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The problem of Interference between Discontinuities of the First Order

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ABSTRACT

The article discusses the problem of determining the differential characteristics of discontinuities, waves and currents behind them. In this paper research history of gas-dynamic discontinuities' differential properties is discussed. The concept of weak discontinuities (discontinuous characteristics, discontinuities of first order) is analyzed. The differential conditions of dynamic compatibility, connecting curvatures of discontinuities with non-uniformities of the flow before and after them are given. The typical problems of interference between discontinuities of first order are provided: interaction of the shock with a weak tangential discontinuity and discontinuous characteristics, refraction of weak discontinuity on a tangential discontinuity, interference of weak discontinuities between themselves. The article presented typical interference problem of discontinuities of first order: interaction of the shock with a weak tangential discontinuity and discontinuity characteristics, refraction of weak discontinuity on a tangential discontinuity, interference of weak discontinuities between themselves. The practical importance of first order problems of interference of discontinuities is shown, because the discontinuity in first derivatives can lead to the formation of shock waves within the smooth flow - the so-called "suspended shock wave."

KEYWORDS
Gas-dynamically discontinuity; a weak discontinuity; discontinuous characteristics; the method of characteristics

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Introduction

Purpose of the work is to review the study history of differential properties of gas-dynamic discontinuities (GDD) and of weak discontinuities (discontinuous characteristics), as well as their interference, development of a new variant of the method of characteristics of second-order precision and the concept of weak discontinuities (discontinuous characteristics, first-order discontinuities).

GDDs can be of zero order Ф0 (a center of rarefaction/compression wave, shock and a sliding surface) (Bulat & Bulat, 2015), on which gas-dynamic-parameter flow parameters (P, v, ϑ) are discontinuous and of the first order, also

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called weak discontinuities (discontinuous characteristics, weak tangential discontinuities) Φ1, on which first derivatives of the gas-dynamic variables are discontinuous (Uskov & Mostovykh, 2012). Discontinuity of streamlines curvature is a weak discontinuity. You can determine features (discontinuities) Φi of gas-dynamic variables space of any order. The term "weak discontinuity" was first introduced by R. D. Adhemar (1904). Note that some authors call weak discontinuities a discontinuous characteristic, since, as shown in the book by R. Courant, and K.O. Friedrichs (1948), weak discontinuities are the characteristics of a differential equations system, which describe the motion of the gas. Courant and Friedrichs showed that strong discontinuities do not coincide with the acoustic characteristics (propagation lines of small perturbations in the gas flow) whereas weak GDDs always do. Consequently, small perturbations in the area behind strong GDD influence its parameters and the geometry and properties of weak GDD are completely determined by the gas stream before it.

The need to obtain ratios between characteristics of strong discontinuities such as flow acceleration along a streamline or curvature of streamlines with derivatives from gas-dynamics variables on both sides of a strong discontinuity has been associated mainly with three objectives:

– The study of the behind the curved shock waves

– Calculation of the interaction of strong and weak discontinuities

– The origin of discontinuities in a smooth flow.

The importance of latter problem is illustrated by a known fact - when building the profile of the Laval nozzle it is necessary to ensure the smoothness of generating line up to the second order inclusive (Silnikov, Chernyshov & Uskov, 2014). Discontinuity of nozzle wall curvature, for example, in conjunction of toroidal critical section with the main polynomial area results in the formation of suspended shock at the nozzle wall, which sharply increases the thermal load and may result in the burnout of the wall.

Materials and Methods

The linear partial differential equation with two independent variables is used:

$$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\partial t}+C\frac{\partial^2 u}{\partial t^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial t}+Fu=0 \tag{1}$$

wherein A, B, C, D, E, F – functions x and t, and u - unknown function.

The propagation of disturbances is studied by decomposition the unknown function into double Fourier integral

$$u(x,t)=\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}d\omega\cdot u_{k\omega}\exp[i(kx-\omega t)] \tag{2}$$

where k and ω satisfy the dispersion equation

$$-k^2A+2k\omega B-\omega^2C+ikD+F=0 \tag{3}$$

The characteristic is determined by rapidly varying disturbances, the coefficients A, B, C, D, E, F can be considered constant, and the wave number k and frequency ω - infinitely large, then the dispersion equation is simplified

$$-k^2A+2k\omega B-\omega^2C=0\,. \tag{4}$$

If some disturbance δu is added to the solution u(x, t), it then will propagate only along the characteristics determined by the dispersion equation, at a velocity of

$$a = \frac{\omega}{k} = \frac{B \pm \sqrt{B^2 - AC}}{C}. \quad (5)$$

If *B2 – AC>0* then there are two different characteristics and the original equation is hyperbolic. Since the velocity of disturbance propagation is *dx/dt*, then the characteristic equation can be written as *V(x,t)=dx/dt*, i.e. the slope of the characteristic is equal to the local velocity of disturbances propagation.

If there is a line defined in space (Figure 1), the flow parameters along which is defined with the distribution of vector velocity inclination angle *θi* and of Prанndtl-Mayer functions

$$\omega = \frac{1}{\sqrt{\varepsilon}} \arctan\sqrt{\varepsilon(M^2 - 1)} - \arctan\sqrt{M^2 - 1}. \quad (6)$$

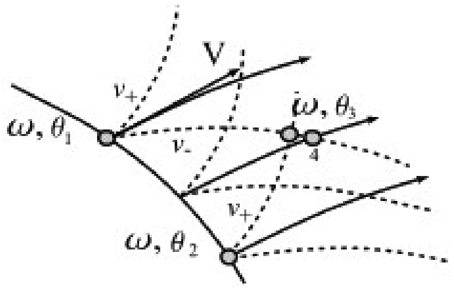

Figure 1. Illustration of weak discontinuities method

From each point on the curve (1-2-3) the flow line are coming out. If M> 1, then from these same points two characteristics of different families v+, v- are coming out. Flow lines are also the characteristics are also called entropy.

The equations of supersonic flow of an ideal gas in the natural coordinate system s-n using the Prandtl-Meyer functions

$$\sqrt{M^2 - 1}\frac{\partial \omega}{\partial S} - \frac{\partial \vartheta}{\partial n} = \delta \frac{\sin \vartheta}{y} \quad (7)$$

$$\sqrt{M^2 - 1}\frac{\partial \vartheta}{\partial S} - \frac{\partial \omega}{\partial n} = -\frac{N_3}{\Gamma(M)}, \quad (8)$$

where,

$$\Gamma(M) = \frac{\gamma M^2}{(M^2 - 1)^{1/2}} \quad (9)$$

Adding and subtracting these equations, we obtain a new system

$$\sqrt{M^2 - 1}\frac{\partial}{\partial S}(\omega \pm \vartheta) \pm \frac{\partial}{\partial n}(\omega \pm \vartheta) = \delta \frac{\sin \vartheta}{y} \pm \frac{N_3}{\Gamma(M)} \quad (10)$$

projecting new equations on the characteristics direction and eliminating the derivatives along the normal to flow lines, we obtain the conditions along characteristics along the right discontinuity (χ = 1):

$$\frac{d}{d\nu_+}(\omega+\vartheta)=2\frac{\partial}{\partial S}(\omega+\vartheta)\cos\alpha+\left(\frac{N_3}{\Gamma(M)}-\frac{\sin\vartheta}{y}\right)\sin\alpha \quad (11)$$

$$\frac{d}{d\nu_+}(\omega-\vartheta)=\left(\frac{N_3}{\Gamma(M)}+\frac{\sin\vartheta}{y}\right)\sin\alpha\,; \quad (12)$$

along the left discontinuity ($\chi = -1$):

$$\frac{d}{d\nu_-}(\omega-\vartheta)=2\frac{\partial}{\partial S}(\omega-\vartheta)\cos\alpha-\left(\frac{N_3}{\Gamma(M)}+\frac{\sin\vartheta}{y}\right)\sin\alpha \quad (13)$$

$$\frac{d}{d\nu_-}(\omega+\vartheta)=-\left(\frac{N_3}{\Gamma(M)}-\frac{\sin\vartheta}{y}\right)\sin\alpha\,. \quad (14)$$

The second equation in the system (12.13) are the well-known conditions on the characteristics of different families written in the traditional form. The first equations (11.13) contain the nonuniformities of the flow. They allow defining the conditions on characteristics taking into account the first derivatives of the gas-dynamic functions. V.N. Uskov suggested formulating DDCC1 on weak discontinuity in a form of theorem on U_χ – function (Bulat, 2014)

$$U_\chi=\frac{\partial}{\partial s}(\omega+\chi\vartheta)\,. \quad (15)$$

Theorem on U_χ - function. Euler equations allow a discontinuity of flow irregularities on characteristic, but U_χ-function on weak discontinuity remains continuous.

Non-traditional conditions on the characteristics (11.13) can be rewritten using U_χ - function. For example, for nonvortical flow:

$$\frac{d\omega}{d\nu_-}=U_-\cos\alpha;\frac{d\vartheta}{d\nu_-}=-U_-\cos\alpha+\frac{\sin\vartheta}{y}\sin\alpha\,; \quad (16)$$

$$\frac{d\omega}{d\nu_+}=U_+\cos\alpha;\frac{d\vartheta}{d\nu_+}=-U_+\cos\alpha-\frac{\sin\vartheta}{y}\sin\alpha\,. \quad (17)$$

Conditions on characteristics can be rewritten in the form, which is explicitly permitted relatively to main flow nonuniformities

$$\frac{\partial\omega}{\partial S}=\frac{1}{2\cos\alpha}\left(\frac{d\omega}{d\nu_+}+\frac{d\omega}{d\nu_-}\right); \quad (18)$$

$$\frac{\partial\vartheta}{\partial S}=\frac{1}{2\cos\alpha}\left(\frac{d\vartheta}{d\nu_+}+\frac{d\vartheta}{d\nu_-}\right). \quad (19)$$

Results

Differential dynamic compatibility conditions

Thus, the derivatives of gas-dynamic variables before and after a weak discontinuity are related to each other and to the curvature of discontinuity. These ratios are called differential dynamic compatibility conditions (DDCC) on the weak discontinuity ($DDCC_1$).

The first results in finding the ratio between derivative of gas-dynamic parameters on both sides of strong discontinuities (DDCC$_0$), obtained in the late 40s - 50s (Lighthill, 1949; Truesdell, 1952) concerned a particular case of a flat or axisymmetric curved stationary shock wave. Somewhat later these results have been summarized (Lighthill, 1957) for the case of problems with high dimensionality.

The analytical solution of the problem of single shock's interaction with weak gas-dynamic discontinuities was obtained by S.P. Dyakov (1957). In this paper we consider the gas flow with an arbitrary equation of state, but it was assumed that the incoming flow was weakly perturbed relatively uniform one, and the surface of shock was slightly different from flat. In this case S.P. Dyakov (1957) managed to build a special coordinate system in which the pressure behind the shock wave satisfies the Poisson equation, and in this coordinate system, to formulate conditions on the derivatives of gas-dynamic parameters. For an arbitrary shock curvature DDCC$_0$ were obtained in 1962 by Shih-I Pai (1962), and then summarized by V.V. Rusanov (1973) for the case of non-stationary flows. In the case of a uniform incident flow Rusanov, in a Cartesian coordinate system, managed to get an expression of differential characteristics of the flow behind the shock through its curvature. In the paper by S. Molder (1979), the single arbitrary curved shocks in a uniform flow of an ideal gas are researched. The ratios obtained for the derivatives of gas-dynamic variables behind the shock allowed to describe a small vicinity behind a strong discontinuities using decomposition of various gas-dynamic variables: pressure, density, modulus and incline angle of velocity vector. Most of ratios between derivatives on both sides of a strong discontinuity mentioned above were rather cumbersome. As a result, the problem of interference between strong and weak discontinuities in gas dynamics were either solved by the method of small perturbations, or obtained as a special case of problems of strong discontinuities interference.

Considering the ratio limit on a shock with $J \to 1$, V.N. Uskov obtained ratios, remarkable by their simplicity and ease of use, between the non-uniformities of flow N_i before the shock and after it (Mostovykh & Uskov, 2011).

$$N_i = c_i \sum_{j=1}^{5} A_{ij} N_j \tag{20}$$

The main gas-dynamic non-uniformities of flow N_i:

$$N_1 = \frac{\partial \ln P}{\partial s}, N_2 = \frac{\partial \vartheta}{\partial s}, N_3 = \varsigma \frac{\partial \ln P_0}{\partial n} \tag{21}$$

where P - pressure, θ – incline angle of velocity vector, P_0 - total pressure, ζ - vorticity, n - the length of a normal to streamlines, s - the length of an arc along a streamline. N_1 - flow nonisobariy (pressure gradient) along the direction, projected on a current line. N_2 - curvature of the streamlines. For the purpose of generality $N_4=\delta/y$ (δ=0 for plane flow) and $N_5=K_\sigma$ (curvature of the shock) were added to the equations. Coefficients A_{ij}, c_i were published in (Uskov, 1987) and are studied in detail by A. L. Starykh.

Problems of interaction between weak discontinuities

Apparently, the first general solution of interaction of a weak discontinuity with one-dimensional shock wave (Figure 2) was obtained by G. B. Whitham (1977).

Similar solutions for refraction of a plane stationary shock wave on a weak tangential discontinuity were received by A.L. Adrianov (2000). A great job of summarizing the results was performed by A. V. Omelchenko. In his paper (Omelchenko, 2002), he deduced a simple ratio between derivatives on non-stationary one-dimensional shock wave.

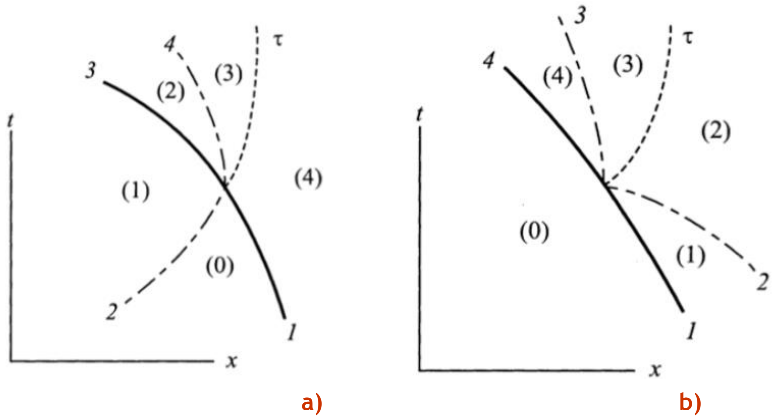

Figure 2. A one-dimensional unsteady interaction of a weak discontinuity and shock waves. (a) - a weak counter discontinuity, (b) - a weak catching discontinuity. 1 - shock wave, 2- weak discontinuity, 3 - reflected weak discontinuity, 4 - shock waves moving with changed acceleration, τ- reflected weak discontinuity

V.N. Uskov (1980) in his doctoral dissertation in obtained explicit analytic solutions of the problems of first order for a single shock, regular reflected shock from the curved wall, triple configurations of shocks, the interaction of two shock of one direction and of opposite directions, shock's refraction on a tangential discontinuity, shock's interaction with the counter and catching weak discontinuity (Figure 3a), refraction of shock and weak discontinuity on weak discontinuity (Figure 3b), the interaction of weak discontinuities of one direction and of different directions with themselves (Figure 3c).

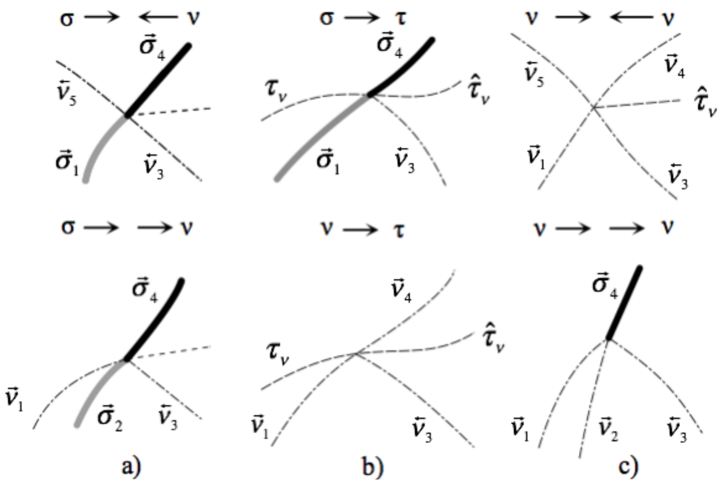

Figure 3. Interaction of weak discontinuities. a) the intersection of a shock and a weak discontinuity; b) the refraction a shock and a weak discontinuity on weak discontinuity; c) interaction between weak discontinuities. σ_i - shocks, τ_v - weak tangential discontinuity, v_i - weak discontinuity (discontinuous characteristic)

Discussions

Equations on the characteristics are, in geometric sense, equivalent to the Newton equation for particles moving along the characteristics (Uskov, et al., 2014). In the classical formulation the characteristics are introduces as the directions, along which the small disturbances propagates, so they are also called lines of influence.

Alternative representation of conditions on characteristics using U_χ - function has an important advantage. U_χ - function remains continuous during passing through the discontinuous characteristics. Consequently, the conditions written in this form "do not notice" the weak discontinuities, which seriously increases the method's accuracy and simplifies the construction of numerical algorithms. In the traditional characteristics method for the proper operation it is required to locate and track their formation on suspended shocks (Katskova et al., 1961). The system (2) and equation derived therefrom allow, any spatial point, to calculate the curvature of the flow line, curvature of two characteristics, Mach number gradient along the flow line, Mach number gradient along the characteristics.

In 1989 P.V. Bulat (1989), basing on solutions obtained by V.N. Uskov (2014) for interaction of weak discontinuities with shocks and between themselves, developed a characteristics method of second order (Bulat, Zasukhin & Uskov, 1989), which were called the method of weak discontinuities (MWD). In some special cases the author managed to reduce the solution to analytical and ordinary differential equations (Bulat et al., 1990; 2000). The numerical implementation of the method was much easier and convenient than the previously developed characteristics methods of second order (Panov, 1957).

On the basis of weak discontinuities method and DDCC1 P.V. Bulat and V.N. Uskov developed pseudo-one-dimensional nozzle theory, which takes into account the curvature of the nozzle and the curvature of the shock waves. The jet boundary, the incident shock wave in overexpanded jet (Figure 4) are build, the dependence of boundary curvature of shock's jet on the edge of the nozzle is researched (Bulat et al., 1993). Later M.V. Chernyshev researched all differential characteristics of the flow in overexpanded jet in the vicinity of nozzle edge, revealed specific values of nonisobarity ratio, researched specific points on the incident shock.

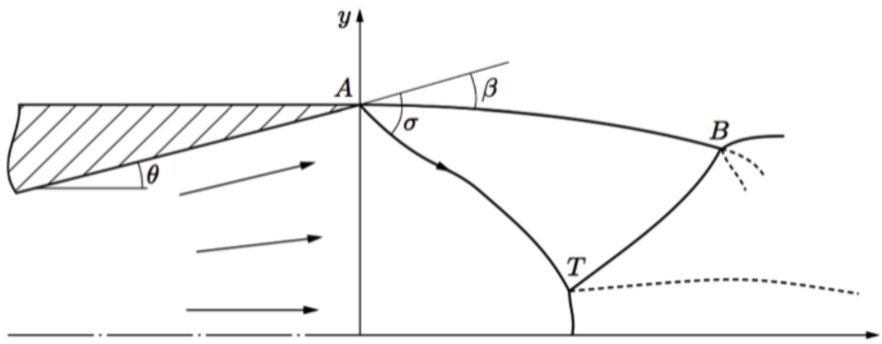

Figure 4. Overexpanded jet. A - nozzle edge, B - intersection point of the shock reflected from the axis with the jet boundary, AB - jet boundary, T - triple point, AT - incident shock

Then P.V. Bulat et al. (2002) found similar solutions for underexpanded jet. He researched the formation of suspended shock inside it, obtained the dependences of jet boundary curvature and of the suspended shock's formation point on a nonisobarity ratio. He also described the flow in the interaction area of shock, reflected from the axis of symmetry, with the mixing layer at the jet boundary, as well as the flow before the Mach disc. In all cases the obtained computation time is much lesser than while using the traditional characteristics method.

Conclusion

Many problems of gas dynamics in mathematical terms are reduced to solving a system of quasi-linear equations in partial differentials. The solution is obtained by integrating the system of ordinary differentials along specific directions, which are called characteristics. On the basis of these ratios the article provides analysis of the problems of one-dimensional shock wave's interaction with colliding and catching weak discontinuities. As an example of using the obtained results in applications of gas dynamics we consider the problem of shock wave propagation through the channel of variable section. The resulting solutions of interaction of strong and weak discontinuities are related to stationary plane flows. As it is known, from the kinematic point of view these tasks are equivalent to interaction of one-dimensional non-stationary waves and discontinuities (strong and weak).

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Disclosure statement

No potential conflict of interest was reported by the authors.

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