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A Method of Compositional Fractal Analysis and its Application in Islamic Architectural Ensembles

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ABSTRACT

The purpose of the study is to examine the ensemble of Islamic architecture and its artistic expression due to its compositional characteristics (shape of the dome, minarets, etc.). The paper presents two methods: a visual fractal analysis and a dimension fractal analysis to verify the applicability of compositional fractal analysis for consideration of the spatial coherence of fractal characteristics (landscape plan, section, elevation, floor plan and ornamental motif). Using the same methodology, we analyze the consistency of fractal characteristics of objects Poi-Kalyan and Bibi-Khanym in Uzbekstan, taking into account their restoration and reconstruction, as well as famous ensemble of architecture, the Taj Mahal in India.

KEYWORDS
Islamic architecture; Taj Mahal; Poi-Kalyan, Bibi-Khanym; compositional fractal analysis

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Introduction

Fractal theory has gained a wide popularity since 1977 with the release of the book by French-American mathematician B.B. Mandelbrot (1983) entitled "The Fractal Geometry of Nature". With the heuristic potential of the concept, as well as a vivid presentation of the results of his discovery, B.B. Mandelbrot's contribution can be easily extended beyond the area of pure mathematical research. Since that publication, other researchers have showed that the concept can be applied to other areas of science and interdisciplinary research such as nonlinear dynamics (Bovill, 1996; Karperien, 2007), the theory of self-organization (Burrough, 1981; Barnsley, 1988; Peitgen, 1988; Hutchinson, 1981; Sala, 2002; Brown & Witschey, 2003), etc.

The application of fractal dimension in architectural research can be found elsewhere. For example, the B.B. Mandelbrot's (1983) box-counting method of

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fractal analysis was applied by W. Bechhoefer & C. Bovill (1994) and B. Hillier (1996) to compare the indigenous buildings and natural landforms in Amasya, Turkey. In the same way, fractal dimensions were applied to analyze several elevations of Robie House in Chicago and Unity Temple by architect Frank Lloyd Wright to compare the complexity between them (Lorenz, 2002). M.J. Ostwald & J. Vaughan (2013) applied the same fractal methods to analyze the architecture of Kazuyo Sejima to compare the characteristic complexity of nature and architecture, and then to compare the characteristic complexity of opaque and transparent building facades (Ostwald, 2013). The method was further developed by M. Batty and P. Longley (1994) to study the visual characteristics of urban spaces.

Now we turn to a new perspective, offered by the current research, and try to uncover some of the principles of the harmonic composition of the ensemble in Agra, relying on modern methods of fractal analysis. This study will further develop the box-counting method of fractal analysis in order to measure not only the level of complexity of architectural characteristics of the Taj Mahal, but also to study the consistency of the architectural characteristics in this ensemble by comparing the coherence between its elements (city plan, landscape, elevation, section, ornamental motive and plan of building).

As our further research will address Islamic architecture, it should be important to note that in the Arabic language, the term "fractal" means “separation”, “suppression or part of the whole”.

Methods

We use a compositional fractal analysis, i.e., a complex analysis including two methods: visual fractal analysis and a dimension fractal analysis to verify the applicability of compositional fractal analysis to consider the spatial coherence of fractal characteristics (landscape plan, section, elevation, floor plan and ornamental motif).

Visual analysis

A visual analysis of the architectural monument is observation of the buildings to identify their self-similar elements which, in fact, are the fractal patterns or identification of mathematical fractal forms.. This is an important stage of the study because the visual analysis helps to achieve some important tasks. First, it will determine whether the fractal models are present (like Sierpinski's "napkin" or the Menger sponge, etc.)in the plans, elevations, and other significant parts of the building. These fractal properties aids to study the building itself. If fractal forms are found in some of the repeated elements, such as cascades, arched forms and belts, then these can be compared with known fractal shapes (e.g., Julia sets, Koch snowflake, etc.).

Here we can rely on simple techniques in architectural composition analysis theory: The plan of the building, its section, and usually the central facade, transmit the fullest architectonics of the building. There are many examples of later alterations in a building, which may significantly change its view, but it retains its constructive and architectonic structure.

Visual examination of the building can be conducted by architects and viewers with great artistic sensitivity, who can reveal the main artistic characteristics of the building. This accompanies not only with an incomparable aesthetic pleasure, but it can also encourage a fractal analysis research that leads and inspires rigorous mathematical calculation in fractal geometry.

Fractal dimension analysis

The method of fractal dimension analysis is based on the research of W. Lorenz (2002) and C. Bovill (1996), who proposed the box-counting dimension method (DB) for calculation of the level of fractality in architectural forms. Here, the calculations of fractal dimensions by box-counting method have been carried out by using a standard software program FracLac for image J, which allows one to make fractal analysis of projections of building (Karperien, 2007). Using DB, it is easy to determine the fractal dimension of a complex image that cannot be described by other traditional methods. The method is based on the fact that the fractal dimension visually expresses the degree of "roughness" and "irregularity" of the structure, which defines the degree of complexity of the object.

The method is used as follows: a grid of a certain size (S1) is superimposed on the image and the number of cells including details of the image, which can then be calculated (N for s1). Then the size of the grid is reduced (S2) and again the number of cells is counted (N for s2).

The fractal dimension between two scales is then calculated by the relationship between the difference of the number of boxes occupied and the difference of inverse mesh sizes. This calculation can be expressed mathematically by the equations:

$$DB(1-2)=\left[\log\left(N(s2)\right)-\log\left(N(s1)\right)\right]/\left[\log\left(\frac{1}{s2}\right)-\log\left(\frac{1}{s1}\right)\right], \qquad (1)$$

$$DB(1-2)=\left[\log\left(N(s2)/N(s1)\right)\right]/\left[\log\left(s1/s2\right)\right], \qquad (2)$$

where S is the size of the grid, and N the number of cells that overlap with the image details (Lorenz, 2013; Bovill, 1996).

The algorithm of the method used in this study will be as follows:

Step One. Calculation of the fractal dimension of the monument projections (plans, elevations, etc.) using the box-counting method. This will be done by overlaying the graphic image with multi-scaled sets and counting the number of cells that have captured details of the image. The fractal dimension(FD) of each architectural projection of the monument can be determined to give the optimal result of fractal dimension in the projection as$1<FD<2$.These results will be reflected in the Tables.

Step Two. Justification of the deep coherence between various projections of the ensemble and, at the same time, justification of the coherence between the building and its environment. For that it is necessary to determine the consistency of multi-scaled levels of fractality between the main projections of the building (between façade, plan, sections, elevations, landscape plan, etc.).

For a more objective analysis of the consistency between two levels of fractal graphs the Correlation (or "CORREL" function in Microsoft Excel software) will be used.

The equation for the correlation coefficient is as follows:

$$CORREL(X,Y)=\sum\left[\left(x-x^{bar}\right)\left(y-y^{bar}\right)\right]/\sqrt{\left[\sum\left(x-x^{bar}\right)^2\sum\left(y-y^{bar}\right)^2\right]}, \qquad (3)$$

where x^{bar} and y^{bar} are sample valuesof X (array1) and Y (array2) ("AVERAGE" (array1) and "AVERAGE" (array2) in the software of Microsoft Excel (2016).

This function shows the level of correlation between the graphs of the various data sets. Here, the character of the fractal dimension in different scales (which usually can be clearly seen in diagrams) is very important to compare the fractal coherence between the different projections of the building.

Data, Analysis, and Results

Architectural characteristics of the Taj Mahal

Here, we present a visual analysis of a famous ensemble of architecture, the Taj Mahal in India. It is necessary to begin with a visual analysis that can likely disclose the artistic characteristics of the ensemble.

The history of the Taj Mahal is well documented (Panda, 2012). It was erected by Akbar's grandson in the period of the great Mughals during a remarkable blossoming of Indian architecture. The central element of the ensemble is a magnificent tomb of the beloved wife of Shah Jahan, Mumtaz Mahal, and it is known that after the Shah's death, he was buried in the underground crypt of the building (see Figure 1).

Figure 1. Taj Mahal in India

The shining white temple is opened for the viewer under the deep, dark coverage of the entrance arch. This architectural reception refers to the ancient idea of the confrontation between light and darkness, the earth and the order of the cosmos in all the world's religions and philosophies (Panda, 2012). Usually viewers devote little attention to large pattern of ornamental motifs on the dome of the gate, but those who do pay attention to it obviously have a good observation and profound knowledge. This pattern was also built by the same constructed arched motif. In addition to the aesthetic characteristics, this pattern presents a spiritual role in its simple graphical formula revealing the metaphysical formula of Cosmo-genesis. Thus, the central part of the ceremonial gate, i.e., the architectural overture to the sanctuary, is the best place for it.

The whole temple complex is full of symbolic images that can be confirmed by many examples; that is not the aim of this article. However, to confirm the fidelity of our approach, we present another example. From the central gate, the attractive, common form of the Taj Mahal is opened and the attentive eye soon settles upon another feature: four beautiful minarets, which have a deviation in the vertical axes. This deviation, according to the past research, was explained as a complex engineering solution to increase protection of the main building in case of earthquake. This explanation is reasonable, but does not completely reveal the obvious genius of the architect, neither in creating a self-similar remarkable minaret, nor in devising their expressive decline.

Can note, one of the most common decorative motifs in the Taj Mahal is the floral ornamentation that covers many architectural forms. It agrees with the aesthetics of the Mughal paradise and its floral elements. The red blossoming flower is the most common ornamental motif, which occurs not only in the details of the ensemble, but in the hands of Shah Jahan in the famous miniature (Fig. 2a, 2b). The divergent petals and the movement of their deployment recall the deviation of minarets. Now it is easy to present the central part of the ensemble with its surrounded minarets as a huge flower. This could be recognized as the best way to symbolize the great love of the Shah for his beloved wife. The visual experience of micro-plans of a flower and the whole temple complex with its deviated minarets were joined together to suggest a type of self-similarity that can be repeated in the compositional design of the monument (Burckhardt, 1999; Starodubova-Enikeeva, 2004; Panda, 2012).

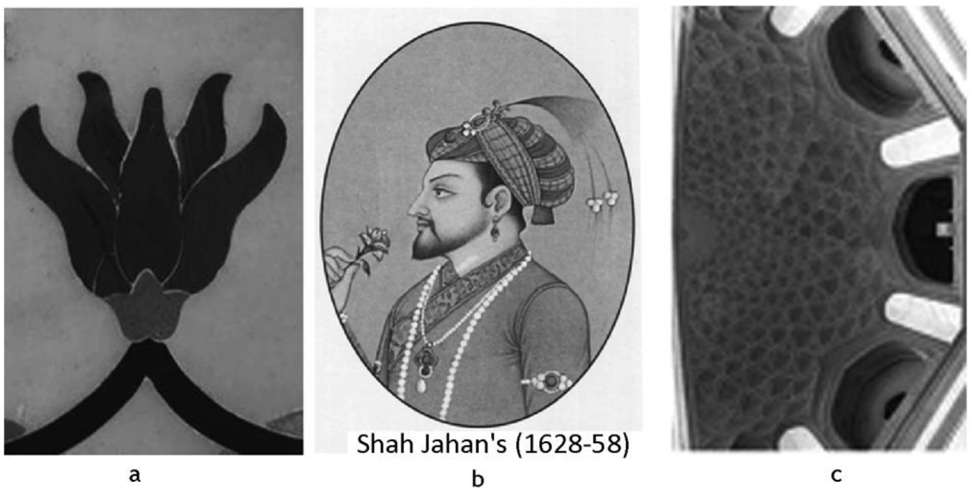

Figure 2. (a) A fragment of ornamental motif, (b) Shah Jahan's miniature, (c) Ornamental motif inside dome space

We emphasize once again that the multi-scale perception was the basic approach used to study the fractal characteristics of the Taj Mahal. We can say that the viewer intuitively conducts a fractal analysis of the Taj Mahal. The second approach demonstrates how rigorous scientific fractal analysis leads to the same results: it confirms the self-similarity and shows the strong harmony and consistency of all parts in the ensemble.

Now, the practical part of the article uses the compositional fractal analysis method (the two steps described above) to subsequently analyze the Taj Mahal's projections (Fig. 3): the city plan that surrounded the Taj Mahal, the landscape plan, elevation, section, ornamental motif inside the central dome (Fig. 2c), and the building plan to study the coherence between these projections and environment.

1. The main projections of the Taj Mahal have high fractal dimensions: city plan, landscape plan, elevations, sections and ornament (Table 1).

2. The level of consistency (correlation (Correl)) between the projections of the monument (between city plan, landscape plans, elevations, sections and ornament) is high; however, we see a low fractal consistency between the building plan and other projections, except the section (Table 2 and Diagram 1).

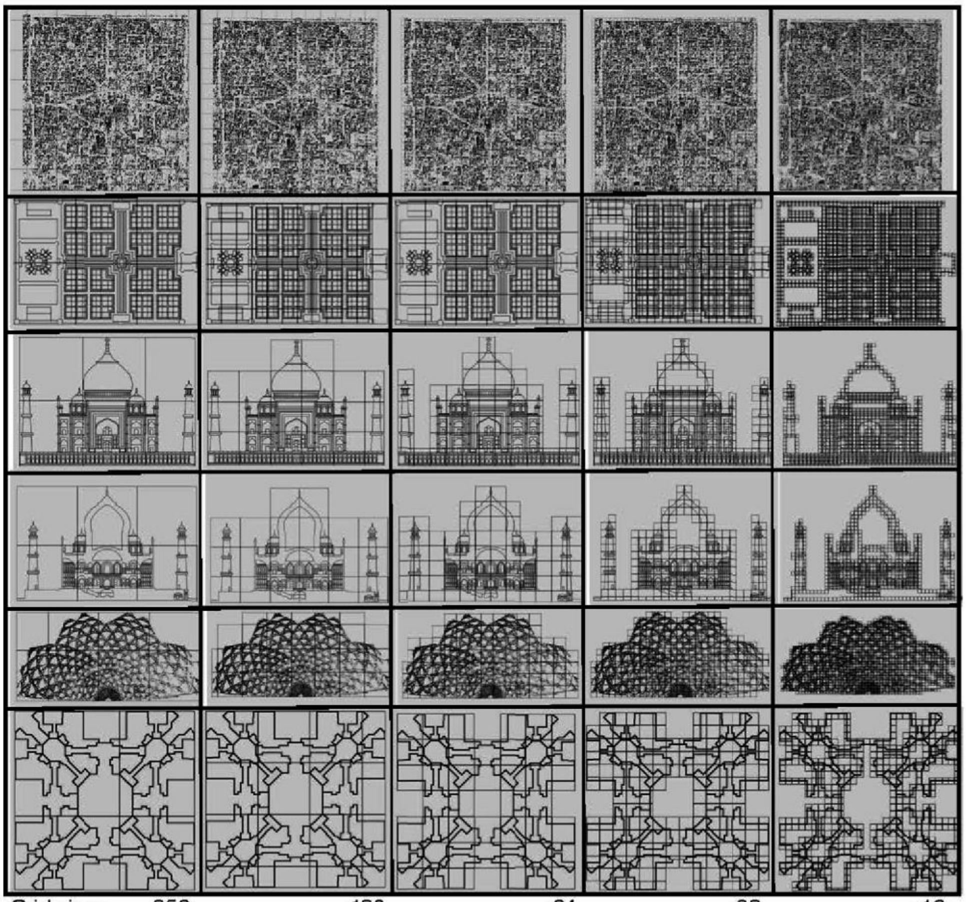

Figure 3. Fractal analysis of the Taj Mahal's projections

Table 1. Level of fractal dimension in projections of the Taj Mahal

Fractal dimension between		Fractal dimension					
Large grid scale	Small grid scale	City plan	Landscape plan	Elevation	Section	Ornament	Building plan
256	128	1.814	1.618	1.486	1.405	1.527	2.000
128	64	1.883	1.675	1.518	1.494	1.676	1.459
64	32	1.928	1.808	1.626	1.475	1.783	1.872
32	16	1.904	1.806	1.653	1.516	1.864	1.573
Average of fractal dimension		1.88	1.80	1.57	1.61	1.81	1.70

Table 2. Level of consistency between the projections of the Taj Mahal

Projections of Taj Mahal	Consistency between the projections of Taj Mahal	level
Between city plan & landscape plan	0.91	high
Between city plan & elevation	0.84	high
Between city plan & section	0.80	high
Between city plan & ornament	0.89	high
Between city plan & building plan	0.41	low
Between landscape plan & elevation	0.99	high
Between landscape plan & section	0.72	high
Between landscape plan & ornament	0.95	high
Between landscape plan & building plan	0.22	low
Between elevation & section	0.71	middle
Between elevation & ornament	0.96	high
Between elevation & building plan	0.22	low
Between section & ornament	0.88	high
Between section & building plan	0.84	high
Between ornament & building plan	0.49	middle

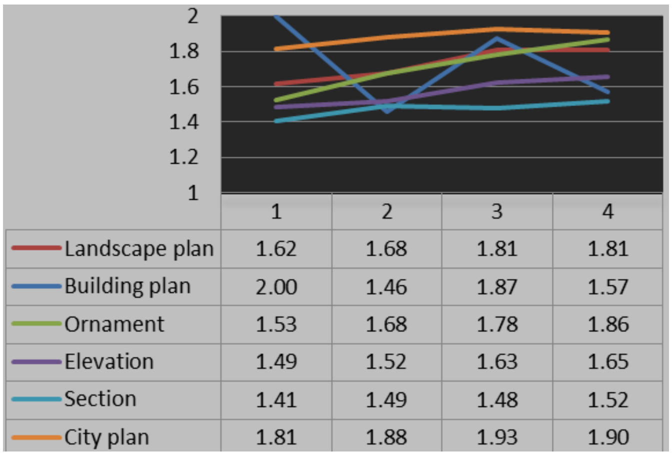

Diagram 1. Consistency between multi-scaled levels of fractality in each projection of the Taj Mahal and consistency between its projections

This result can be confirmed by the fact that the building plan has a special structural function, if compared to the other projections: it is not a visual element, and it was designed based on Euclidean geometry. Then, the section has a high degree of consistency with the plan itself and the other projections;

the section of the Taj Mahal is an internal "elevation" of the building and vertical "plan" together. Thus, it probably has high consistency with the plan of the building and its elevation, as well as with an ornament and landscape plan.

An important conclusion is that there is a high fractal coherence of both the urban and architectural levels. These reveal the integrity and unity between the ensemble and its environment, and the strong artistic expression of the Taj Mahal ensemble as well.

Architectural characteristics of Poi-Kalyan and Bibi-Khanym Mosques

Here, the method of compositional fractal analysis is applied to analyze the coherence of fractal characteristics of plan and elevation of Poi-Kalyan and Bibi-Khanym Mosques in Uzbekistan after their restoration and reconstruction. These results are shown in (Figs. 4 and 5).

The architectural ensemble Poi Kalyan (Poi Kalonansambli) is the central architectural monument of Bukhara in Uzbekistan. It is the second largest mosque in Central Asia, after the Bibi Khanum in Samarkand. It has the traditional style of Timurid architecture and faced mosaic tiles. The ensemble consists of three buildings that were constructed in the XII - XVI centuries: Kalon Minaret, Kalyan Mosque and Miri Arab. The mosque was built in place of a destroyed mosque: “Qarakhanid.” The construction of the mosque was completed in 1514. This ensemble was reconstructed and rebuilt after several fires and wars. Bibi Khanum Mosque (bibikhanump) was built in 1399-1404 as an architectural ensemble in the center of Samarkand. It has rich decoration of tiles, carved marble, etc. The composition of the ensemble reveals the traditional Persian architectural style.

Figure 4. Poi-Kalyan Mosque in Uzbekistan

Figure 5. Bibi-Khanym Mosque in Uzbekistan

The fractal analysis of projections of Poi-Kalyan Mosques and Bibi-Khanym are presented in Figures 6 and 7.

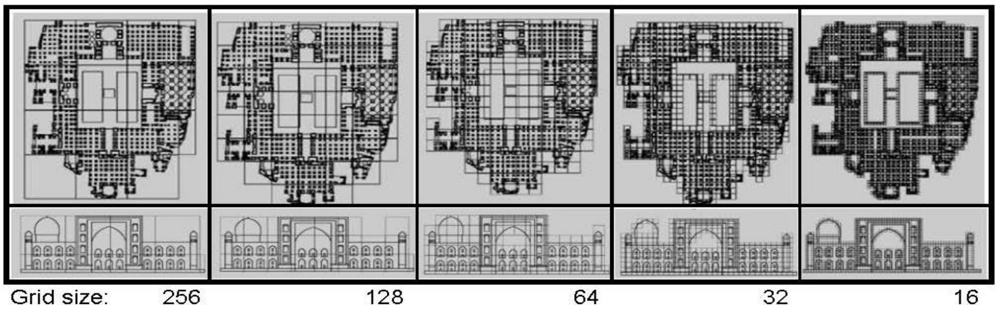

Figure 6. Fractal analysis of Poi-Kalyan's projections

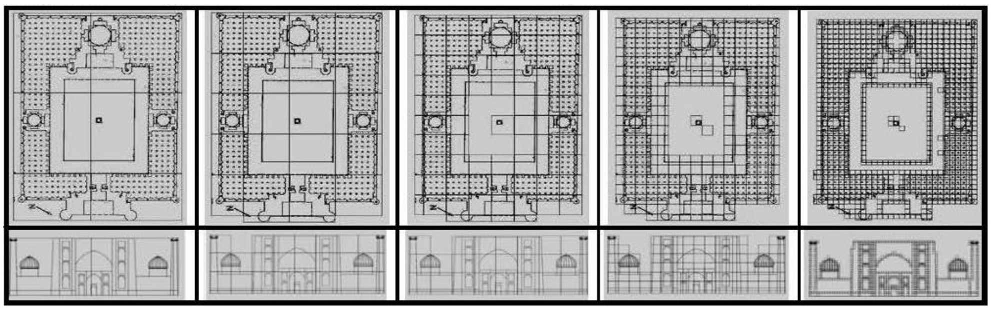

Figure 7. Fractal analysis of Bibi-Khanym's projections

From these results, it can be concluded that:

1. Like the main projections of the Taj Mahal, the plans and elevations of Poi-Kalyan and Bibi-Khanym Mosques have high fractal dimensions (Table 3);

2. The level of consistency between the projections of the monuments (between their plans and elevations) is high for Bibi-Khanym; however, there is a low fractal consistency between the plan and elevation of Poi-Kalyan (Table 4) as seen in Diagrams 2 and 3 as well;

3. We can connect this result with the reconstruction and restoration of the ensemble plan, which may affect the compositional consistency between the different architectural characteristics of the ensemble.

Table 3. Level of fractal dimension in projections of Poi-Kalyan and Bibi-Khanym Mosques

Fractal dimension between		Fractal dimension Poi-Kalyan Mosque		Bibi-Khanym Mosque	
Large grid scale	Small grid scale	Plan	Elevation	Plan	Elevation
256	128	1.737	1.824	1.875	1.737
128	64	1.856	1.843	1.844	1.807
64	32	1.767	1.618	1.724	1.626
32	16	1.768	1.429	1.710	1.512
Average of fractal dimension		1.78	1.67	1.79	1.67

Table 4. Level of consistency between the projections of Bibi-Khanym and Poi-Kalyan Mosques

Projections of Poi-Kalyan & Bibi-Khanym Mosques	Consistency between the projections of Poi-Kalyan Mosque	Level	Bibi-Khanym Mosque	Level
Between plan & elevation	0.33	Low	0.89	High

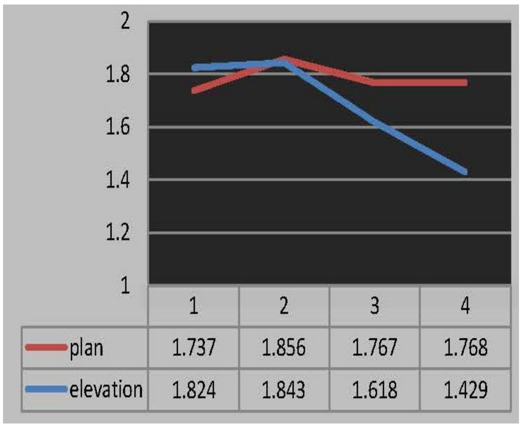

Figure 8. Consistency between multi-scaled levels of fractality in each projection of Poi-Kalyan Mosque and consistency between its projections

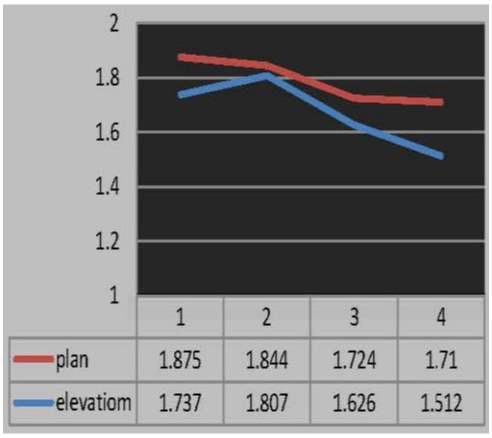

Figure 9. Consistency between multi-scaled levels of fractality in each projection of Bibi-Khanym Mosque and consistency between its projections

Discussion

Some special studies were able to show how the mind of the viewer can create a discussion about the integrity and harmony of the total picture, and how to connect this picture to previously accumulated knowledge and experience; how the visual impressions of the viewer encourage awareness of semantic analysis G. Bikram (1986), R. Panda (2012) and G. Riether & D. Baerlecken (2012). Even without these studies, we can also visually discover the secrecy of art in an ensemble by its relationship with the philosophy and psychology present in the chromatic composition of its decoration. Some obvious thoughts that may arise in the viewer's mind when approaching the Taj Mahal may include the pointed arch motifs, deducted in the outline of the ensemble. Upon a closer approach, these arched motifs are repeated *ad infinitum* in the decorative parts along the walls in several scales, the general lines of the main gate of the complex, the platform of the monument, and the landscape that surrounds it. The arched motif is a very active theme in defining the semantic characteristics of the ensemble.

It is also interesting to compare the complexity of the Taj Mahal, Poi-Kalyan and Bibi-Khanym Mosques with other architectural ensembles, both ancient and modern ones. It should be noted that the measure of the characteristic complexity of an image, elevation or plan, i.e., the fractal dimension D, is being between 1.0 and 2.0, with lower values (say D = 1.1) denoting an object with minimal complexity and high values (say, D = 1.9) suggesting a visually complex object. In this sense, the above ensembles have a high complexity, since their fractal dimensions are well above D = 1.5 (D=1.57-1.81 for the Taj Mahal, D=1.67-1.79 for Poi-Kalyan and Bibi-Khanym Mosques, see Tables 1 and 3). Interestingly, the other Islamic monuments, such as the Kılıç Ali Paşa Mosque and the Sűleymaniye Mosque have a similar complexity, demonstrating the fractal dimension from 1.5812 to 1.6989 and from 1.6592 to 1.7975, respectively (Ostwald & Ediz, 2015), indicating the common trends in the Islamic art from the viewpoint of fractal dimension. Examples of other famous ensembles indicate poorer complexity.

For instance, the values of fractal dimension for the ancient Maya settlements were found to be in the range of D=1.20-1.38 (Brown and Witschey, 2003), houses of the Murcutt's rural domestic architecture has been described as $1.192<D<1.309$. At the same time, K.C. Wen and Y.N. Kao (2005) point out that the fractal dimensions of Les Maisons Domino House plans designed by modern architect Frank Lloyd varies within 1.749 to 1.907.

Conclusion

To sum up, in this work, the method of compositional fractal analysis has been applied to identify the degree of coherence between the ensemble and its environment. Particularly, the method allowed us to identify the consistency (or inconsistency) between visual and non-visual projections of monuments, like ensembles of the Taj Mahal, Bibi-Khanym or Poi-Kalyan. In general, relying on the fractal theory, it was possible to put forward a comprehensive research task, i.e. to prove that the perceived harmony and artistic expression of the Taj-Mahal, the unity of all of its parts and the refinement of the surrounding landscape can be explained by a high level of fractal consistency between several spatial projections in elevation, plan, section, landscape plan, and the old plan of the city which surrounds the monument and shapes the cultural landscape.

Thus, based on the fractal characters of any ensemble the above method enables one to conclude on consistency of the ensemble before and after reconstruction or restoration, in other words, to determine the impact of reconstruction or restoration itself. This opens new artistic expression of the ensemble.

We believe that the researchers who use the methodology of fractal analysis have broad prospects in the study of architectural monuments. Particularly, this allows us not only to investigate more deeply the aesthetic aspects of the monuments, but also to identify common structural patterns that underlie the organization of nature, art and human perception. However, this trend, to our opinion, needs further development.

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Disclosure statement

No potential conflict of interest was reported by the authors.

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