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The Elements of Substitution Thinking and Its Impact On the Level of Mathematical Thinking

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ABSTRACT

In current teaching trends, in addition to the modernization of teaching, an effort to make the education more effective appears. In the field of mathematics, the effectiveness of education is closely linked to the development of pupils' logical thinking. It is necessary to find the elements of mathematics, which constitute the connection between particular mathematics fields and teach them purposefully. The substitution is not a natural way for pupils to solve equations and inequalities and therefore it is necessary to develop substitution mind throughout the study of mathematics. The pupils with developed substitution thinking perform better in solving tasks from different areas of mathematics. The ability to transform the task according to developed substitution thinking, allows pupils to use the knowledge of mathematics in the interdisciplinary subject relationship effectively. A sign of advanced substitution thinking is a discovery of way how to solve problems. It also allows the learner to atomize the task and thus solve the task in the elementary steps. The mathematical confidence gained through developed substitution thinking becomes a pupil's internal motivation factor to tackle the most demanding tasks and not only mathematical ones.

KEYWORDS
Substitution, substitution thinking, atomisation, effectively

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Introduction

Currently mathematics penetrates not only into subjects of scientific, economic and technical fields but also into everyday practical human activity. Mathematics can be said to be an effective mean for finding solutions of problems and answers many questions in the areas mathematics penetrates. Mathematics teaching should be an invitation for pupils to the creation of maths. "Teaching mathematics means to do mathematics, mathematics is an

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action and so it can provide a realistic approach to teaching mathematics, which proceeds from real impulses and problems from the world of the pupils. A pupil must become a rediscoverer, a creator of mathematics, which develops, not only his mathematical activity but his mathematical thinking as well"(Fulier, Šedivý, 2001, p.6). From this point of view, we split the study of mathematics into two stages. The first stage is about learning basic skills and acquiring the basic knowledge of the various parts of mathematics. The second stage aims at effective usage of the first stage in solving problems, especially while training high-school or secondary school graduates. As the strongest motivation is the ability to achieve the objective, it is necessary to create mathematics so that pupils always have an objective to reach, and do not acquire the feeling that mathematics are just an amount of techniques, definitions and logical operations. "The motivating force of an equation is that it is a puzzle, a call to action. The desire to solve the puzzle is the most important moment of the whole solution process" (Hejný, 1989, p. 193).

The desire to solve a given mathematical puzzle requires above all a creative approach, using already acquired knowledge and skills. While teaching mathematics it is appropriate to encourage a creative attitude of the pupil and constantly maintain it in tension of discovery. A pupil should not be only a passive observer of the teacher's solutions showcase when going over a new thematic unit. Teacher should select the appropriate model examples and thoughtful process solutions so that the pupil can become a mathematical researcher, who discovers a new field of mathematical knowledge. Pupils should learn how to use acquired mathematical knowledge and skills effectively. Otherwise, it becomes demotivating for a pupil to be only a passive observer "dissipating" in the maze of calculations. On the basis of analysing pupils' work it can be said that the problem is not in discouraging math but in an inappropriate way of teaching mathematics. Teaching mathematics should be an invitation to think and find a solution. Probably the most difficult task of the teacher is to convince a pupil that task solving is his/her ability, even if it is a type of task still unknown. A teacher should be prepared to overcome the basic difficulties in the minds of pupils, which often arise when looking at mathematics as the amount of tasks and the procedures that must be learned to solve them. This most fundamental difficulty arises from inappropriate teaching of math and causes a pupil to be convinced that he himself would not have been able to solve the tasks. "Modern School assumes full developing of pupils' activities and their independence" (Malacká, 2015). A pupil needs a teacher, who will bring him/her into the secrets of maths and will teach him how to use particular mathematical skills already acquired. A pupil who is taught this way will acquire the mathematical confidence that he/she will need to find solutions to maths tasks.

The pupils of all types of schools often miss the links between the particular mathematical skills they have acquired during their studies of mathematics. Solving mathematical problems in many cases consists of several elementary tasks to be solved gradually until coming to the result. The particular steps of solution can be of different areas of mathematics.

Example 1: Determine the relative position of the lines *p (A, B) and q (C, D) where A [-1, -2], B [-1; 1], C [1, 1], [2, 3].*

The first elementary task is to solve the coordinates of the normal vectors of the two lines. The second elementary task is to write a general equation of defined lines. These first two tasks belong to the analytic geometry. The next task is to solve a system of equations arising from the general equations of lines. This role is a matter of algebra. The final task is to determine the relative positions of lines according to the number of solutions of systems of linear equations. It is a new understanding of the outcome of a system of two solutions of linear equations.

Such a division of task solution can be called "task atomization". Each "atom" of task solutions is already an acquired mathematical skill, which is transported to the solution of the specified tasks. Such a method of solving tasks removes the mechanical rote learning of acquired procedures of task solving as a whole. On the contrary, the described method of task solving requires, after solving one of the elemental task, to think about next solution procedure. But not the way it was solved during the maths lesson, which is basically the mechanical rote learning of acquired procedures. The task leads to thinking about what kind of task I have to solve at the moment and also the way in which it is necessary to interpret the result obtained to be the answer to the solving the task.

Operations with rational numbers belong among the basic skills in mathematics. Later the pupils learn the operations with rational expressions - an expression of unknown in the denominator. In both cases the same algorithms are used. However, the fact that pupils have mastered operations with fractions, operations with unknown in the denominator are solved with a great difficulty. Let us analyze the way of teaching these two math skills on the example of summation of rational numbers.

Before teaching the operations with rational numbers, pupils get familiar with the algorithm of finding the least common multiple of natural numbers. In this procedure decomposition on product of prime divisors is used. When adding up two fragments, both fractions are adjusted to a common denominator, which is the least common multiple of the original denominators. When adding two rational expressions the same procedure is used. Nevertheless, pupils familiar with adding the fractions have problems to learn adding the rational fractional expressions with unknown in the denominator. A detailed analysis of their thinking when fixing angle expressions showed that they miss the connection, connection with adding the fractions sufficiently. It was therefore necessary to analyze these two mathematical skills and find the missing intermediary in the pupils' thinking. When adding fractions mostly small numbers are used. In this case, the common denominator is searched using an algorithm: we will find out which multiple of larger denominator is a multiple of smaller denominator and it is then common searched denominator. When adding the angle expressions with unknown denominator, the denominators are fixed on the product and the searched common denominator is created from individual members of product. At the same product terms it is taken the term with a higher exponent. For pupils they are two separate algorithms and it is not clear to them why they are said that the rational fractional expressions add up as well as rational numbers. On the other hand, it is often not clear for a teacher why pupils consider adding the angle expressions difficult and adding the fractions easy.

The mentioned algorithms miss the connection and that is searching for the least common multiple by fragmentation on terms of prime divisors, and in the case of rational angle expressions it is about the polynomial factorisation of the root factors. The research, that was made with the pupils of elementary and secondary schools as a part of dissertation thesis, has shown significant improvement in pupil achievement when, at first, they had to solve a couple of tasks of the following type:

Example 2: Add up the fractions$\frac{31}{60} + \frac{13}{45}$ (Gonda, 2010).

Both denominators must be factored into the product of prime divisors$\frac{31}{2^2.3.5} + \frac{13}{3^2.5}$, the common denominator contains all prime divisors with a larger exponent (we use the knowledge of searching the least common multiple). Thus, the common denominator is $2^2.3^2.5 = 180$. When we compare the original denominators with the common denominator when written in the form of a product, it is clear that the numerator of the first fraction is multiplied by number 3 and the numerator of the second fraction is multiplied by $2^2 = 4$. In both cases there are the missing members of the product of the original denominators with respect to the common denominator. After such preparation we can go on to the teaching of adding up the angle expressions.

Example 3: Simplify the expression $\frac{5}{2x^2+6x} + \frac{x}{x^2-9}$.

We modify the denominators of both angle expressions into the product:$\frac{5}{2x(x+3)} + \frac{x}{(x-3)(x+3)}$.

The common denominator will be the expression containing all members of each expression, if the decomposition contains the same members, we choose the member with the higher exponent. Therefore, the common denominator is the expression $2x(x+3)(x-3)$. From a methodological point of view, it is the preferred form to $2x(x^2-9)$. Comparing the common denominator with original denominators we determine that it is necessary to multiply the numerator of the first fraction by the expression $(x-3)$ and the numerator of the second fraction by the expression $2x$.

On the base of these examples it is clear for pupils that when we add up the fractions and rational angle expressions, in principle, we follow the same procedure. The missing connecting link is just the decomposition into the product of prime divisors. These interconnecting elements do not arise in the pupils minds automatically, therefore it is necessary for the teacher to create them during the learning process. Adding up the fractions and expressions with an unknown in the denominator is not the same. We talk about analogous procedures that can be factored into the same steps – "atoms" of solving: the factorization of denominators into product, the creation of the common denominator of the product members, multiplying the old numerators by differential terms. In this case, it is still necessary transformation ability. Transforming the task of the angle expressions field into the field of operations with rational numbers. The pupil should be aware that both tasks are about adding up the fractions and intuitively assume the same mathematical rules. Then it is enough to import learned procedure from the field of rational numbers into the field of rational angle expressions.

Linking atomization and transformation of task

Developing the ability of pupils to transform the task from one field into another one of mathematics we consider to be one of the basic mathematical competences of pupils. The task transformation frequently occurs when the substitution was used in the task solution.

Example 4: On the set R solve the equation: $sin^2x - cos^2x + \sqrt{3}\sin x - 2 = 0$.

Solution: In order to have only one goniometric function (which we can then determine from the equation), we express the cosine function of the sine function according to the formula: $cos^2x = 1 - sin^2x$, so the equation after fixing will be $2sin^2x + \sqrt{3}\sin x - 3 = 0$.

After the introduction of substitution $\sin x = y$ we obtain a quadratic equation $2y^2 + \sqrt{3}y - 3 = 0$,

which roots are numbers $y_1 = \frac{\sqrt{3}}{2}$ and $y_2 = -\sqrt{3}$ y_2. After replacing into the substitution equation we receive a pair of basic goniometric equations $\sin x = \frac{\sqrt{3}}{2}$ and $\sin x = -\sqrt{3}$.

The roots of the first of them are numbers $x_1 = \frac{\pi}{3} + 2k\pi$ and $x_1 = \frac{\pi}{38} + 2k\pi$; the second equation has no root on the set R. The result of our equation is set

$$K = \bigcup_{k \in Z} \left\{\frac{\pi}{3} + 2k\pi; \frac{2}{3}\pi + 2k\pi\right\}$$

In the example above there is the standard use of a substitution. The equation was transformed from the goniometric to the quadratic by introduction of substitution. But it is possible to look at the use of substitutions from a different point of view. By introducing substitution, we imported already acquired skills of solving quadratic tasks into solving goniometric ones. And solving the equation above can be atomized. The adjustment of goniometric expressions, solving quadratic equation, solving basic goniometric equation. It is important to understand that the introduction of substitution is not the only step of task solution. We regard as “atoms”, particular steps of tasks, just basic calculating algorithms.

In example 1 two mathematical areas are automatically linked. Two general equations were created and they created a system of equations. Through testing it was established that pupils realized that they were creating a system of equations, only if the general equations were written one below the other. If the general equations were in different places of the blackboard, the discovery of the next solution step was sporadic. Just after getting the general equations of the two lines there is a place for the teacher to warn pupils that there is a connection between the two branches of mathematics, followed by a new interpretation of solving the system of equations.

Example 3 shows the need for algorithm transformation from one area to another area of mathematics. The research showed that up to 86% of pupils considered adding up the fractions and adding up the terms with unknown in the denominator as two different mathematical procedures. 94% of pupils reported that during the maths lessons they met with the fact that adding up the fractions and expressions with unknown in the denominator is basically the same. It is obvious that pupils miss the connecting element pointed out in the solutions of Example 2. Based on the research it can be recommended even at primary schools, where elementary algorithms are primarily taught, to point to possible links to solve the tasks.

In Example 3 the substitution is the interconnection element. It is a step that is made when calculating thanks to certain mathematical top view. The substitution is not a calculating algorithm that, under certain circumstances, is used in task solution, despite the fact that one can find different substitution schemes that pupils often learn by heart mechanically. The substitution is a step that results from thinking about the various phases of the task solution.

Thanks to examining the way pupils, prepare for a math test, it was found out that 78% of them calculate the whole examples at home as the preparation for the test. The rest of them only recall the basic ideas of solutions. The second group then said that they are aware of affinity of various tasks solving. The first group claimed that the preparation for the test took them a lot of time because they had to memorize a large number of procedures.

Creating a link between different parts of mathematics in the minds of the pupils is no easy matter, and for most of them, this necessary link will not be created at all. By examining their ways of thinking and dealing with the tasks we found out that pupils perceive task solving as a whole, as a calculating algorithm. In such an approach to learning mathematics, we can see a gradual deterioration of the learning outcomes in mathematics of pupils. And it is because of the following reasons. After reading the task the pupil does not think about analysis of the problem, but tries to assign the learned procedures to the task. Another problem is that a pupil learns analogous tasks as two separate ones without realizing the common parts of solving tasks. The result is a pupil who can more or less successfully deal with the same tasks as those solved during the maths lessons. There we have a pupil who memorizes offered procedures. If this is the way pupils prepare for maturita exam or entrance exams for college he or she achieves very poor results, as demonstrated in our study. If the teacher purposefully creates connections between mathematics it encourages pupils to exploratory approach the solving of tasks. This kind of pupil has no tasks solutions in mind from previous maths lessons but the amount of basic algorithms – “atoms” , from which the pupil tries to put together the solution of the problems. He or she does not solve the task as a whole, but deals with the particular parts of it, and after solving each sub-task analyzes the scope for further action. After solving a sufficient number of elementary tasks, the pupil gets the final solution to the task. If the pupil thinks in this way, he or she achieves excellent results in the study of mathematics. Teacher training is needed for this kind of teaching.

In mathematics, we often encounter the term “replace” and in its various forms: supersede or substitute. On the basis of the detailed analysis of the pupils' thinking it can be concluded that the implementation of “replacement” is not as simple a mathematical operation as it seems at first sight. The mathematical meaning of the term “substitute” is very deep and wide. The content of this notion in the minds of pupils should be build gradually. In the first stage, pupils meet the form “substitute”. The most often in geometric problems, when in the equation, for example, the calculation of the rectangle, they substitute the specified values. Even at these tasks it is appropriate to start building the mathematical content of 'substitution'. It is advised to inform pupils that by substituting they replaced the general dimensions of the rectangle with the specific numerical values. Another important step is determining the value of a rational expression for a specified values of variable s contained in the expression. At this stage it is essential that the pupils learn that all the same variables are replaced by the same numerical value. The third important step is solving the system of equations with substitution method. At this stage the pupil in fact meets for the first time a new dimension of the term replaced. When using the substitution method of solving system of equations, the unknown is replaced by term. This term is not specified in the task, but it needs to be created. It is advisable to point out the analogy of the substitution method with replacing the numerical value. The pupils should understand that, in both cases, it is a question of "replacement". The importance of this phase of the construction of the mathematical content of the term "replace" also lies in the fact that we face, perhaps for the first time, the transformation task. Solving a system of two equations with two unknowns is transformed by "replacement" to the solving of one equation with one unknown. At this stage the pupil already begins to learn transforming the assigned task to the task that he or she can deal with easily. The next step is mainly transforming nonalgebraic equations to algebraic equations. At this stage the pupils get familiar with the substitution as one of the options for solving nonalgebraic equations. Again, the principle of the substitution is the replacement of the variable or term with another term in the defined equation. The aim of substitution is either simplify the task or transform. The difficulty of this phase of the construction of the mathematical content of the "substitution" is not only a need to find substitution expression or to create it, but also to recognize the appropriateness and in many cases the necessity to solve the task by appropriate substitution.

"The ability to use appropriate substitutions is to a certain extent a demonstration of maturity of the researcher's mathematical erudition. We have to say that the method of substitution is one of the basic methods in higher mathematics" (Hejný, 1989, p.211). Reaching such an advanced mathematical erudition of pupils is not an automatic process but it is a purposeful work of teachers in teaching mathematics. Based on the research we can summarize that if the substitution is presented to pupils as a way of solving some nonalgebraic equations and inequalities, it becomes for him a term without a foundation. He or she sees the substitution as a new approach, a new algorithm that has to be learned. The aim of substitution is mainly as task simplifying, and to get the task that can be solved more easily by replacing. During the research, 81% of pupils said that using substitution is difficult for them. In fact, there is no problem in the substitution difficulty itself but in an undeveloped "substitution" thinking of pupils. The substitution occurs primarily in determinative mathematical tasks.

Developing the Ability of Transforming Task by Teaching Substitution

The determinative mathematical tasks are historically the oldest mathematical problems. While solving the mathematical concepts, methods, algorithms and entries were born. They are the tasks that require finding, calculating, constructing etc. of all mathematical objects of a specific type having the desired properties. They include classical tasks such as solving equations and inequalities and their systems, investigation set of points and constructional tasks. We can solve determinative mathematical tasks according to direct and indirect methods.

The most striking indirect method of solving determinative tasks is to transform it to another task that we can solve. Very often the substitution is used in this process. The essence of substitution is the replacement of a complex expression with another symbol. At the same time there is the simplifying or transformation to a task that the pupil already knows. The substitution difficulty is to find or create an expression that must be replaced by another symbol (Odvárko, 1990).

The following two examples illustrate the simplistic and transformation function of substitution.

Example 5: On the set R solve the equation $\sqrt{2x^2-3x+7}=4x^2-6x-1$.

Solution: At first we determine the domain of the equation. The expression under the square root must be non-negative, therefore, we deal with the inequality $2x^2 - 3x + 7 \geq 0$. The all real numbers suit this inequality. The domain of the defined equation is D = R. If we solved the equation in standard way - with squaring, we would get algebraic equation of the fourth degree. Its solution would be lengthy, for a lower grade grammar school pupil it would be insurmountable. (Irrational equations are discussed in the first year of secondary school and algebraic equations of higher degrees even in the fourth grade.) After the "discovery" of substitution $y = 2x^2 - 3x$ the equation is simplified to shape $\sqrt{y+7}=2y-1$. This equation is solvable even for 1st year pupils. At first we determine additional conditions. For each solution of the equation is true $y+7\geq 0 \wedge 2y-1\geq 0$. Therefore all the results belong to the interval $y \in \langle \frac{1}{2}; \infty)$. After rising the power and simple adjustment and we obtain quadratic equation $4y^2 - 5y - 6 = 0$, the solution of which is the number $y = 2$. After replacing in substitution equation, we have again quadratic equation $2x^2 - 3x - 2 = 0$, the results of which are $-\frac{1}{2}$ and 2. The result of the equation is set $K=\left\{-\frac{1}{2};2\right\}$.

Example 6: On the set R solve the equation $6.\log x + 1 + \log x + (1 + \log x)^2 + (1 + \log x)^3 + \ldots = 0$

Solution: First, we show the solution that we recommend, if used as a model example, while we cannot forget to remind the curriculum about logarithms. Therefore, it should not be included as the first! After adjustment we obtain:

$1 + \log x + (1 + \log x)^2 + (1 + \log x)^3 + \ldots = -6.\log x$, on the left side there is the infinite geometric series. To get the series convergent there $|1+\log x|<1$ must pay. The given condition is satisfied for $x \in (0{,}01;1)$. For the given values of the variable, the endless series on the left can be replaced by its sum and we get the equation $\frac{1+\log x}{-logx}=-6\log x$

After adjustment we solve the equation $6.log^2x-\log x-1=0$. Using the substitution $y=\log x$ we move on to the quadratic equation $6y^2-y-1=0$. Its roots are numbers: $y_1=\frac{1}{2}$ and $y_2=-\frac{1}{3}$.

After returning to the substitution equation we get a pair of roots $x_1=\sqrt{10}$ and $x_1=\sqrt{10}$ and $x_2=\frac{1}{\sqrt[3]{10}}$, while the root x_1 is not satisfactory for the convergence of an endless series. Therefore, the equation has only one root $K = \left\{\frac{1}{\sqrt[3]{10}}\right\}$.

If this example is added to the control work, we find out, that many pupils immediately get rid of logarithms and introduce the substitution in the form $y = \log x$ or $y = 1 + \log x$. In that case, the differential equation is transformed to the form

$$6y + 1 + y + (1 + y)^2 + (1 + y)^3 + \cdots = 0$$

or

$$6(y - 1) + y + y^2 + y^3 + \cdots = 0.$$

None of the obtained equations requires knowledge of logarithms any more.

In the available literature the method of substitution is defined as a transformation method for task solution. After detailed study of it, the substitution can be described as a method in the first phase consisting of the creation of the substitution expression, the replacement of the formed expression with another expression. It produces the task simplification or transformation of the solving task to the task of different areas of mathematics. In the second phase a return to the original variable through substitution relationship occurs and this causes the task to transform into the original field of mathematics again, to the field in which the task was given. The above steps of substitution can be seen as the basic substitution algorithm.

By detailed analysis and examination of solving tasks associated with the use of substitution we can also get a different perspective on the substitution and find its third attribute. If we use such a logarithmic equation when dealing with substitution, the task begins and ends as a solution of a logarithmic equation. Using the appropriate substitution can be seen as importing the most common quadratic equation in solving a logarithmic equation. As the informatics would say: the substitution calls and imports already created algorithms of task solving. Although the substitution method has its basic algorithm that pupils must learn, it is not possible to use it in rote. The reason is that it is not always possible to find out from the assignment whether the task will or will not be solved by substitution and it is not possible to clearly identify the solving phase of the task, when to use the substitution. After all, many tasks are solvable in different ways. During research we met with the fact, that if the substitution were not given sufficient attention, the pupils came to the conclusion that the substitution should be used only if the equation had the following scheme: $a(V(x))^{2n} + b(V(x))^n + c = 0$. Subsequently, the substitution $y = (V(x))^n$ can be used, which transforms a given equation to the quadratic equation. 68% of pupils could use the substitution only in the case that if the task assignment directly corresponded to the scheme. This scheme has a justification in solving problems. It is a scheme which should be considered as a possible partial result when dealing mainly nonalgebraic equations and inequalities. Thus, the proper use of substitution requires the pupil's ability to split the task solution into subtasks – to atomize the task. By gradual achievement of partial solutions, we will obtain the complete solution of a given task. The subtask, that is creating substitution relationship by targeted modifications, must be easily handled by the pupils. The reward is the simplification of the task solution. Substitution creates connections between different parts of mathematics and thus helps to view mathematics as a whole.

It teaches pupils to move from one unit to another within the same task. Since substitution requires a creative approach, finding a suitable substitution does not let the learned techniques apply automatically. But even here we can find the schemes of the process. These schemes are more instructions, that now the substitution is suitable, but the actual substitution equality should always be discovered.

To achieve the objective, in this case the efficiency of substitution, it is necessary to use the appropriate method. We consider the heuristic method the most appropriate of didactic teaching methods. The basis of the method is that pupils are actively involved in the discovery of new knowledge. They do not solve the entire problem but only the individual stages. When solving the tasks involving the substitution the discovery approach is needed. Being able to split the task solution into stages and after finishing each, find the way forward. And the next step of solution can be associated with the discovery of appropriate substitution. If pupils find more options for next solutions they should be able to choose the most efficient option. When applying the heuristic method we recommend choosing examples that provide several solution options, and we let the pupils find them. A teacher can complete other possible solutions. Choosing another way of solutions depends on the pupils. If they do not choose the most efficient method, after solving the task their way, we solve the task once again according to the most efficient way and we compare the two procedures. If it is suitable we solve the task without the substitution, in order to illustrate the effectiveness of this method.

The advantage of the heuristic method is that it allows mutual enrichment of pupils about their knowledge, ideas how to solve the task. It is therefore appropriate if someone suggests a solving procedure, we should invite him to explain his proposal. This reasoning is of great importance not only because of possible enrichment of the other pupils, but it can help the teachers get a view into the thinking of pupils. We can correct erroneous knowledge by following discussions on draft solutions. We recommend to discuss each proposal and give reasons why it is appropriate, or why the procedure is mathematically inadmissible.

In order to show the direction of the solution and its possible approach for a pupil to avoid useless procedures there are some recommended techniques. If such recommendations represent a system whereby we can come from the formulation of the problem to its solution, then we call them heuristic guides - heuristics. We realize that, in contrast to algorithmic rules, heuristics do not guarantee reaching a solution. Carrying out each sub instruction of heuristics requires activities linked by understanding their effect, as one pupil succeeds and another one does not. Heuristics do not replace professional knowledge and skills, but allow using them better in the creative process. And this is also the aim of teaching substitutions for pupils to approach the task solution creatively and at the same time learn to use acquired knowledge and skills effectively (Turek, 2008).

We recommend using the universal heuristic instruction DICRR for teaching the substitution. The authors of this general heuristic manual are Zelina and Zelinová (1990). The following examples illustrate the application of heuristics teaching substitutions.

Example 7: On the set R solve the equation $\left(\frac{2x+5}{x-1}-4\right)\left(\frac{2x+5}{x-1}+6\right)=11$.

Solution:

Step 1: D - Define the problem

Here we perform the classification of the equation and realize what is the aim of solving the equation. The classification of the equation is needed so that we can select the right solutions algorithm. Here, pupils often commit errors when they begin to solve the task without classifying the equation. They just start to solve the task purposelessly.

In our case it is the algebraic equation with unknown in the denominator, which after adjustments leads to a quadratic equation. Our goal will be to find such values of unknown x that after substituting into the given task the equation will change to the equality.

Step 2: I - Be informed about the problem

In this section, we recall the basic knowledge about dealing with a particular type of equation, what steps are needed to do in this type of equation. We make an estimate of the number of solutions. In this step we check if pupils have mastered the knowledge and skills necessary to solve the task. The unknown is in the denominator, so it is necessary to determine the conditions under which the equation makes sense. On the set R the quadratic equation has at most two solutions depending on the value of discriminant. To calculate the roots we use the relationship $x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, it is therefore necessary to modify the equation:

$ax^2+bx+c=0$.

Step 3: C - create solutions, ideas, hypotheses

The most creative part of the entire manual. The first two steps test the theoretical preparedness of the pupil. In this section it is necessary to know how to use the knowledge creatively and effectively. The teacher acts as the moderator and recorder of the proposed solutions. With suitable questions he directs and motivates pupils to seek other options. If pupils do not find all possibilities the teacher can add his own suggestions of solution. The first possible solution is to fix the expressions in brackets to a common denominator and by following equivalent adjustment to fix the equation to the desired form $ax^2+bx+c=0$.

With regard to the form of the entry, even the simplifying substitution $\frac{2x+5}{x-1}=y$ can occur. However, we want to bring pupils to the substitution $\frac{2x+5}{x-1}+1=y$. Therefore we ask if it is possible to use the substitution $\frac{2x+5}{x-1}-4=y$. Then, together with pupils, we look for similar but more effective substitution and so we will get to the above substitution.

Step 4: R - Rate ideas, solutions

In this section, we evaluate different proposals for the equation solutions. The main criteria is the solution efficiency as using any way, we will solve quadratic equations. Step by step we go through the various proposals for equation solution and try to estimate the difficulty and the length of the solutions. At the same time we try to uncover potential solution difficulties. For the time consuming way we consider fixing the expressions in brackets and then the equivalent adjustment. As the substitution $\frac{2x+5}{x-1}+1=y$ results in equation

$(y-5)(y+5)=11$,

we chose this solution way as the most appropriate one.

Step 5: R - realize selected solutions in practice

The last thing we have to do is to solve the given equation by our chosen way. Finally, we go over the solution again and pick up the parts of solutions which are useful in the future. The result of the equation

$$(y-5)(y+5)=11$$

is a pair of numbers y = ± 6. By replacing to substitution equation we solve the pair of equations $\frac{2x+5}{x-1}+1=6$ and $\frac{2x+5}{x-1}+1=-6$. The equation roots $\boldsymbol{K}=\left\{\frac{10}{3};\frac{2}{9}\right\}$ can be get by their solving.

In teaching basic algorithms needed to solve particular types of equations we use mainly the reproductive method. It is highly recommended to put emphasis on creating the link: type of equation - algorithm of solution as the use of heuristics requires certain skills that we use in the first and second step of heuristics DICRR. The pupils have a tendency to solve the task immediately by a learned algorithm. We consider this as the cause of their failure in dealing with demanding tasks and tasks with which solution they did not meet While an unknown task is often considered also that one that is formulated in different way. The heuristic approach to task solving does not offer the immediate task solution, but leads to the previous analysis of the problem and to gathering the necessary knowledge. So it leads the pupils to reflect on the initial thinking about the task and to determine the objective of the solution and then to select the path to achieve the objective. We think that the use of heuristic method for teaching substitution enhance the development of creative approach to solving equations and inequalities as well as it will provide pupils with quality conditions for solving problems in other areas of mathematics and real life.

Substitution can be considered a parallel thought process for solving the task. It's It is constant thinking during the task solution, if it is possible somehow to simplify the solution of the task or how it would be possible to import already acquired knowledge of task solutions to similar problems. It teaches the solver to think about the possible knowledge import from other areas of mathematics.

The objectives of mathematics education at primary and secondary schools can be split into two levels. The first level is to teach pupils basic mathematical algorithms used in solving mathematical tasks. The second level is to teach pupils effectively to use acquired mathematical competencies to solve more difficult tasks. The substitution requires to master objectives of first level and significantly contributes to the second level. It turns out that pupils with advanced substitution thinking perform significantly better in coping with demanding tasks.

Basic Elements of Substitution Thinking

The ultimate goal of teaching mathematics is not to teach pupils some procedures to be used according to the type of the assignment, but to teach them to look for solution of the task using already acquired mathematical knowledge and skills. It is right to teach them how to think further during task solving, not to allow them to count automatically memorizing the learned process. It is advisable to teach them to see in advance, visualize how the task solution will develop after making this or that step. A very common mistake is that pupils are thinking about how the task was solved in the classroom.

Creative approach to mathematics, and not just to mathematics, consists in finding answers to questions. Those questions should not be directed only to the past but above all to the future. Properly developed substitution thinking leads to creative task solutions, asking the right questions and looking for answers. Pupils with developed substitution thinking know that it is not important to remember the process of solutions but principles of solutions.

As essential elements of the substitution thinking may be considered:

1. The ability to split the task into elementary parts - atomize task solving. In many cases, the pupil after reading the task assignment tries to assign any of the learned ways of solving to the task, and this way leads to a final task solution. If he fails to assign the learned process of solving to the task he does not even start solving the task. The most common reason given in the questionnaire was: "We did not solve such a task in mathematics lesson." Pupils, whose substitution thinking was developed are not primarily looking for the total solution of the task but they determine the partial objectives of solutions. They realize that after reaching their stated objective of solution they will stand before a new, often easier task than was in the assignment, and, at the same time, they came closer to total task solving. Such an approach also leaves creative energy insight. They perceive the task solution as their own creative problem and at the same time they built a mathematical confidence to be able to tackle the new tasks with which they did not meet in mathematics lessons.

2. *To realize, that solving math problems might not be monothematic* - it is the ability to combine information from several parts of mathematics in solving one task. In achieving the above mentioned first level of mathematics objectives the pupils get the idea that the solutions of mathematical task are monothematic. They learn, for example, to make basic operations with rational numbers, editing rational expressions, solving linear and quadratic equations etc. To acquire the necessary basic skills, they deal with multiplicity of monothematic tasks. And at this stage of mathematics education it is recommended to remind pupils of, for example, for solving linear equations they use the knowledge and skills they have acquired when they fixed the expressions. A monothematic view of task solving is restrictive in further study of mathematics. Based on the advanced substitution thinking the pupil knows that in solving tasks the knowledge and skills from different areas of mathematics can be, and often it is also necessary, to import knowledge and skills. A non-monothematic view to solving the tasks is closely linked with the ability to atomize the task.

3. *To find effective ways of task solving*- to simplify or transform the role. While practicing numeracy a pupil often meets with the term "simplify" right in the assignment. The results of such tasks are shorter writing assignments. If there is no word "simplify" in the assignment, the pupils often "forget" to simplify during task solving. They do not realize that simplify means to create a mathematical object with which the work is more effective. The pupils with advanced substitution thinking approach solving the task creatively and try to

optimize the task solution, to find the easiest way to solve the specified task. Using the substitutions is a powerful tool in optimizing the task solution.

4. *To identify the primary problem of the phase of task* solution and then solve it. In applying this element of the substitution thinking the pupil solely focuses on the problem that he set. Subsequently, the pupil faces the new task, from which the difficulty was "removed". It was the difficulty that impeded on the way to an overall task solution. Such solution phasing splits it into the autonomous several stages, which, in many cases, might be atomized. With this attribute of the substitution thinking the pupil is able to determine his own goal, which is usually different from the objective to be achieved by the assignment.

5. Being able to import the knowledge and skills from different areas of mathematics in solving problems - pass on the basic idea of solutions to different type of problem. The pupil who has got this mathematical competence is able to solve mathematical tasks easily. The pupil is not limited by barriers of mathematics, to which the task according to assignment belongs. He realizes that in mathematics we often meet with different names for the same calculating operations. The pupil with developed substitution thinking understands, for example, that to look for intersections of the quadratic function with the coordinate axis x means to solve the quadratic equations. And the obtained roots are x-coordinates of the searched intersections. Many maths problems can be effectively solved by such a mathematical detached view, which assumes the combination of different areas of mathematics in the mind of the pupil.

6. *To choose the most effective task solution among several options*. The purpose of introducing the substitution is to simplify the task, make the solution more efficient. Often the introduction of the substitution is not required, and it is up to the choice of the solver whether or not the substitution will be used. The pupils who are taught to prefer the substitution, know, that thanks to the substitution they can simplify the task, and they are supposed that the substitution algorithm is perfectly mastered by them. During lengthy calculations the question emerges in the mind of the pupils: "Would it not be possible to solve the task in an easier way?" They are used to searching and creating not only count mechanically. Then, they often find new, more effective solutions. They can even surprise the teacher with these solutions. Many of them can imagine in their mind where the particular paths of the solutions lead. This idea makes it easier to select another process of the task solution. Thanks to this advanced mathematical imagination, the pupil is competent to see the substitution expression before it is actually created by modifications.

There we can see the illustration of the pupil with advanced substitution thinking to the task solving here. In the maths lesson, when teaching the inequalities with an unknown in the denominator, the following example of the inequality $\frac{x+3}{x-4} \leq 0$ will be solved. The pupils will learn how to solve the inequality in quotient form by using the method of zero points. The pupil with advanced substitution thinking can successfully solve even the following inequalities: $\frac{2x+3}{x-8} \leq 4$, $\frac{5x+3}{3x-4} > \frac{5-x}{3x-1}$. He is used to thinking about the task assignment and looking for ways of solutions, not just assign the procedure to the assignment. He realizes that the simplest way is to compare fractions with zero, because then he just has to know the final sign of a fraction and that is, in fact, the method of zero points. On the basis of these considerations, it is clear for him to nullify one of the inequality sides. So he changed the task to the identical one with sample example from the maths lesson. If he meets the inequality in the product form $(x^2-2x+3)(x^2+8x+15)>0$, he realizes that, in fact, he solves the same problem as in the inequality in the quotient form. Even the product is good to compare with zero and the determining is the final sign of the product. So, analogously, he will use the method of zero points.

When the pupil with advanced substitution thinking is supposed to solve the inequality $(x^2-x-1)(x^2-x-7)<-5$, while discovering the solutions, there will take place the sequence of following considerations in his mind. Firstly, he thinks about annulling the right side and about the following use of the method of zero points. Without the multiplication of brackets of the left side he finds out that the absolute term of the polynomial would be 7 and after nullifying the right side he, at the secondary school level, will fail to fix polynomial to product. Then it is obvious for him that the multiplication of brackets, where the unsolvable fourth degree polynomial will be produced, is not the way to solve the inequalities. Other considerations suggest the possibility of simplifying the tasks and he knows that the way to simplification is the way of appropriate substitution. The pupil will realize that if there are the polynomials of first degree in the brackets, after removing the brackets the quadratic inequality will be produced and the pupil knows its solution. These considerations will lead him to the introduction of substitution e.g.: $x^2-x=y$, that opens for him the way of the solution of the inequality. In solving this task, the pupil had to identify the primary problem and then solve it effectively.

At teaching non-algebraic equations and inequalities, the pupils learn that one possible solution is to use the substitutions when there is an expression and its second power in the assignment. The logarithmic equation $\log_4 x^3-\frac{4}{\log_4 x^2}=8$ does not fit into the scheme. It is possible to fix the equation to the learned scheme by simple adjustments and then use the substitution. The pupil with developed substitution thinking is able to use the substitution immediately. He can mathematically see that

$$\log_4 x^3 = 3\log_4 x \text{ and similarly } \log_4 x^2 = 2\log_4 x$$

hold on the basis of one of the logarithm sentences. Based on this “seeing” fixing he will introduce the substitution $\log_4 x = y$, and so he transforms the task into algebraic equation, which solution he already knows. The way of solution of this example shows the diversity of pupils' access to the solution of a new type of task. Most pupils matching the learned procedures try to remove the fraction as the first step. They miss the need to create equal logarithms and they face the problem what to do with the term $\log_4 x^2 . \log_4 x^3$. The solver, that identifies the problem of the inequalities of logarithms in the equation, will solve it, becomes a successful solver, because he was able to find another type of task solution. The first type of pupil solves the problem of how to multiply logarithms with different arguments. It is a problem that he created himself and its solution does not exist.

If the pupil succeeds in fully developing the substitution thinking, he gets the upper perspective in solving mathematical tasks. He is aware of the fact that on the basis of the task solution in the maths lesson, he can solve many other tasks with the right approach. Developed substitution thinking is not just the use of the substitution; it is basically a way of thinking, of how to find the ways to solution of the specified tasks. This way of thinking is based on mathematical freedom. The pupil realizes that when dealing with the task he can make any mathematically permissible operation that will allow him to achieve the partial objective that he set. In many cases he does not think about the direct solution of the task, but about its possible transformation into another task which solution he already knows. In fact, he thinks how to come to the solution of the known task and the substitution does not have to be always used. The substitution thinking goes beyond the substitution and can be considered as an important element of developed logical thinking. Therefore, the development of purposeful substitution thinking in pupils is a great contribution to the achievement of their high mathematical erudition.

Disclosure statement

No potential conflict of interest was reported by the authors.

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