagram category can be seen as a subset of the variable use category, as evidence of understanding the generality requirement with respect to notating geometric diagrams for general claims dictates the use of variables for all aspects of the diagram that can vary. Additionally, students who exhibit limited understanding of the generality requirement with respect to mathematical justifications use examples as their sole justification for claims. Thus, analysis of student work should consider these dimensions in concert with one another. As other researchers (e.g., Healy & Hoyles, 2000; Weber et al., 2020) have argued, the justifications students use in their written arguments, on their own, do not necessarily reveal students' understanding of the generality requirement or of proof more broadly. Instead of focusing on one specific aspect of students' work to assess their understanding of the generality requirement, I contend that researchers should look across multiple dimensions and types of tasks in order to present a nuanced picture of student understanding. As demonstrated through analysis of Lexi's work, it is through looking across dimensions and types of tasks that patterns in understanding emerge, as well as specific aspects that could be addressed through future instruction.

I used Lexi's work throughout the teaching experiment to provide examples of what it might look like for a student to demonstrate understanding or limited understanding of the generality requirement. These illustrations are not meant to be exhaustive or representative of a typical student. Given that the data used to construct the framework came from a small group of students who all participated in the same teaching experiment designed to develop their understanding of the generality requirement, future research should investigate students' understanding in different settings and at varying stages in the learning process. Through this process, researchers may also be able to identify additional aspects of student work on proof-related tasks that have the potential to reveal students' understanding of the generality requirement.

Features of the Teaching Experiment that Impacted Evidence of Lexi's Understanding Shown in the Data

While it is not possible to draw direct connections between specific features of the instructional sequence that resulted in Lexi's understanding due to the fact that only one iteration of the teaching experiment has been conducted, there are aspects of the instruction that appear to have played an important role in developing her understanding of the generality requirement (see **Appendix A** and Conner and Otten (2021) for more information about the instructional tasks). For example, the first task that students engaged in involved considering the validity of the questions, "do all quadrilaterals tessellate?" and "do all regular polygons tessellate?" Selecting two questions that were parallel in structure but differed in validity allowed for the second question to serve as a pivotal counterexample for students (Stylianides & Stylianides, 2009b) and supported a transition in our conversations from determining whether the statements were true to determining why only some polygons tessellate. The focus on understanding why also helped to shift students' conversations from talking about specific examples to posing hypotheses regarding which classes of shapes will tessellate (moving towards considering the general case) (Conner & Otten, 2021). Finally, the second question highlighted the limitations of empirical arguments since students were surprised to find that their initial belief regarding the validity of the claim was wrong. Although students experienced additional opportunities to consider the explanatory feature of proofs and reflect on the limitations of empirical arguments, the first task showed promise in motivating a need for learning how to construct deductive arguments.

More broadly, the use of student-centered instruction and minimal teacher direction about specific aspects that should or should not be included in a proof made it possible to gain insights into Lexi's understanding without wondering whether she was merely repeating what she had been told. For example, the importance of providing minimal instruction on what should or should not be included in a proof was evident in Lexi's work during the final interview, where she clearly tried to incorporate the classes' list of features of "good proofs" when proving the false claim, "all rhombuses are similar". Just like it is possible for students to memorize a set of procedures that allow them to correctly solve problems without understanding why the procedure works, is it equally possible for students to memorize what should and should not be included in a proof without understanding why. Additionally, Lexi's use of diagrams and understanding of variables within the diagrammatic context was evident in this study because she was responsible for constructing diagrams to accompany her arguments rather than being provided ones, as is custom in traditional contexts (Herbst & Brach, 2006). Given that high school geometry textbooks tend to provide few opportunities for students to construct proofs in the introduction to proof chapter (Otten et al., 2014b) and tend to include direct instruction with the teacher and textbook retaining most of the mathematical authority (e.g., Cirillo, 2011; Herbst & Brach, 2006; Otten et al., 2017), more research is needed to assess students' understanding of the generality requirement within a proof context that is more commonly found in secondary classrooms.

Specific features of the interview protocols also afforded opportunities to reveal insights into Lexi's understanding that were not necessarily evident during the sessions. Asking Lexi to orally explain her arguments after completing her written response to the proof task allowed for insights into the purposes behind the examples used and often resulted in her providing more details than what was present in her written argument. Asking students to orally explain their proofs as well as construct a written argument has been found to be similarly beneficial in studies involving both middle and secondary students (Campbell et al., 2020; Stylianides, 2019). As shown through Lexi's work on the tasks in session 13 and the final interview, it is possible for a student to learn how to use variables and diagrams in the proving process in a way that might suggest attention to the generality requirement, while also interpreting them as a specific example or capable of generating specific examples. These insights would not have surfaced had I only analyzed her written work and was not able to ask follow up questions in an interview setting.

Additionally, the construct-evaluate task sequence used across the interviews, wherein Lexi evaluated provided student responses for the same task after constructing her own written response, allowed for determining alignment (or not) between her work on the two tasks and revealed additional insights into the generality requirement. The usefulness of the construct-evaluate task sequence was similarly found by Stylianides and Stylianides (2009b) and Weber et al. (2020) and highlights the importance of using multiple types of tasks to elicit students' understanding of proof. While multiple studies (e.g., Bieda & Lepak, 2014; Healy & Hoyles, 2000; Stylianides & Stylianides, 2009a) have asked students to evaluate provided arguments as a way of gaining insight into their understanding of proof, this type of task is not commonly found in secondary geometry textbooks (Otten et al., 2014a). Future research could investigate the practice of having students evaluate peer arguments or provided (fictitious) student arguments as a way to assess their understanding and help them develop understanding about the key aspects of proofs.

IMPLICATIONS FOR RESEARCH AND PRACTICE

Through the use of the student understanding of the generality requirement in proof tasks framework, the field can continue to deepen our own understanding of how students make sense of the generality requirement and its implications for different facets of the reasoning and proving process. In particular, the framework can support work in recognizing which aspects of the generality requirement are easier for students to demonstrate understanding of than others and begin to design instructional supports that acknowledge the complexity of this component of proofs. Furthermore, the framework serves as a bridge across multiple existing bodies of research, showing how students' justifications, examples, diagrams, and variable use within constructed arguments, alongside their evaluation of provided arguments, can provide a nuance, and more complex picture of students' understanding of the generality requirement than what can be seen through focusing on a single component. While it is understandably not possible to attend to this many facets of students' work in larger scale studies, the present study demonstrates that focusing on multiple aspects of student work can reveal greater understanding of the generality requirement than what might be evident within a single facet (e.g., students' justifications in written arguments).

In order to develop a nuanced understanding of the generality requirement, students likely need increased opportunities to consider and prove general claims than what is provided in current textbooks (Otten et al., 2014a), alongside multiple opportunities to reflect on the domain of the claims (Mason, 2019) and the specific roles that diagrams and examples play in the proving process. Attending to the generality within a mathematical claim requires students to see the general within the particular (Mason & Pimm, 1984) and recognize that the words all, every and any indicate the impossibility of an example that satisfies the criteria in the hypothesis but contradicts the conclusion. Shifting students' attention towards the generality requirement is an important, yet undervalued part of teaching students to construct and make sense of proofs. Professional development or curricular supports that help teachers enact ambitious teaching that responds to students' ideas in a way that draws increased focus to this aspect of proof in their classrooms is needed. Within such supports, teachers would likely not only need curricular resources that provide opportunities to consider the generality requirement, but also supports to develop their knowledge of the learning goal (e.g., understand that the domain of the claim dictates the way that geometric diagrams are notated and interpreted), understanding of the subject-matter ideas involved in the instructional tasks (e.g., the difference between notation methods that are based on mathematical convention and those that are dictated by the generality requirement) and knowledge of possible student responses within the task (e.g., awareness that students may interpret the use of variables as placeholders rather than as varying quantities) (Stylianides & Stylianides, 2014).

Implications from Looking at Lexi's Work Chronologically

Analysis of Lexi's understanding across different moments in the study suggests that just because a student can construct arguments that adhere to the generality requirement in their justifications when constructing and evaluating mathematical arguments does not necessarily mean they can apply this understanding to other aspects of their arguments, such as their use and interpretation of variables and diagrams. Notably, Lexi consistently used and interpreted variables as placeholders despite exhibiting understanding of the generality requirement across the other dimensions and excelling in her regular mathematics classes. Students' difficulty in using variables appropriately when proving general claims may point to limited understanding of the generality requirement, limited understanding of variables, or both. Thus, there is a need for proof instruction that seeks to expand students' understanding of the different roles variables take on during the proving process rather than assuming that students already possess the background knowledge needed to effectively use variables in the new context. Alternatively, teachers could support students in learning how to use generic examples rather than variables in their arguments, which has the added ability of helping students see the general within a specific example (Mason & Pimm, 1984).

Tracing Lexi's understanding chronologically highlights the finding that she began constructing arguments that adhered to the generality requirement before explicitly referencing the generality requirement in her evaluations of provided arguments. During the second interview, Lexi's justifications when evaluating provided arguments suggest that her transition away from using examples as justification was motivated by the belief that proofs should explain why the conjecture was true rather than an awareness of the generality requirement. Hanna (2000) argues that explanatory proofs should be prioritized in the classroom with students who are first learning how to prove as "the fundamental question that proof must address is surely 'why?'" (p. 8). Lexi's work across the study, alongside findings from studies such as Bieda and Lepak (2014), offer a second reason why explanatory proofs should be prioritized in K-12 classrooms: namely, explanatory proofs can support students in recognizing the limitations of examples as justification for general claims since examples, on their own, rarely explain why a claim is true.

Finally, the summary of Lexi's understanding across the study (**Table 3**) highlights the influence the task, context, and other factors have on the types of arguments students produce and the evidence of understanding the generality requirement it reveals. For example, Lexi constructed a single counterexample when disproving the claim that all right triangles are similar during session 12 but then produced a class of counterexamples when disproving the claim that all rhombuses are similar during the final interview. A couple of factors could have influenced the change in response. First, Lexi completed the task about right triangles with three of her classmates, allowing for a conversation about their response to the task that was not possible during the final interview when she completed the task on her own. Second, the single counterexample Lexi produced for the right triangle claim came about in order to resolve her group's discussion about the possible acute angle measurements in right triangles. In contrast, Lexi did not appear to have a similar question about the angles of a rhombus; instead she only wondered out loud whether a square was also a rhombus before writing her argument. From a future research perspective, the differences in Lexi's responses across the tasks highlights the value in assessing students' understanding of proof using multiple types of tasks and varied contexts. The importance of considering various features of the task and context extends the argument made by Dawkins and Karunakaran (2016), who contend that researchers must consider the mathematical content used in proof tasks when making claims about students' understanding of proof.

CONCLUSION

The results from this study provide a nuanced perspective of the generality requirement with respect to proof. By analyzing Lexi's oral and written work across multiple points in time, this study demonstrated that students may not necessarily develop understanding of each facet of the generality requirement simultaneously. Instead, students may become gradually aware that the generality requirement not only influences the types of justifications used when proving true and false mathematical claims, but also influences the way that examples can be used in the proving process and dictates the use of variables when proving claims for infinite classes of numbers or notating aspects of mathematical diagrams that can vary. That said, the findings of the present study are limited in that they cannot reveal the range of understandings different students can have for the generality of proof due to the focus on one student who had experienced prior successes in mathematics and participated in lessons that explicitly focused on this aspect of proof. Given the central role the generality requirement plays in the proving process, this framework offers a way for researchers to gain a more robust assessment of students' understanding of this component of proof, information which can in turn be used to develop and refine instructional tasks designed to improve the teaching and learning of proof in K-12 classrooms.

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APPENDIX A

Table A. Overview of the instructional sequence

Session	Classroom task	Rationale for task
1	"Do all quadrilaterals tessellate?"	Initial task provided opportunity to test examples, using copies of paper quadrilaterals, where each of the cases seemed distinct from one another due to differences in their appearance. The task also offered a rationale for *why* the claim was always true that extended across every possible case.
2	Create step-by-step directions that explain how to tessellate *any* quadrilateral.	Task supported students in looking for features common to all quadrilaterals that result in them forming a tessellation. This task also helped transition students away from the guess-and-check approach that was common in session 1.
3	"Do all regular polygons tessellate?" "Do you still think that all quadrilaterals tessellate? If no, explain why. If yes, is there something special about quadrilaterals that make it so that they will always tessellate?"	First task served as a pivotal counterexample (Stylianides & Stylianides, 2009) in that it aligned with students' procedure for tessellating quadrilaterals but was true for only some cases. Thus, it helped to motivate a need to understand why all quadrilaterals tessellate. Second question, used as a written reflection prompt followed by class discussion, emphasized the scope of the original claim and continued to press on the question of why quadrilaterals will always tessellate.
4	Teacher-researcher summarized the first three sessions then explained why all quadrilaterals, but not all regular polygons, tessellate.	While discussing an informal proof for why all quadrilaterals tessellate, I aimed to situate proof as a way of understanding why a claim is always true and how we know that it will always be true for every quadrilateral.
5	Circle and spots problem and monstrous counterexample (Stylianides & Stylianides, 2009)	Tasks were used to further raise awareness of the limitations of using examples as justification when proving that a general claim is always true.
6	Introduce generic examples through a number trick: https://nrich.maths.org/2280 Students proved: "9*11 equals 1 less than 10^2, 3*5 equals 1 less than 4^2. Will this pattern always be the case?"	Task aimed to support students in understanding how to use a geometric diagram in the proving process; namely, by only attending to features that hold true for all cases within domain of claim. Second task aimed to support students in using and interpreting variables within the proving context, where they act as varying quantities.
7 8	Students constructed diagrams for six quadrilateral theorems (Cirillo & Herbst, 2011)	Task introduced students to the format of conditional statements and allowed for discussion around different notation methods and what can/cannot be assumed to be true based on a geometric diagram.
9 10	Develop definition of similar polygons; based on sequence in Kobiela and Lehrer (2015)	Task developed students' understanding of definitions while establishing content knowledge needed for sessions 11 and 12.
11 12	Students posed conjectures of the form "all ___ are similar", drafted an argument for squares, critiqued peer arguments, revised their arguments, then discussed proof as a whole class. Students continued proving/disproving remaining conjectures in session 12.	Task developed students' understanding of proof by engaging in multiple aspects of the reasoning-and-proving process. The whole class proof discussion in session 12 was used to introduce/revisit specific characteristics of proofs (logical structure, notation for diagram, need to justify each claim in argument).
13	Students individually proved the exterior angle theorem using the process described in sessions 11-12. Task was posed using two examples and the question, "is this a coincidence?"	Task continued to develop students' understanding of proof by engaging in the reasoning-and-proving process. Individual written work was used to assess students' current understanding of proof and ability to construct proofs. Task was posed in this manner to minimize the mathematical vocabulary and evoke uncertainty about the claim's validity (Buchbinder & Zaslavsky, 2008)
14	Students developed shared criteria for features of "good proofs"; based on Boyle et al. (2015).	Task assessed students' conceptions of proofs and allowed for reflected on key ideas from the teaching experiment.