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A Case Study Examining Links between Fractional Knowledge and Linear Equation Writing of Seventh-Grade Students and Whether to Introduce Linear Equations in an Earlier Grade

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ABSTRACT

As a part of the larger study, a case study of two seventh grade students, Peter and Willa, was conducted. To examine links between their fractional knowledge and algebraic reasoning, the students were interviewed twice, once for their fractional knowledge and once for their algebraic knowledge in writing linear equations that required explicit use of unknowns. Peter and Willa's fractional knowledge influenced the linear equations they wrote to represent quantitative situations. In particular, the findings showed that a reversible iterative fraction scheme is important to understand reciprocal relationships between two quantities in writing a basic linear equation of the form ax=b. Also, considering fractions as multipliers on unknowns is important to write linear equations. Implications for the possibility of introducing linear equations in an earlier grade and how to support the introduction are suggested.

Keywords: fractional knowledge, linear equation writing, reciprocal reasoning, quantitative reasoning

INTRODUCTION

Since the early 1990s, mathematics educators have been interested in elementary students' early algebraic reasoning (Bastable & Schifer, 2008; Carpenter, Franke, & Levi, 2003; Kaput, Carraher, & Blanton, 2008). Algebra researchers found that early experience with problems involving unknown quantities was helpful for elementary students' success with algebraic operations such as generating equations (Bodanskii, 1991; Carraher, Schliemann, Brizuela, & Earnest, 2006; Dougherty, 2008). For example, in line with this trend, the Korean national mathematics curriculum changed so that equation solving and proportional expressions, which used to be introduced in the seventh grade by junior high teachers, were taught in the sixth grade by elementary teachers (MEST, 2009). This revision was controversial in Korea because while some math educators were worried that early introduction of equations would overburden both elementary students and their teachers (Han, 2010; Seo, Yu, & Jeong, 2003), other math educators thought that early introduction of equation solving would improve students' chances of success in subsequent algebra courses (Carpenter et. al., 2003; Han, 2010). Through this case study including two students, I investigate the possibility of introducing linear equations in an earlier grade and suggest how to support it without overburdening students.

This paper reports a part of a larger task-based, clinical interview study on the relationships between students' fractional knowledge and equation writing (Hackenberg & Lee, 2015). This paper focuses on two of these students, seventh graders Peter and Willa. The purpose of this paper is (a) to demonstrate that fractional knowledge is closely related to establishing algebra knowledge in the domain of writing linear equations; and

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(b) based on this, to give suggestions for the possibility of learning linear equations in an earlier grade. The research questions for this paper are: (1) How do Peter and Willa reason with fractions as quantities and develop linear equations involving unknowns?; (2) What factors possibly influence the writing of two linear equations based on reciprocal relationship? and; (3) What suggestions can be given to the possibility of introducing linear equations in an earlier grade?

LITERATURE REVIEW

Traditionally algebra, including unknowns, variables, equations, functions, etc., has been introduced in seventh grade because of psychological assumptions about developmental constraints and readiness (Sfard, 1995). However, a recent trend in mathematics education research provides evidence that elementary school students can incorporate algebraic concepts and representations into their reasoning (Bodanskii, 1991; Carpenter et al., 2003; Carraher et al., 2006; Empson, Levi, & Carpenter, 2011). Regarding this possibility, in this section, I focus on differences and similarities between algebra and arithmetic and then on the notion of early algebra.

Algebra and Arithmetic

To explain the case for the early learning of algebra, Van Amerom (2003) discussed the similarities and differences between algebra and arithmetic in linear relationships and equation solving in terms of quantities, interpretation of letters, operation, and the concept of equality. Regarding quantities, arithmetic covers computation with known numbers, while algebra focuses on reasoning about unknown or variable quantities. In reference to the interpretation of letters, symbols, and expressions, arithmetic letters indicate abbreviations or units but algebraic letters refer to variable and unknown numbers. With regard to operation, in arithmetic 5+7 is interpreted as a command to add 5 to 7 while in algebra, 5+7 indicates a structure (e.g. a + b) of the number 12. In terms of the concept of equality, in arithmetic, an equal sign signifies the result of a computation whereas in algebra it represents initial equivalence. Despite these differences, Van Aemrom (2003) found that an informal approach towards algebra based on word problems could minimize the discrepancy between arithmetic and algebra.

Early Algebra

Early algebra researchers have taken the position that students' algebraic knowledge can be developed from students' implicit knowledge of numbers, and being exposed to algebra in elementary grades can help students become successful in algebra (Carpenter et al., 2003; Carraher et al., 2006; Kaput et al., 2008). That is, experience with generalizing arithmetical and quantitative relationships and symbolizing the generalizations with natural language and diagrams in algebraic activities, can support elementary students' understanding of the use of conventional symbol systems through syntactically guided reasoning on generalization (Kaput, 2008; Smith, 2003). Also, early algebra researchers (Carraher, Schliemann, & Schwartz, 2008) commented that early algebra is different from learning algebra early in three ways: (1) early algebra should build on contexts of problems; (2) early algebra should introduce formal algebraic notations gradually; and (3) early algebra should be closely connected with existing topics of early mathematics. As an example of early algebra research, Empson et al. (2011) found that elementary school students explicitly or implicitly used the fundamental properties of operation such as distributive property and equality in an equal sharing problem context involving the concepts of division and fractions, which are basic topics of early mathematics. From their findings, they proposed a term, *relational thinking*, in order to indicate the connection between early algebra and algebra. Empson and colleagues consider relational thinking as "a critical precursor to learning algebra with understanding" (p. 426).

In addition, some studies found that young students can learn to use letters to indicate unknown quantities (Bodanskii, 1991; Carraher et al., 2006, 2008; Dougherty, 2008; Van Amerom, 2003). For instance, Carraher and colleagues (2006) developed the concept of unknowns with third grade students by using a variable number line. That is, the researchers introduced a number line including *N* as a location after several lessons about the changes of quantities on number lines. Then the successive five locations from *N* by units of 1, 2, 3, 4, 5 on the right side were identified as *N*+1, *N*+2, *N*+3, *N*+4, and *N*+5. After that, the researchers asked students to represent the problem situation by using variables, when given that "Mary and John each had the same unknown amount of money on Sunday and then on Monday each received $3 from their grandmother." According to the researchers, about 70% of students used *N* to represent the initial amount of money and *N*+3 to indicate the amount of money on Monday. That is, most students were able to use a letter to represent unknown quantity.

In a similar vein, Van Amerom (2003), in her research, encouraged students in grade 5, 6 and 7 to develop their own informal strategies for solving word or story problems including obvious links between arithmetic and algebra, and then she supported students' development of algebraic way of thinking. Through this intervention, she found that elementary school students were able to understand the meaning of algebraic letters and to solve linear equations by developing informal strategies while solving word problems. Considering that arithmetic and algebraic knowledge are related to each other, and elementary school students are able to learn the concept of unknowns, in this case study of two seventh grade students I investigate the possibility of introducing linear equations in an earlier grade than they are currently taught.

CONCEPTUAL FRAMEWORK

In this section, I present five main ideas used in data analysis: a quantitative approach to fractional knowledge and algebraic reasoning; operations and schemes for building fractional knowledge; students' multiplicative concepts; whole numbers and fractions as multiplicative operators; and reciprocal reasoning.

A Quantitative Approach

A quantitative approach that is, a focus on operating with quantities and their relationships (Thompson, 1995), was taken in this study. Quantity is defined as a person's conception of the measureable attribute of an object or phenomenon, which involves a unit and iteration of the unit (Thompson, 1994). In the present study, I aimed to approach fractions using a model of lengths. For example, thinking about three-fourths does not mean solely recognizing a symbol of three over four, but rather taking one-fourth (corresponding to a unit) three times (corresponding to iteration) in a rectangle or segment. I also used a quantitative approach in the algebra interview. When one engages in quantitative reasoning with unknowns, reasoning about relationships among quantities is the foundation for writing equations, for example, considering unknowns as quantities that have yet to be measured.

Operations and Schemes for Fractions: the Iterative Fraction Scheme and the Splitting

An operation is a mental action. Schemes are composed of operations, and they are a way of describing goal-directed behaviours in which people engage (von Glasersfeld, 1995). In this study, for students who have constructed an iterative fraction scheme (IFS), three-fifths is literally one-fifth three times, and seven-fifths is one-fifth seven times. That is, there is a multiplicative relationship between unit fractions and any fraction they can make. Research shows that students who can construct an IFS also can reverse the operations of that scheme (Hackenberg, 2010). For example, consider presenting students with a bar (rectangle) that is seven-fifths of another bar and asking students to make the other bar. To solve this problem, students can split the given bar into seven equal parts and take one of those parts five times to form the second bar. Students who make a solution like this have constructed a reversible iterative fraction scheme (RIFS).

Constructing both an IFS and an RIFS requires an operation called splitting (Hackenberg, 2010; Norton, 2008; Steffe & Olive, 2010). The splitting operation is a unification of partitioning and iterating. Partitioning indicates dividing a whole into some number of equal parts, and iterating means repeating a unit or a part of a unit to get larger amount (Steffe, 1991). For example, in order to find a new bar that is three times the length of a given bar, one can iterate the given bar three times to get the new bar and also partition the new bar into three equal parts to get the given bar. Solving the problem like this refers to that students consider partitioning and iterating as combined.

Students' Multiplicative Concepts

Researchers have shown that constructing an IFS requires particular multiplicative concepts. Students' multiplicative concepts are based on how they produce and coordinate units. The first multiplicative concept involves coordinating two levels of units in an activity. The second multiplicative concept involves coordinating two levels of units prior to an activity and also coordinating three levels of units in an activity. The third multiplicative concept involves coordinating three levels of units prior to as well as in an activity.

The two students in this case study had constructed the third multiplicative concept as described in prior research (Hackenberg & Tillema, 2009; Steffe, 1994). This meant that they could coordinate three levels of units prior to operating. For example, prior to partitioning, both students could view a one-foot length as a unit of three equal parts, each containing five parts, and they could operate further with that unit structure. Researchers have found that students' multiplicative concepts influence their fractional knowledge (Hackenberg, 2010; Steffe & Olive, 2010).

Whole Numbers and Fractions as Multiplicative Operators on Unknowns

Coordinating three levels of units prior to operating is necessary but not sufficient for conceiving of fractions as multiplicative operators. Let's say that the height of a father is three times the height of his young son. If students are asked to write an equation for this situation, they can write equations such as F=3S or S=(1/3) F, in which F represents the height of the father and S refers to the height of his son. In this example, students who represent "three times the height of the son" as "3S" may have developed a concept of *a whole number as a multiplicative operator on an unknown*. In the same way, students who represent "one-third the height of the father" as "(1/3) F" may have developed a concept of *a fraction as a multiplicative operator on an unknown*. If one-third of a whole amount includes the idea of taking that part three times to produce the whole, then the whole multiplied by one-third must produce one-third of the whole.

Reciprocal Reasoning

Reciprocal reasoning means being able to flexibly switch between taking either of two quantities as the referent to measure the other quantity (Hackenberg, 2010). For example, the money that person A has is two-thirds of the amount of money that person B has. If students can reason reciprocally, they know that the amount of money that A has is two-thirds of the amount of money that B has, and also that the amount of money that B has is three-halves of the amount of money that A has. In order to reason reciprocally, conceiving of fractions as multiplicative operators on knowns and unknowns is necessary (Hackenberg, 2010), and further reciprocal reasoning is required to write linear equations with different referents (Driscoll, 1999).

METHODS

Participants

Two students, Peter and Willa, from a middle school in a Midwestern town of US were selected via classroom observations, consultation with students' teachers, and one-on-one, task-based selection interviews to assess students' multiplicative concepts (Hackenberg & Tillema, 2009; Steffe, 1992). In addition to selection interviews, the students completed a written fractions assessment to triangulate claims about their fractional knowledge.

The two seventh grade students were assessed to have constructed the third multiplicative concept. When a co-researcher and I observed their class, they were using a standard algebra textbook and classroom activities seemed to focus on manipulations of standard algebraic notation. I interviewed them, and a colleague was a witness-researcher (Steffe, & Thompson, 2000). Although the case study is based only on two hour-long interviews, these interviews revealed thorough snapshots of their thinking, not everything about their fractional knowledge and algebraic reasoning, but a small yet critical subsets of their idea, enough to provide some insight into an issue related to the possibility of introducing linear equations in an earlier grade without cognitive overload.

The two students were selected because I had interviewed both of them in the larger study and determined they had the same multiplicative concept. Also, because the interviews were conducted in the first half of the fall semester, they had just become seventh graders in the same classroom. Studying them at this point in time could be informative for considering the possibility of learning linear equations in an earlier grade as shown in the 2009 Korean Mathematics Curriculum revision that moved the introduction of linear equations from the seventh to the sixth grade (refer to Han, 2010). Specifically, investigating the relationships between fractional knowledge and linear equation writing might shed light on ways to successfully execute the earlier introduction of linear equations.

Data Collection and Analysis

Each student participated in two 45-minute, semi-structured, video-recorded interviews, focusing on fractions and algebra respectively. I conducted the four interviews with a co-researcher as witness, who took notes on the interactions between the students and me and sometimes intervened to probe students' reasoning which I missed. Both students had the fractions interview first, followed by the algebra interview in seven (Willa) or eight (Peter) weeks. The interview protocols were refined in a prior pilot study, and quantitative

Table 1. Peter and Willa's Fractional Knowledge

No.	Fraction Interview Questions	Peter & Willa's Fractional Knowledge
F1	Could you draw a separate candy bar that is 9/7 of the given bar?	Both constructed IFS
F2	(a) This candy bar is 3/5 of another one. Could you draw the other candy bar? (b) This candy bar is 4/3 of another one. Could you draw the other candy bar?	Both constructed RIFS
F3	Tanya has $84, which is 4/7 of David's money. Could you draw a picture of this situation? How much does David have?	Both initially showed some difficulty in RIFS with composite units, although Peter eventually demonstrated RIFS and Willa did not demonstrate it.

situations were used as a basis for all problems. The interview problems were designed so that the reasoning involved in the fractions interview was a foundation for solving problems in the algebra interview. In the fractions interview, unknowns could remain implicit in a student's thinking while in the algebra interview unknowns were explicitly asked about. For example, in the fractions interview, students were asked to draw a picture of a situation in which one party had $84, which was four-sevenths of the other party's amount of money, and then figure out the latter amount. In the algebra interview, students were asked to draw a picture of and write equations for a similar situation in which both monetary quantities were unknown.

Two digital video cameras were used to record each interview: One recorded interaction between the interviewer and student, and the other recorded the student's written work. These videos were mixed electronically into one file for analysis. Then to analyse data, I repeatedly viewed Peter and Willa's video files while taking notes and then created detailed notes comprising transcriptions, interpretations, and conjectures about the participants' reasoning. Through this process, I made a model of the fraction schemes that Peter and Willa had constructed and their reasoning about unknowns and writing equations (Steffe & Thompson, 2000). The models were refined by discussing them with the co-researcher. The refined models helped me identify relationships between Peter and Willa's quantitative reasoning with fractions and algebraic reasoning in the area of writing and solving equations.

FINDINGS

Peter and Willa's Fractional Knowledge

The following interpretations of Peter and Willa's fractional knowledge are based on their responses to three questions of the fractions interview, referred to as F1, F2, and F3 problems (see **Table 1**). In the fractions interview, both Peter and Willa showed that they had constructed an IFS and an RIFS. However, Willa had not yet fully constructed an RIFS with composite units although she showed potential progress through interaction with me. In contrast, Peter eventually demonstrated an RIFS with composite units via interaction with me even though he had, like Willa, initially struggled.

Peter and Willa each constructed an IFS and an RIFS. In F1, when asked to draw a bar (rectangle) that was nine-seventh of a given bar, both students partitioned the given bar into seven equal parts and drew a separate bar that was the same length at the given bar. Then they each added two more parts to the bar (see **Figure 1**). When asked how much one part was, both students said "This [a piece from the new bar] is one-seventh on the top [original] bar and this one is one-ninth on the bottom [new] bar." When asked if it was possible for the same piece to have different names, they said, "Yes, the piece can be both one-seventh and one-ninth."

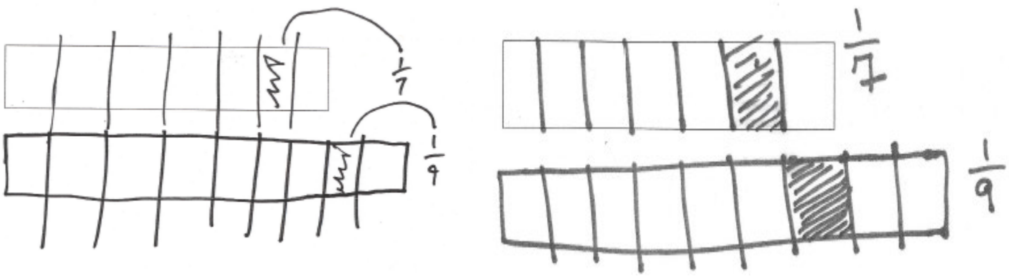

Figure 1. Peter (left) and Willa's (right) pictures for F1

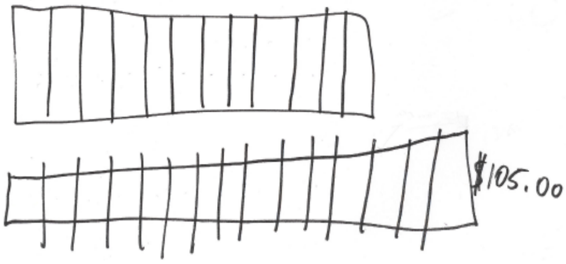

Figure 2. Peter's initial picture for F3

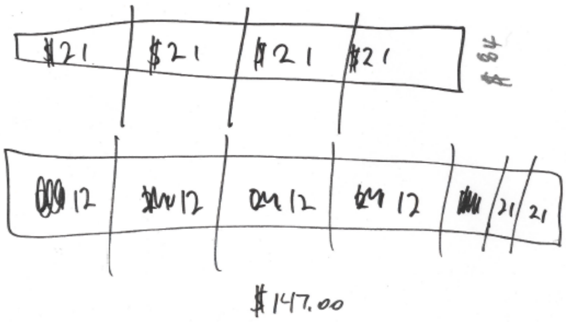

Figure 3. Peter's revised picture for F3

In F2, students were given a bar that was three-fifths of a second bar they were to draw. Both Peter and Willa partitioned the given bar into three equal parts and drew two more parts of this size onto the given bar. In addition, in an extension of F2, the students were given a bar that was four-thirds of a second bar they were to draw. Both Peter and Willa partitioned the given bar into four equal parts and drew a new bar that consisted of three parts of that size. So this data excerpt shows that Peter and Willa think of nine-seventh as one-seventh taken nine times, of three-fifths as one-fifth taken three times, and of four-third as one-third taken four times.

However, both Peter and Willa initially experienced some difficulty in solving problems that required them to use their RIFS with composite units. In the F3 problem, when asked to draw a picture of the situation in which Tanya's $84 is four-sevenths of David's money, Peter drew a bar partitioned into 12 equal parts for Tanya's money and a second bar partitioned into 15 equal parts for David's money, saying "84 divided by 7 is 12 so you can make 11 segments here and then you add three more to get David's amount of money." Then when asked to find how much money David has, he decided that David's money was $105 (84+7•3) because he thought each piece of Tanya's bar was worth $7 (see **Figure 2**).

So I posed a non-composite unit situation like that of F2 problem: This given rectangle is four-seventh of another rectangle; can you make the other rectangle? Peter partitioned the given bar into four equal parts and added three of those parts to make the seven-seventh bar (see **Figure 3**). When we returned to F3 problem, Peter divided 84 by 4 to determine that one part of Tanya's bar was worth $21, and he seemed convinced that was the size of one of Tanya's four parts. Finally, he found that David's money amounted to seven units of $21.

Initially, Peter did not use his RIFS because he did not split four-sevenths into four equal parts and iterate one of these parts seven times to make seven-sevenths. Instead, Peter began with twelve-twelfths and

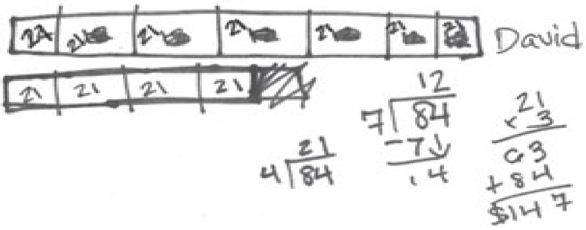

Figure 4. Willa's picture for F3

Table 2. Peter and Willa's Algebraic Knowledge

No.	Algebra Interview Questions	Peter's Knowledge	Willa's Knowledge
A1	There are 5 (or 3) identical candy bars and each candy bar weighs some number of ounces. If 'h' refers to the weight of one bar, how much does 1/7 (or 1/5) of all the candy weigh?	Consistently considers whole numbers and fractions as multiplicative operators on unknown	Considers whole numbers as operators consistently but fractions as operators on unknowns only some of the time
A2	Theo has a stack of CDs some number of cm tall. Sam's stack is two-fifths of that height. Can you draw a picture of this situation? Can you write an equation?	Demonstrates the use of RIFS with composite units. Generates the second equation which is in reciprocal relationship with the first equation	Does not generate the second equation which is in reciprocal relationship with the first equation

indicated 15 of those parts as David's money. However, going back to a problem like that of F2 was helpful for him to rethink the composite unit situation, and finally he demonstrated the use of an RIFS with composite units.

In responding to F3 problem, Willa first drew a big bar and partitioned it into seven equal parts to represent David's money (see **Figure 4**). Then she drew another bar consisting of four pieces to represent Tanya's money. To find David's money, she divided 84 by 7 to get 12. Then she wrote $12 each piece of the bar representing David's money (note that this is crossed out in **Figure 4**). When I asked her whether a piece of Tanya's bar represented $12, Willa divided 84 by 4 and changed $12 to $21, saying "There is $21 [indicating each of Tanya's pieces]. So that equals $84 and then he has three more pieces. David has to have more money because she has just four-sevenths of his money." However, she did not immediately determine David's money. So I restated my understanding of her solution to this point, and Willa figured out that a piece of David's bar represented $21. Finally Willa found the amount of David's money, $147, by computing $21 times 3 and then adding it to Tanya's $84.

In this solution, Willa did not seem to use her RIFS, either. That is, she did not split four-sevenths (representing Tanya's money) into four equal parts and iterate one of these parts seven times to make seven-sevenths. Instead, she began with seven-sevenths (representing David's money) and identified four of those parts as Tanya's money. Approaching the problem in this way means that her IFS and RIFS were not activated, although Willa showed that she had constructed these schemes in previous questions.

Peter and Willa's Algebraic Knowledge

The evidence for the following interpretations of Peter and Willa's algebraic knowledge comes from two questions of the algebra interview, referred to as A1 and A2 problems (see **Table 2**). In the algebra interview, both Peter and Willa showed that they considered whole numbers as multiplicative operators on unknowns. However, they demonstrated differences in the use of fractions as multiplicative operators and reciprocal reasoning.

When asked to draw a picture of one-seventh of five candy bars, Peter said to me that it would be impossible because there were only five candy bars. So I scaled down the given numbers and paraphrased the initial problem to a sharing context such as how much a person got if three people fairly shared two candy bars. Peter solved this problem by partitioning each candy bar into three equal parts. When I asked him how much of both candy bars each person got, he said, "two candy bars each divided by three people and since two times three is six, each person gets two parts of the candy bars." When asked to name the fraction that represented, Peter said that a person would get two-thirds of one bar and two-sixths of both bars. Also, he said that a person

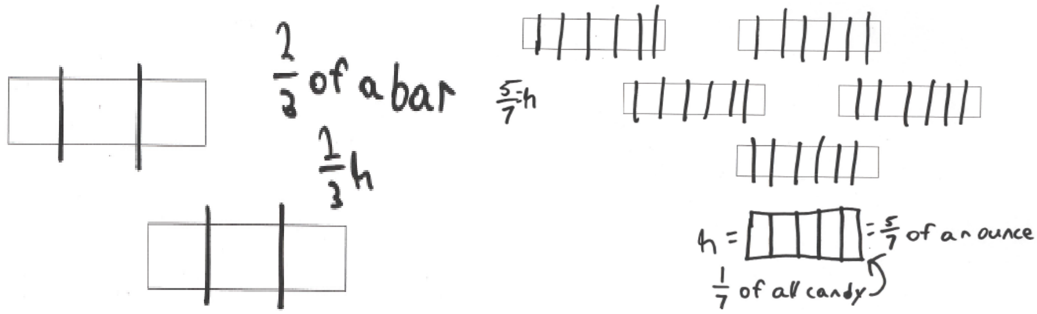

Figure 5. Peter's pictures for A1

would get "2/3 h" weight when asked write expressions for this situation. Then, he tried to solve the original question. He partitioned each bar into seven equal parts and said that each person will get five-sevenths of an ounce (he thought of an ounce as "h" in this problem) as he did in the previous problem (see **Figure 5**). The data excerpt shows that Peter considered a whole number as a multiplicative operator on an unknown by partitioning the bars into a whole number of parts that he could divide by the number of people sharing the candy. Also, when asked how much bar a person would get, he answered by using a fraction as multiplicative operator on unknown such as 2/3 h or 5/7 of an ounce.

When I asked Willa to draw a picture of a one-seventh of five identical bars, she also answered, "I am not sure how to do it." So I asked her a modified version of the problem by scaling down the given numbers and using a sharing context. When asked how to share three candy bars fairly with five people, Willa partitioned each bar into five equal parts and shaded three parts from the first bar. When asked to write expressions for this situation, she wrote $\frac{3h}{5}$. However, her explanation of the expression did not match the expression itself. That is, in explanation, Willa brought up another equation such as, "here 3h/5. Every person will get three-fifths of a candy bar. That brings three-fifths. (She wrote 3/5). I think it will be 3/5 h." So I asked whether the two expressions (3h/5 and 3/5 h) were same or not. She answered, "No, this is like….this one is 3 times *h* divided by 5. This one is 3 divided by 5 times *h*" Then when asked whether they gave her the same picture or different pictures, she answered that they were different. When asked about how she saw 3h/5 in her picture, she answered that three candy bars were divided by five people and thus her picture indicated 3h/5. However, right after her response, Willa said, "I don't know. Maybe these are same. I don't know how to draw this one [3/5 h]. I think this is the same. I am confused. They [3h/5 and 3/5 h] are not same equation but I think they are same."

In this data excerpt, Willa was not sure whether "multiplied by 3 and divided by 5" was the same as the fraction, 3/5. That is, Willa seemed to have derived understanding of the two expressions from different thinking processes, but she did not appear to be sure whether the two expressions represented the same quantity. In addition, this data excerpt showed that it could be more natural for Willa to use whole numbers as multiplicative operators on an unknown (e.g. 3h ÷ 5) rather than fractions as multiplicative operators on an unknown (e.g. 3/5h).

In working on the A2 problem, when asked to draw a picture of situation in which Theo has a stack of CDs some number of cms tall and Sam's stack is two-fifths of that height, Peter first drew five short line segments for Theo's stack and two short line segments for Sam's stack (see **Figure 6**). When asked to express the situation with an equation, he wrote $S=T\cdot\frac{2}{5}$, where 'S' stood for Sam's stack and 'T' for Theo's stack. When asked for another equation in terms of Sam's stack, Peter wrote $S+\frac{3}{5}=T$, saying "Since you know Sam's stack 2/5, you can just say Sam's stack plus three-fifths which will create five and one will be just the answer." I suggested that he check his equations with numerical numbers. When I asked him to find Theo's stack given that Sam's stack height was 10 cm tall, Peter used his picture and determined that Theo's stack height was 25 cm tall, saying "Since this is two, you just divide 10 by 2 and then take them times five." Then Peter was asked to check his equations with these numbers. He found that his second additive equation did not work (He wrote 10+3/5=25 and 10+3/5·10=25 as seen in **Figure 6**).

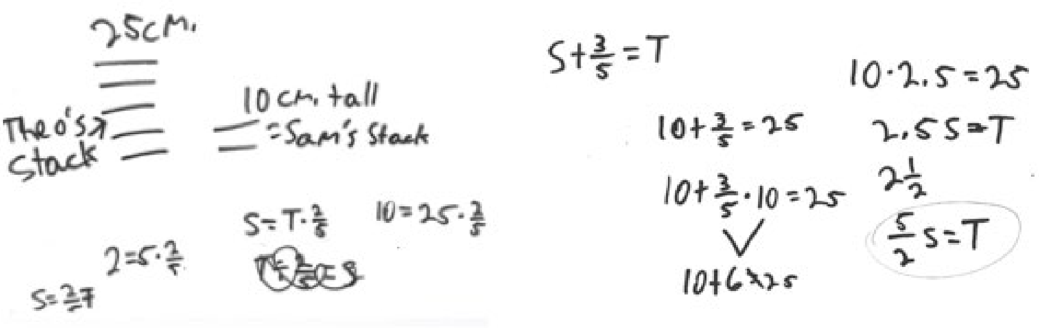

Figure 6. Peter's pictures for A2

I suggested that he think about something to do Sam's stack to get Theo's as he did just before with a small change like adding on some fraction or multiplying. As soon as he heard my suggestion, Peter said that 10 times 2.5 equals 25. When asked what would be an equation for the original situation (not the example), Peter wrote 2.5S=T. When asked to change the decimal to a fraction, he first wrote $2\frac{1}{2}$ and then changed the mixed number into an improper fraction ($\frac{5}{2}$) when requested to do so. Finally, he got the equation $\frac{5}{2}$•S=T. When asked if the two equations had some relationship to each other, he said "they were same because this equation [5/2·S=T] is just this one [S=T·2/5] multiplied $\frac{5}{2}$ by reciprocal fraction."

This data excerpt shows that Peter could consider fractions as multiplicative operators on unknowns and also observe that the two equations he produced were related to each other by a reciprocal fractional relationship. Also, Peter showed the use of an RIFS when he determined the height of Theo's stack from that of Sam's stack (10 cm). That is, Peter divided 10 by two (which seemed to correspond to partitioning the two-fifths of Sam's stack into two equal parts) and multiplied by 5 to get Theo's stack height (which seemed to correspond to iterating one-fifth five times). In other words, Peter used Sam's stack height as the referent to determine Theo's stack height with the numerical examples. In this regard, Peter seemed to be able to do reciprocal reasoning if he consciously perceived what he did with the numerical examples. However, although there is not enough evidence to claim that he had constructed reciprocal reasoning quantitatively when he generated the second equation (5/2 S = T), through interacting with me, he could determine the second equation by using a numerical example.

In working on A2, Willa drew an appropriate picture immediately: a rectangle for Theo's stack height that was partitioned into five equal parts and a shorter rectangle for Sam's stack height that spanned the first two of those parts. When asked to write equations for the height of Sam's stack in relation to Theo's, Willa wrote "S=Sam" and "t=theo." Then she wrote $\frac{2}{5}$ t=S. When asked to write another equation, she wrote 2• ($\frac{t}{5}$) =S. When asked to write an equation in terms of Theo's stack, Willa produced the equation $\frac{5s}{2}$= T. When asked to explain the equation by referring to her picture, Willa said that she got Theo's five-part stack by taking Sam's two-part stack five times (10 pieces) and dividing them by two. I asked her to check her equations by using numerical examples. Willa checked her three equations with s=10 and T=25 and found that they worked.

This data excerpt shows that although Willa developed two correct equations of $\frac{2}{5}$ t=S and $\frac{5s}{2}$= T, her work on this problem does not demonstrate that she consistently considered fraction as multipliers on unknowns: Willa used 2/5 as a multiplicative operator on an unknown, but she did not use 5/2 as a multiplier on an unknown. In this respect, Willa's work on the A2 problem corroborates that she regularly conceived of whole numbers as operators on unknowns but she did not always use improper fractions as multipliers on unknowns. Also, this data excerpt allows me to propose that Willa had yet to construct reciprocal reasoning because she did not come up with another equation based on the reciprocal relationship between two quantities by using five-halves or two and half as a multiplicative operator of an unknown.

DISCUSSION AND CONCLUSIONS

In this study, I described Peter and Willa's fractional knowledge and algebraic reasoning in writing linear equations. In terms of their fractional knowledge, I assessed that both had constructed a splitting operation, an IFS, and an RIFS. However, when they solved the F3 problem, neither initially used his/her RIFS with composite units. That is, Peter and Willa did not split four-sevenths into four equal parts and iterate one of those parts seven times, although Peter eventually used his RIFS with composite units through interaction with me.

In terms of their algebraic knowledge, I found that Peter regularly used whole numbers and fractions as multipliers on unknowns but Willa did not always use fractions as multipliers on unknowns although she regularly used whole numbers (e.g. 3h/5 and 5s/2=T) and proper fractions (e.g. 3/5 h and 2/5 t=S) as multipliers. That is, Willa showed some indication that she used fractions as multipliers in the A1and A2 problems. However, the indication was not consistent across all of her work in that she did not use five-halves as a multiplier on an unknown when asked to come up with another equation in the A2 problem. In addition, Willa did not come up with second equation using a reciprocal relationship with the first equation while Peter figured out second equation using a reciprocal relationship although he took some time and used some numerical examples to do it.

The difference that Peter and Willa showed in writing two linear equations based on reciprocal relationship seems to be related to two factors although I could not conclude that there were no other factors: (1) Whether to activate an RIFS with a composite unit; and (2) whether to consistently consider fractions as multipliers on unknowns. In the fractions interview, Willa did not activate her RIFS with a composite unit, while Peter eventually activated it throughout the fractions interview. This activation of an RIFS in the fractional problem situation appears to influence the activation of reciprocal reasoning in the algebraic problem situation using an unknown. That is, to use an RIFS, students have to split the given bar into the number of the numerator and repeat the number of the denominator to get a whole unit. To activate reciprocal reasoning, they have to essentially do the same operation as they did for an RIFS but also one more thing, switching views to take another quantity as the referent to measure the other quantity. For example, if students are asked to find the second bar when a given bar is four-sevenths of the second bar, they will partition the given bar into four equal parts and iterate one of them seven times to get a seven-sevenths bar by using an RIFS. However, to find two equations for this situation by using reciprocal reasoning, they need to switch their view of the referent: The first bar is four-sevenths of the second bar because they need to take four times of a unit in the second bar (one-seventh) to get the first bar and the second bar is seven-fourths of the first bar because they need to take seven times of a unit in the first bar (one-fourth) to get the second bar. Moreover, in the algebra interview, Willa did not always view fractions as multiplicative operators on unknowns, while Peter consistently considered fractions as multiplicative operators. These findings support prior research showing that in writing and solving a basic linear equation of the form $ax = b$, considering fractions as multiplicative operators on unknowns and understanding reciprocal relationship between two quantities are important factors (Driscoll, 1999); and that these important factors are affected by students' fractional knowledge (Hackenberg, 2010; Lee & Hackenberg, 2014).

Returning to the possibility of learning linear equations in an earlier grade, the findings of this research may provide some insights. First, regarding the question of whether learning linear equations in elementary curriculum cognitively overburdens students, I would say that some students, particularly those who have not yet constructed the third multiplicative concept, might be overburdened by trying to accomplish these goals by the end of the sixth grade. However students like Peter and Willa, who have constructed the third multiplicative concept, might not be overburdened if they are supported in developing their fractional knowledge and algebraic ideas at the same time. Such students probably could accomplish basic equation solving in conjunction with developing fractional knowledge by the end of their sixth grade year.

Second, concerning the issue of whether teaching equations in the elementary curriculum overburdens teachers, I would also say that teachers will not be overburdened if they try to develop students' fractional knowledge and equation writing together based on a quantitative approach. For examples, currently many countries tend to require that teachers teach all fractional knowledge prior to introducing equation solving (Choy, Lee, & Mizzi, 2015; Mizzi, Lee, & Choy, 2016). With this sequence, moving equation solving into the sixth grade might increase the burden on teachers by putting more topics into the curriculum. In a sense, the sequential approach seems reasonable in that students can build fractional ideas before using them to build algebraic ideas, which require conceptualizing unknowns, i.e., thinking about quantities that they do not yet know the measure of, as well as operating on and notating these unknowns. Reasoning with unknowns is not explicitly required to build fractional knowledge.

However, using cases like Peter and Willa to reflect on the idea that learning fractions and learning algebra might proceed in concert with each other implies that teaching both ideas together might be one way to reduce teachers' burden. That is, fraction problems based on a quantitative approach (similar to the tasks used in

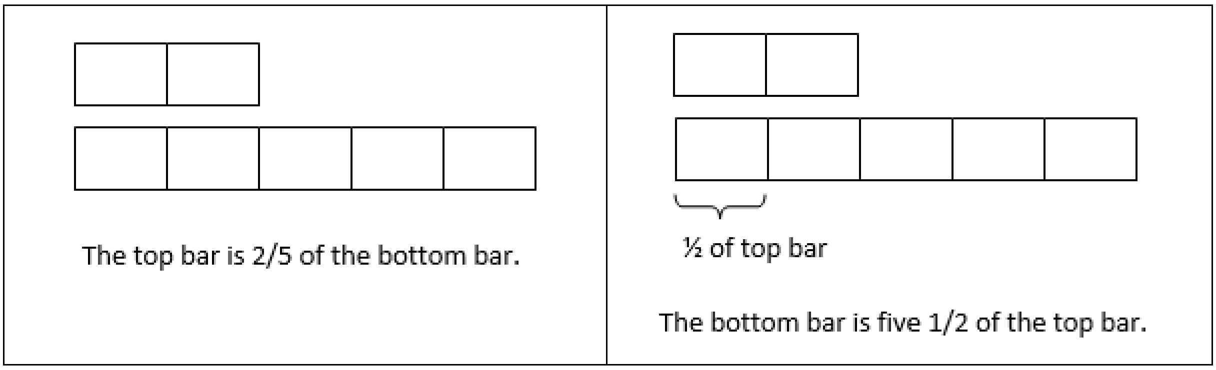

Figure 7. An example of measuring two quantities using different referent units

this research) can be a site for students to develop their understanding of equation writing and solving. For example, in this study, I posed word problems that brought together fractional and algebraic reasoning and provided probing questions that reoriented Peter to try to solve a problem with composite units in a way that expanded use of his RIFS, which was related to developing the second equation based on reciprocal relationship.

Outcomes of this study suggest that the decision to introduce linear equations in an earlier grade than they are currently taught should depend on whether students have achieved the necessary preparation rather than on psychological developmental constraints attributed to sixth grade students. That is, if students are prepared to activate an RIFS in composite unit situations and to consistently view fractions as multipliers on unknowns, they will be able to learn writing and solving linear equations in the sixth grade.

Then how can we help students develop an RIFS and a view of fractions as multipliers on unknowns? First, to support students' construction of an RIFS, teachers or teacher educators need to emphasize the construct of measurement in dealing with fractions. That is, beyond defining a fraction m/n as m parts out of all n equal-sized parts that constitute the whole, they should provide opportunities for students to consider any fractions as a whole number multiple of a unit fraction. For example, students need to engage in measuring how many times a unit fraction $1/n$, fits into the fraction by iterating the unit fraction m times. Also, students need to clearly understand the relationship between parts and a whole in terms of the measurement construct, such as considering that a whole, m/m can be created by iterating a unit fraction, $1/m$, m times. In addition, to encourage the reciprocal reasoning involved in an RIFS, students need to experience measuring two quantities using different referent units (refer to Lee, 2017). That is, when students are given two fractional bars as seen in **Figure 7**, they should be able to describe the relationship of the two bars in two ways: (1) When considering the bottom bar as a referent unit, the top bar is two-fifth of the bottom bar, (2) when considering the top bar as a referent unit, the bottom bar is five halves (or two and half) of the top bar.

Second, to help students develop their view of fractions as multipliers on unknowns, teachers or teacher educations need to give them opportunities to imagine *unknown quantities* as potential results of measuring fixed extensive quantities prior to actually measuring them (refer to Hackenberg & Lee, 2015). That is, teachers or teacher educators can encourage students to visualize the quantitative relationships by having them draw a picture using rectangular bars to represent unknown quantities and their relationship.

For example, when we write A × B, we interpret A as the number of groups which have B number of items. That is, 3 × 5 means 3 groups of 5 items. As an extension of this multiplicative concept, 1/3 × 15 will be interpreted that a size for one group is 15 but there is only 1/3 of one group. Thus, we need to subdivide the length of 1/3 into 5 units of equal parts to represent the portion that is 1/3 of a total number of 15 units (**Figure 8**, top left). In a similar way, $1/3 \times x$ indicates subdividing the length of 1/3 into some number of equal parts although how many will span the length has not yet been determined (**Figure 8**, top right). When applying the commutative property of multiplication, 15 × 1/3 will represent 15 groups of 1/3 by iterating 1/3 of a fraction bar 15 times (**Figure 8**, bottom left) and $x \times 1/3$ will show x groups of 1/3 in which 1/3 of a fractional bar is repeated x number of times (**Figure 8**, bottom right).

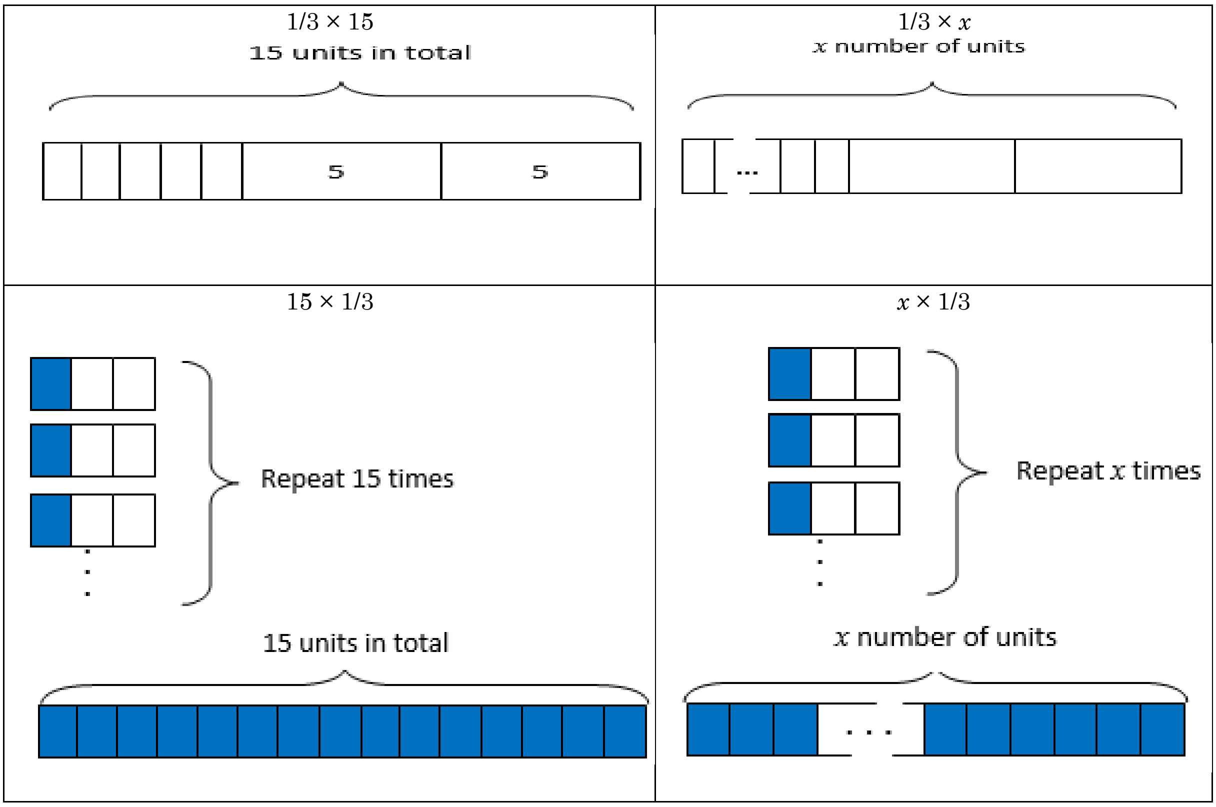

Figure 8. Representations to show viewing fractions as multipliers on a known (left) and on an unknown (right) based on a quantitative approach

This study has some limitations in that the insights and implications are based on a case study of two students. Thus, although it provides some insights into the possibility of teaching linear equations in an earlier grade than they are currently taught, to expand on the results, follow-up studies should include more participants. However, although based on a single case, this outcome shows promise that teaching fractions and equation writing together can create an effective synergy in students' development of fractional knowledge and algebra ideas, a possibility that warrants further investigation.

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